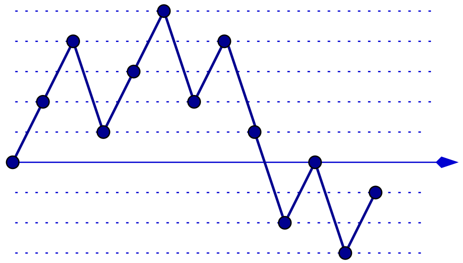
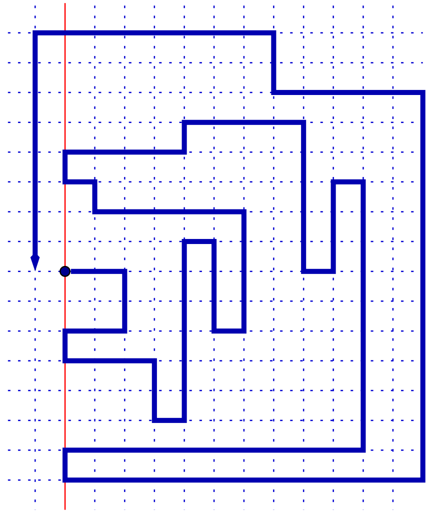
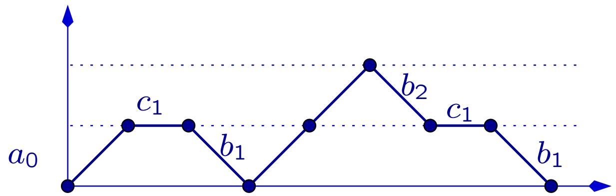
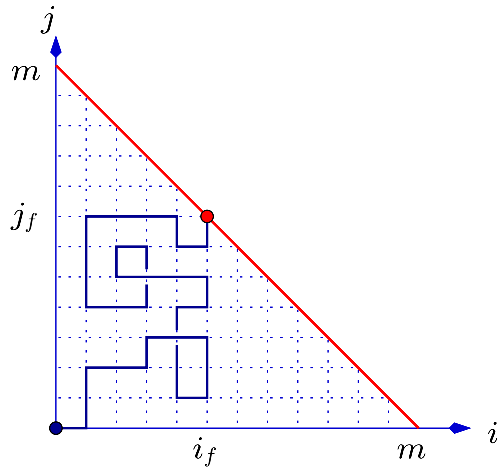
Lattice walks everywhere





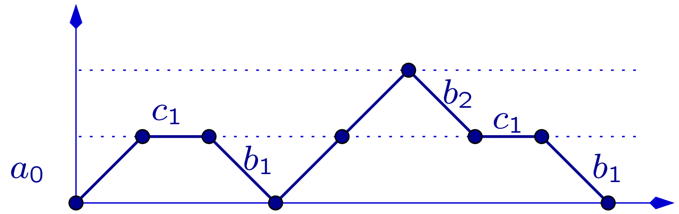




Mireille Bousquet-Mélou, CNRS, LaBRI, U. Bordeaux 1, France

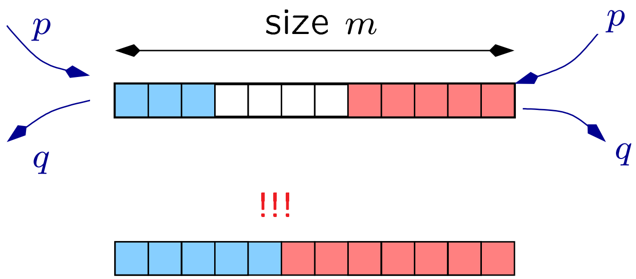
1. Continued fractions [1980–]

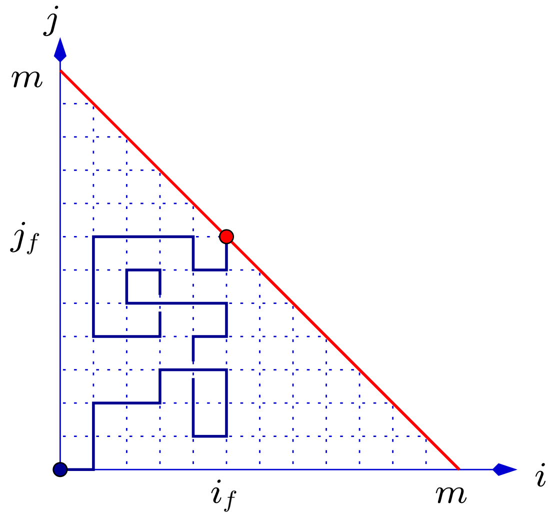
$$\cfrac{1}{1 - c_0 - \cfrac{a_0 b_1}{1 - c_1 - \cfrac{a_1 b_2}{1 - \cdots}}}$$



