Planar maps and planar configurations
Philippe Flajolet and Analytic Combinatorics

Marc Noy (UPC Barcelona)

Random Maps, Coalescing Saddles, Singularity Analysis, and Airy Phenomena 2001

Analytic combinatorics of non-crossing configurations 1999

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The first cycles in an evolving graph 1988
Flajolet, Knuth, Pittel

Random mapping statistics 1990
Flajolet, Odlyzko

Random Maps, Coalescing Saddles, Singularity Analysis, and Airy Phenomena

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Received 10 March 2001; accepted 1 August 2001

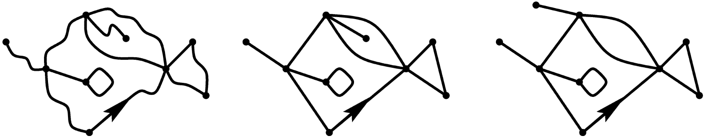

Fig. 1. Three representations of maps. The first two are identical as maps, while the third one is not, although the three underlying planar graphs are identical.

sented in a largely self-contained way (see, e.g., [29, 43] for more). It is intended as a preparation of the technical treatment in the rest of the article. The two basic ingredients introduced concurrently here are: (i) exact power representations for map counts (via the Lagrangean framework) that are to be later exploited by the saddle-point method in Sections 3 and 4; (ii) singularity analysis, which provides direct asymptotic estimates, and is extended in Sections 5 and 6 as well as Appendix A.

A *map* is an embedding of a connected *planar* graph in the sphere, considered upto orientation preserving homeomorphisms. By construction, the complement of the vertices and edges of a map in the sphere is a union of simply connected *faces*. In general, loops and multiple edges are allowed. A map is completely characterized by its underlying graph together with a cyclical ordering of edges around each vertex. Following Tutte [48, 49], we consider *rooted* maps, that is, maps with an oriented edge called the *root*—this simplifies the analysis without essentially affecting statistical properties (see [42] and Section 6). To represent maps on the plane, a point of the sphere must be placed at infinity; by convention, we always choose it so

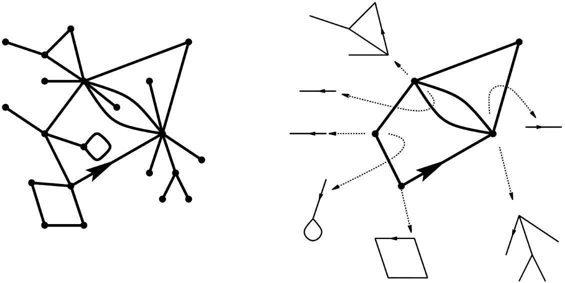

Fig. 2. The decomposition of a map into its nonseparable core and the pending submaps.

dom (connected) maps. The paradigm that we illustrate by a particular example is in fact, of considerable generality as can be seen from Section 6.

2.1. The Physics of Maps

From earlier works [7, 27, 43], it is known that a random map of $\mathscr{M}_n$ has with high probability a core that is either "very small" (roughly of size $k = O(1)$) or "very large" (being $\Theta(n)$). The probability distribution $\Pr(X_n = k)$ thus has two distinct modes. The small region (say $k = o(n)$) has been well-quantified by previous authors, see [7, 27, 43]: a fraction $p_s = \frac{2}{3}$ of the probability mass is concentrated there. The large region is also known from these authors to have probability mass

Question

What is the size X_n of the largest 2-connected component in a random map with n edges?

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Tutte 1960

The number of (rooted) maps with n edges is

$$M_n = \frac{2 \cdot 3^n}{(n+2)(n+1)} \binom{2n}{n}$$

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Gao, Wormald 1999

$$\mathbf{P}(|X_n - n/3| < n^{2/3+\epsilon}) \to 1 \qquad \text{as } n \to \infty$$

The second largest component is almost surely of size $O(n^{2/3+\epsilon})$

Similar result for 3-connected components in 2-connected maps, 4-connected components in 3-connected triangulations...

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Proof Estimating coefficients of high powers of gen. functions

$$[z^n]G(z)^j \qquad j \approx \beta n$$

using Cauchy integrals

Philippe's dictum

Strive to find limit distributions

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Strive to find limit distributions

Question

Which is the limit distribution for the size of the largest 2-connected component in a random map?

Core of a map: 2-connected component containing the root

Two regimes

- Core is small $\rightarrow$ Discrete law
- Core is large $\rightarrow$ Continuous law

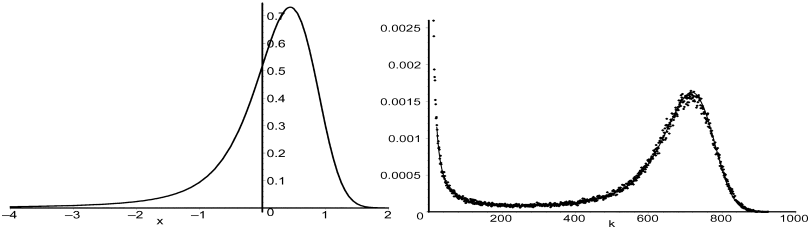

Fig. 3. Left: The standard Airy distribution. Right: Observed frequencies of core-sizes $k \in [20; 1000]$ in 50,000 random maps of size 2,000, showing the bimodal character of the distribution.