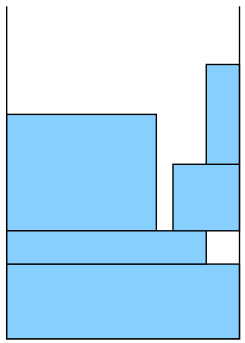
2. A storage allocation scheme [1986]

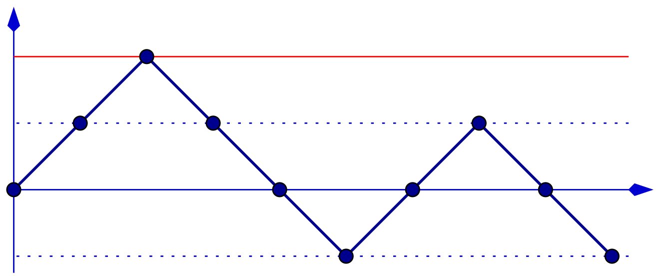
$2(p+q) = 1$





3. A packing problem [1998], with Coffmann, Flatto and Hofri





Two papers

1. Basic analytic combinatorics of directed lattice paths (with C. Banderier, 2002)

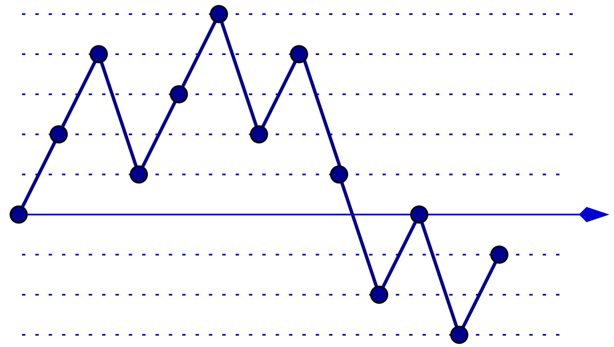
2. The enumeration of prudent polygons by area and its unusual asymptotics (with N. Beaton and T. Guttmann, 2011)

I. Basic analytic combinatorics of directed lattice paths

(with C. Banderier, 2002)

The problem

• Let $\mathcal{S} \subset \mathbb{Z}$ be a finite subset of steps.
Consider walks on $\mathbb{Z}$, starting from 0, that take their steps in $\mathcal{S}$.



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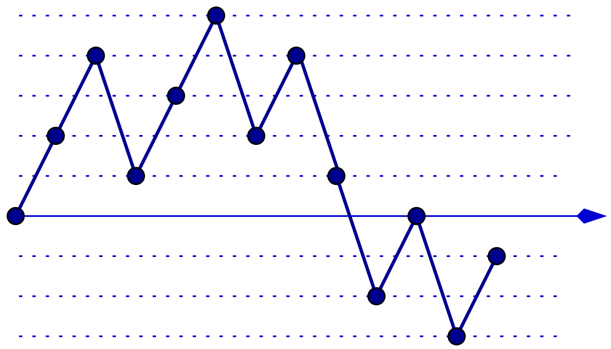
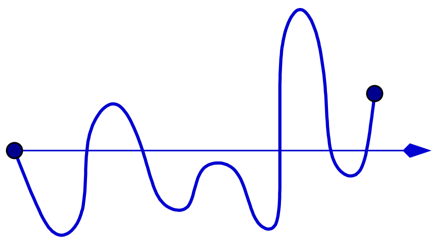


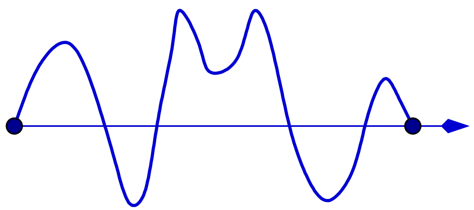
• What is the generating function of...

general walks?