Major properties of the function $\mathscr{A}(x)$ (including the equivalence between the two definitions of (2)) are gathered in Appendix 7. The Airy distribution[2] is a probability distribution, i.e., $\int_{\mathbb{R}} \mathscr{A}(x)\, dx = 1$, and an unusual feature is the fact that the tails are extremely asymmetric:

$$\mathscr{A}(x) \underset{x\to-\infty}{\sim} \frac{1}{4\sqrt{\pi}}|x|^{-5/2} \quad \text{and} \quad \mathscr{A}(x) \underset{x\to\infty}{\sim} \frac{2}{\sqrt{\pi}}\, x^{1/2} \exp\left(-\frac{4}{3}x^3\right). \tag{3}$$

A plot of the map-Airy distribution is presented in Figure 3 (left).

We shall find that the size of the core (when conditioned upon the large region) and the size of the largest 2-connected component of a random map are described asymptotically by an Airy law of this type. Figure 3 (right) exemplifies this with simulation results of core-size: the "bimodal" character of the combinatorial distribution is clearly visible and the convergence of simulation data to the limit Airy distribution curve is already excellent at size $n = 2{,}000$. (Additional simulation data

Composition scheme

$$M(z) = \sum M_n z^n \qquad C(z) = \sum C_k z^k$$

$$M(z) = C(z(1 + M(z))^2) = C(H(z)) \qquad H(z) = (1 + M(z))^2$$

$$M(z) = \sum C_k H(z)^k$$

The number of maps of size n with core-size k is

$$[z^n]H(z)^k$$

$$M(z) = \sum M_n z^n \qquad C(z) = \sum C_k z^k$$

Singularities

$$\rho_M = 1/12 \qquad \rho_C = 4/27$$

Singular expansion

$$M(z) = M_0 + M_2(1 - 12z) + M_3(1 - 12z)^{3/2} + \cdots$$

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Singularities

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Singular expansion

$$M(z) = M_0 + M_2(1 - 12z) + M_3(1 - 12z)^{3/2} + \cdots$$

Critical composition scheme

$$H(\rho_H) = \rho_M$$

Coalescence of singularities $\rightarrow$ non-Gaussian limit law

k:	Left tail $[\epsilon n, (\alpha_0 - \epsilon)n]$	Centre $\alpha_0 n$	Right Tail $[(\alpha_0 + \epsilon)n, (1-\epsilon)n]$
Saddle points:	$\tau < \tau'$	$\tau = \tau'$	$\tau > \tau'$
Method:	simple saddle point (Section 2.1)	double saddle point (Section 2.2)	simple saddle point (Section 2.1)
Type:	$\int e^{-t^2}\,dt$	$\int te^{-t^3}\,dt$	$\int e^{-t^2}\,dt$
Angle:	$\pm\frac{\pi}{2}$	$\pm\frac{2\pi}{3}$	$\pm\frac{\pi}{2}$
Error:	$n^{-1/2}$	$n^{-1/3+\epsilon}$	$n^{-1/2}$

k:	Central region $[\alpha_0 n + an^{2/3},\ \alpha_0 n + bn^{2/3}]$	"Wide" region $[\epsilon n,\ (1-\epsilon)n]$
Saddle points:	$\tau' \approx \tau$	
Method:	nearby saddle points (Section 2.3)	coalescing saddle points (Section 3)
Type:	$\int (x-t)e^{\,xt-t^3/3}dt$	$\int (x-t)e^{\,xt-t^3/3}dt$
Angle:	$\pm\frac{2\pi}{3}$	$\rightarrow$ cubic curve
Error:	$n^{-1/3+\epsilon}$	$n^{-1/3}$

Fig. 5. Top: A broad classification of the methods involved in the classification of tails and center of the core-size distribution. Bottom: Refinements of the saddle-point method applicable to the critical region of the law of core-size.

ensuring a complete local capture of the contribution as well as validity of the quadratic approximation. Here, we adopt $\delta = \log n/\sqrt{n}$

Theorem

For a critical composition scheme

$$\mathcal{M} = \mathcal{C} \circ \mathcal{H} \qquad \text{of type } \frac{3}{2} \circ \frac{3}{2}$$

the distribution of core-size X_n has three asymptotic regimes

Theorem

For a critical composition scheme

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the distribution of core-size X_n has three asymptotic regimes

1. Left region $k = \alpha n$, $\alpha < \alpha_0$

$$\mathbf{P}(X_n = k) \sim c k^{-3/2}$$

2. Central region

$$\mathbf{P}(X_n = \alpha_0 n + x n^{2/3}) \sim n^{-2/3} f(x) \quad \text{scaled Airy law}$$