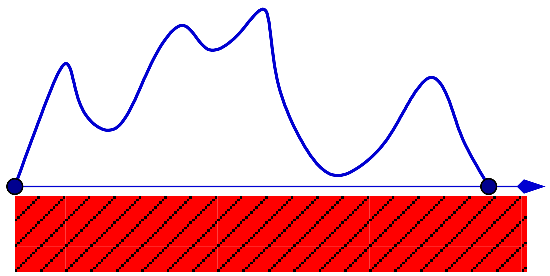
$W(z,u)$

bridges?



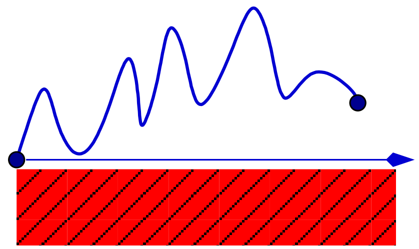
$B(z)$

excursions?



$E(z)$

meanders?



$M(z,u)$

z: the number of steps, or length; u: the height of the final point

Results

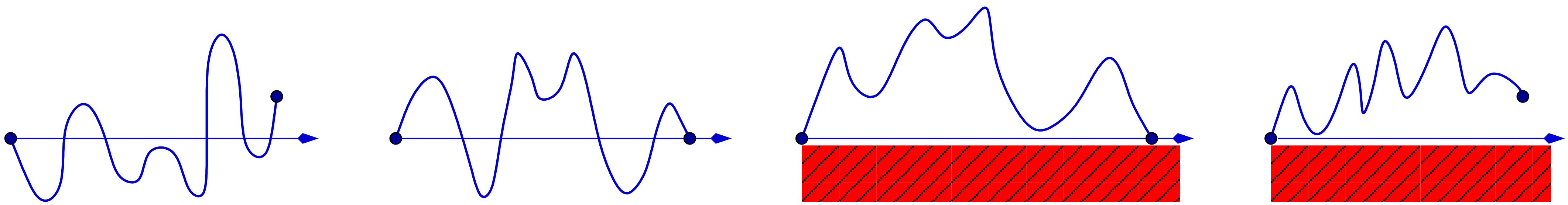
1. Exact expressions of the series [Gessel 80]

+ [MBM-Petkovšek 00], [Banderier, MBM, Denise, Flajolet, Gardy, Gouyou-Beauchamps 02]

2. Uniform asymptotic results

2. Uniform limit laws

general walks | bridges | excursions | meanders



$W(z,u)$ | $B(z)$ | $E(z)$ | $M(z,u)$

Exact expressions of the series: general walks and bridges

Let $-m = \min \mathcal{S}$ and $M = \max \mathcal{S}$, and let

$$P(u) = \sum_{i \in \mathcal{S}} \omega_i u^i = \sum_{i=-m}^{M} \omega_i u^i$$

be the generating polynomial of (possibly weighted) steps.

- The generating function of general walks is

$$W(z,u) = \frac{1}{1 - zP(u)}.$$

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- The coefficient of u^0 in $W(z,u)$ counts bridges. It can be obtained from a partial fraction expansion in u:

$$B(z) = z \sum_{i=1}^{m} \frac{U_i'(z)}{U_i(z)},$$

where $U_1(z), \dots, U_m(z)$ are the m solutions of $1 - zP(U) = 0$ that are finite at $z = 0$.

Exact expressions of the series: excursions and meanders

- The generating function of excursions is

$$E(z) = \frac{(-1)^{m-1}}{z\omega_{-m}} \prod_{i=1}^{m} U_i(z)$$

where $U_1(z), \ldots, U_m(z)$ are the m solutions of $1 - zP(U) = 0$ that are finite at $z = 0$.

- More generally, the generating function of meanders is

$$M(z,u) = \frac{\prod_{i=1}^{m}(u - U_i(z))}{u^m(1 - zP(u))}.$$

Proof: a functional equation and the kernel method

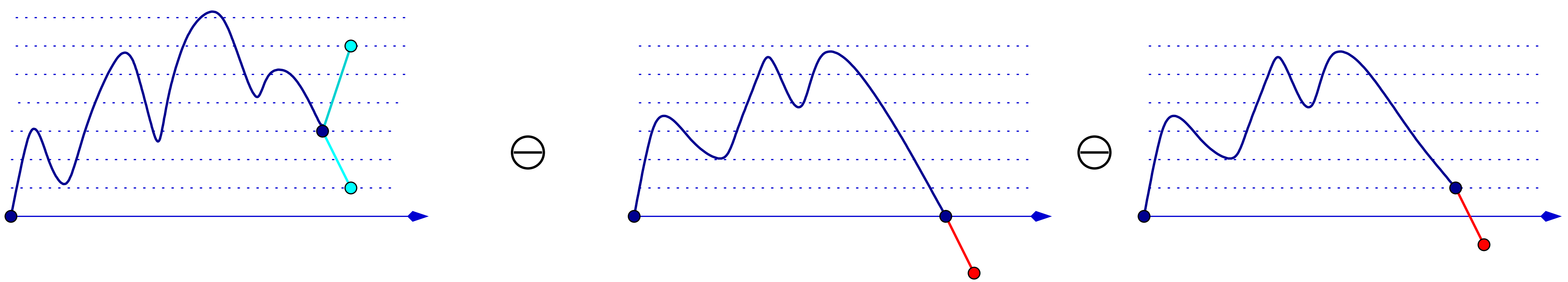
- Step-by-step construction of meanders: If $\mathcal{S} = \{-2, 3\}$, then

$$M(z,u) \quad = \quad 1 \quad + \quad z(u^3 + u^{-2})M(z,u) \quad - \quad z\left(M_0(z)u^{-2} + M_1(z)u^{-1}\right)$$

add an arbitrary step...

... but if the final level is 0 or 1

one should not add a down step



Proof: a functional equation and the kernel method

Equivalently,

$$u^2\left(1 - z(u^3 + u^{-2})\right) M(z,u) = u^2 - zM_0(z) - zM_1(z)u$$

• The left-hand side vanishes when $u = U_1(z)$ and $u = U_2(z)$, where U_1 and U_2 are the two roots of

$$1 - z(u^3 + u^{-2}) = 0$$

that are finite when $z = 0$.

• The right-hand side is a polynomial in u of degree 2, leading coefficient 1, that cancels when $u = U_1(z)$ and $u = U_2(z)$. Hence

$$u^2\left(1 - z(u^3 + u^{-2})\right) M(z,u) = (u - U_1(z))(u - U_2(z))$$

and the expression

$$M(z,u) = \frac{(u - U_1(z))(u - U_2(z))}{u^2(1 - z(u^3 + u^{-2}))}$$

follows.

Two stowaways

- A bijection shows that

$$B(z) = 1 + z\frac{E'(z)}{E(z)}.$$

Hence

$$B(z) = z\sum_{i=1}^{m} \frac{U_i'(z)}{U_i(z)} \quad \Rightarrow \quad E(z) = \frac{\text{cst}}{z}\prod_{i=1}^{m} U_i(z).$$

(The constant is the easily determined using $E(0) = 1$.)

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• An algorithm, based on symmetric functions manipulations, computes an algebraic equation for the excursion generating function

$$E(z) = \frac{(-1)^{m-1}}{z}\prod_{i=1}^{m} U_i(z)$$

In plethystic notation, it expresses the symmetric functions $e_j[e_m]$ in terms of the e_i's.

The platypus algorithm

Asymptotics for bridges and excursions

Theorem. Let $\tau > 0$ be the unique solution of $P'(\tau) = 0$, where P is the generating polynomial of steps.

Then the numbers of bridges and excursions of length n behave asymptotically as

$$\begin{aligned} B_n &= P(\tau)^n \, n^{-1/2} \left(b_0 + \frac{b_1}{n} + \cdots \right) \\ E_n &= P(\tau)^n \, n^{-3/2} \left(e_0 + \frac{e_1}{n} + \cdots \right) \end{aligned}$$

Proof: the saddle-point method for bridges, since

$$B_n = \frac{1}{2i\pi} \int_{\mathcal{C}} P(u)^n \frac{du}{u}$$

As a by-product, this dictates the singular behaviour of the series $U_i(z)$, from which one derives the singular behaviour of $E(z)$.

Asymptotics for general walks and meanders

Theorem. Let $\tau > 0$ be the unique solution of $P'(\tau) = 0$, where P is the generating polynomial of steps.

• The numbers of walks is

$$W_n = P(1)^n$$

• The asymptotic behaviour of the number of meanders (= non-negative walks) depends on the drift

$$\delta = P'(1)$$

⋆ If $\delta > 0$, a positive fraction of walks are meanders:

$$M_n = \kappa P(1)^n + P(\tau)^n n^{-3/2} \left(m_0 + \frac{m_1}{n} + \cdots \right)$$

⋆ If $\delta < 0$,

$$M_n = P(\tau)^n n^{-3/2} \left(m_0 + \frac{m_1}{n} + \cdots \right)$$

⋆ If $\delta = 0$,

$$M_n = P(\tau)^n n^{-1/2} \left(m_0 + \frac{m_1}{n} + \cdots \right)$$

Limit laws for basic parameters

- The number of contacts of an excursion with the x-axis (discrete limit law)

- The position Y_n of the endpoint of a meander

  ⋆ If $\delta > 0$,

$$\frac{Y_n - \mu n}{\sqrt{n}} \to \text{Gaussian}$$

  ⋆ If $\delta < 0$, discrete limit law

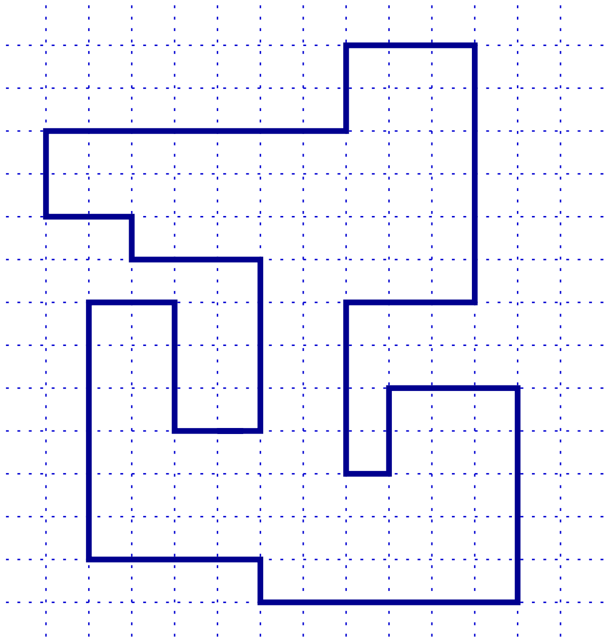
  ⋆ If $\delta = 0$,

$$\frac{Y_n}{\sqrt{n}} \to \text{Rayleigh}$$

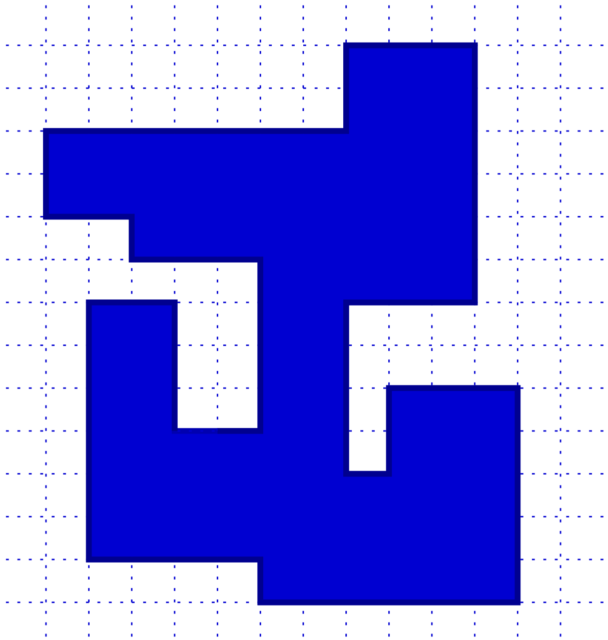
II. The enumeration of prudent polygons by area, and its unusual asymptotics

(with N. Beaton and T. Guttmann, 2011)

Self-avoiding polygons



Self-avoiding polygons



- Awfully hard to count

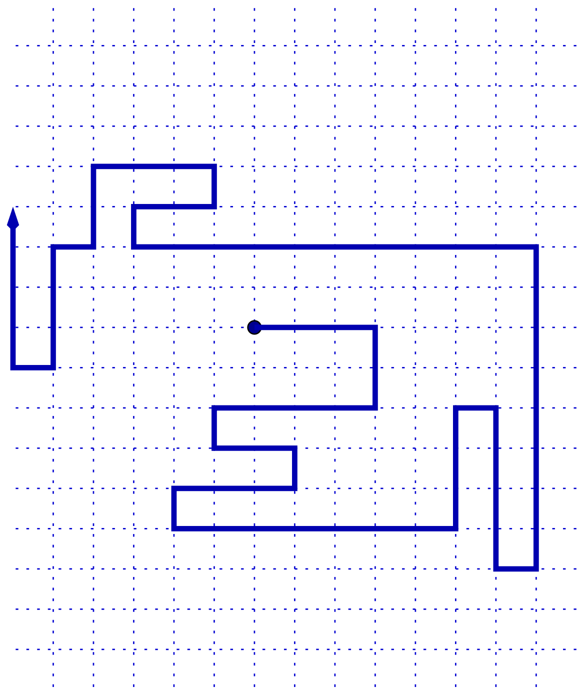
- Conjectured asymptotics: for polygons of area n,

$$p_n \sim \kappa \mu^n n^0,$$

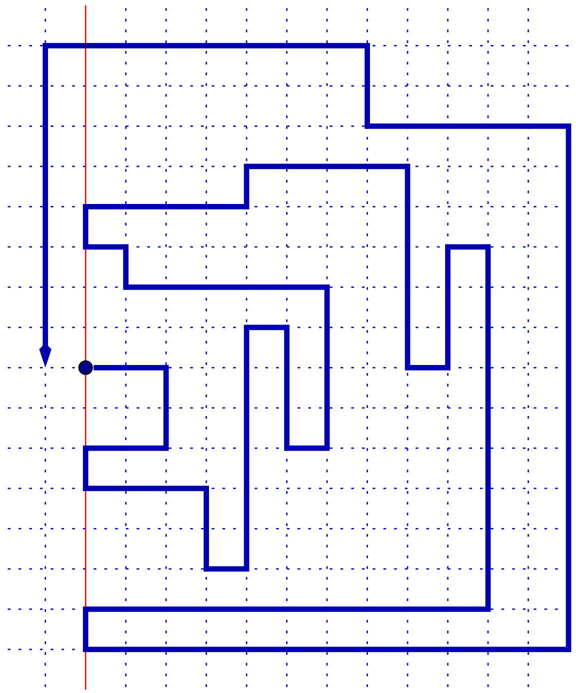
where the exponent depends only on the dimension

An easier family: 3-sided prudent polygons

Prudent walks: a step never points towards a vertex it has already visited



Prudent walk



Prudent polygon

[Turban-Debierre 86], [Préa 97], [Santra-Seitz-Klein 01], [Duchi 05], [Dethridge, Guttmann, Jensen 07], [mbm 08], [Beffara, Friedli, Velenik 10]...

An easier family: 3-sided prudent polygons

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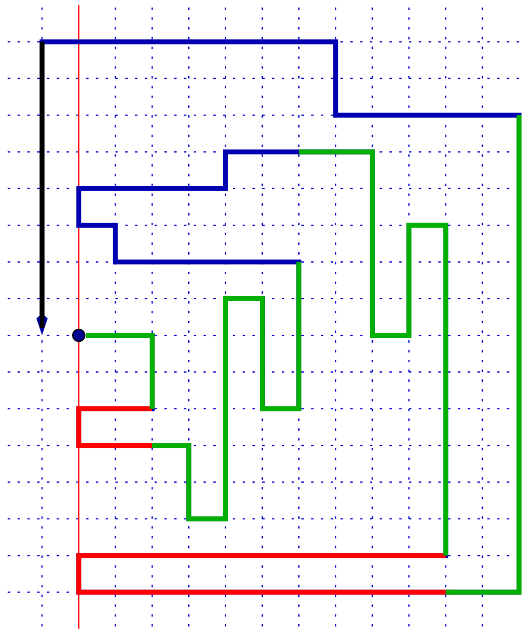

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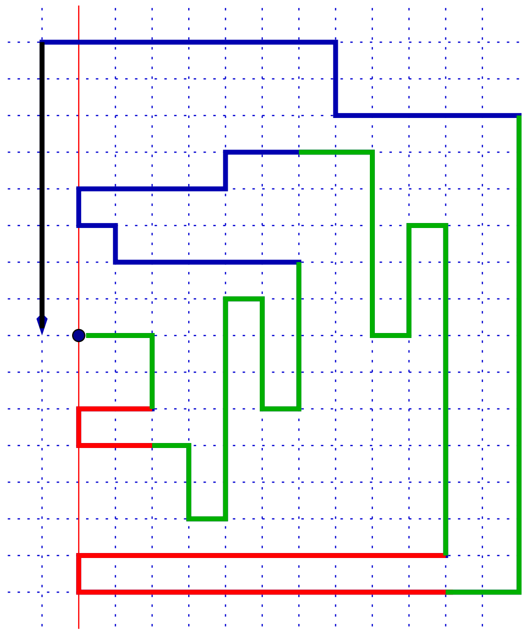
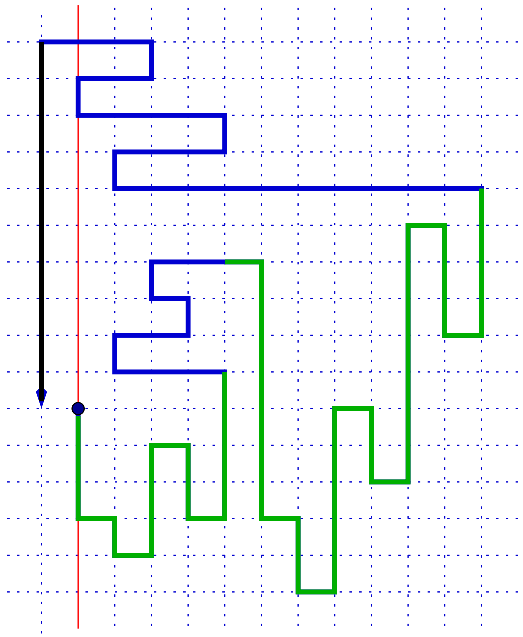


Prudent polygon

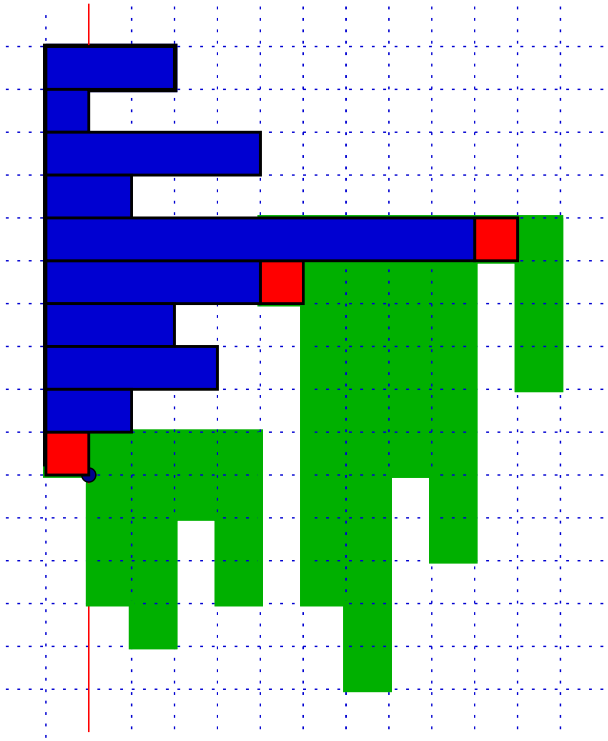


Three-sided prudent polygon

An easier family: 3-sided prudent polygons

Three-sided prudent polygons decompose into bars, bargraphs, and separating cells [...] and their area generating function is

$$P(q) = \frac{2q(3-10q+9q^2-q^3)}{(1-2q)^2(1-q)} + \frac{2q^3(1-q)^2}{(1-2q)^2} \sum_{m\geq 1} \frac{(-1)^{m+1}q^{2m}}{(1-2q)^m(1-q-q^{m+1})} \prod_{k=1}^{m-1} \frac{1-q-q^k+q^{k+1}-q^{k+2}}{1-q-q^{k+1}}$$



Asymptotics

- The series:

$$P(q) = \frac{2q(3-10q+9q^2-q^3)}{(1-2q)^2(1-q)} + \frac{2q^3(1-q)^2}{(1-2q)^2} \sum_{m\geq 1} \frac{(-1)^{m+1}q^{2m}}{(1-2q)^m(1-q-q^{m+1})} \prod_{k=1}^{m-1} \frac{1-q-q^k+q^{k+1}-q^{k+2}}{1-q-q^{k+1}}$$

- A beautiful singularity analysis yields a very unusual asymptotic behaviour in the study of lattice models:

$$\boxed{P_n = (\kappa_0 + \kappa(\log_2 n))2^n n^\gamma + O(\log n\, 2^n n^{\gamma-1})}$$

where the exponent γ is irrational

$$\gamma = \log_2 3$$

and $\kappa(x)$ is a (small) periodic function of x.

Remark. Similar to (but more complex than) the analysis of the expected longest run in a random binary sequence, where the key series is

$$(1-q) \sum_{k\geq 0} \frac{q^k}{1-2q+q^{k+1}}.$$