3. Right region $k = \alpha n$, $\alpha > \alpha_0$

$$\mathbf{P}(X_n = k) = O(A^k), \; A < 1$$

Corollary

The size of the largest component in many varieties of planar maps is asymptotically Airy distributed around $\alpha_0 n$

TABLE 4 Parameters of the composition schemas of Table 3

Maps	Cores	α_0	c	p_ℓ
General, $\mathcal{M}_1$	bridge/loopless, $\mathcal{M}_2$	2/3	$3/2$	2/3
Loopless, $\mathcal{M}_2$	simple, $\mathcal{M}_3$	2/3	$3^{4/3}/4$	2/3
General, $\mathcal{M}_1$	nonseparable, $\mathcal{M}_4$	1/3	$3/4^{2/3}$	1/3
Nonsep., $\mathcal{M}_4$	nonsep. simple, $\mathcal{M}_5$	4/5	$15^{5/3}/36$	4/5
Nonsep., $\mathcal{M}_4$	3-connected, $\mathcal{M}_6$	1/3	$3^{4/3}/4$	16/81
Bipartite, $\mathcal{B}_1$	bip. simple, $\mathcal{B}_2$	5/9	$3^{8/3}/20$	5/9
Bipartite, $\mathcal{B}_1$	bip. bridgeless, $\mathcal{B}_3$	3/5	$(15/2)^{5/3}/18$	3/5
Bipartite, $\mathcal{B}_1$	bip. nonsep., $\mathcal{B}_4$	5/13	$(13/6)^{5/3}\cdot 3/10$	5/13
Bip. nonsep., $\mathcal{B}_4$	bip. nonsep. simple, $\mathcal{B}_5$	5/17	$(17/3)^{5/3}\cdot 3/20$	5/17
Singular tri., $\mathcal{T}_1$	triangulations, $\mathcal{T}_2$	1/2	$(3/2)^{1/3}$	1/2
Triangulations, $\mathcal{T}_2$	irreducible tri., $\mathcal{T}_3$	1/2	$6^{2/3}/3$	729/2048

last schema involves a slight adaptation but clearly resorts to a similar analysis.) In addition, as shown by Table 2, all families of Table 1 obey the Lagrangean framework, Eq. (4), and are thus amenable to the saddle-point methods of Sections 3 and 4 as well.

Theorem 6 (Airy law for varieties of maps). *Consider any schema of Table 4 with parameters α_0, c, and p_ℓ. The probability $\Pr(X_n = k)$ that a map of size n has a core of size k admits a local limit law of the map-Airy type with centering constant α_0, scaling parameter c, and weight p_ℓ: uniformly for x in a bounded interval*

$$\ldots c\mathcal{A}(cx) \ldots$$

TABLE 1 A selection of classical families together with their associated generating functions, $M(z) = \sum_{n\geq 1} M_n z^n$, where M_n is the number of maps in $\mathcal{M}$ that have size n

Maps, size $n \geq 1$	Generating function (first terms)
$\mathcal{M}_1$ general maps, n edges	$M_1(z) = 2z + 9z^2 + 54z^3 + 378z^4 + 2916z^5$
$\mathcal{M}_2$ bridgeless maps, n edges	$M_2(z) = z + 3z^2 + 13z^3 + 68z^4 + 399z^5$
$\mathcal{M}_2$ loopless maps, n edges	$M_2(z) = z + 3z^2 + 13z^3 + 68z^4 + 399z^5$
$\mathcal{M}_3$ simple maps, n edges	$M_3(z) = z + 2z^2 + 6z^3 + 23z^4 + 103z^5$
$\mathcal{M}_4$ nonseparable maps, n edges	$M_4(z) = 2z + z^2 + 2z^3 + 6z^4 + 22z^5 + 91z^6$
$\mathcal{M}_5$ nonseparable simple maps, n edges	$M_5(z) = z + z^3 + z^4 + 6z^5 + 16z^6 + 71z^7$
$\mathcal{M}_6$ 3-connected maps, $n+1$ edges	$M_6(z) = z^5 + 4z^7 + 6z^8 + 24z^9 + 66z^{10}$
$\mathcal{B}_1$ bipartite maps, n edges	$B_1(z) = z + 3z^2 + 12z^3 + 56z^4 + 288z^5$
$\mathcal{B}_2$ bip. simple maps, n edges	$B_2(z) = z + 2z^2 + 5z^3 + 15z^4 + 52z^5$
$\mathcal{B}_3$ bip. bridgeless maps, n edges	$B_3(z) = z^2 + z^3 + 6z^4 + 16z^5 + 71z^6$
$\mathcal{B}_4$ bip. nonseparable maps, n edges	$B_4(z) = z + z^2 + z^3 + 2z^4 + 6z^5 + 19z^6$
$\mathcal{B}_5$ bip. nonsepar. simple maps, n edges	$B_5(z) = z + z^4 + 3z^6 + 7z^7 + 15z^8 + 63z^9$
$\mathcal{T}_1$ singular triangulations, $n+2$ vert	$T_1(z) = z + 4z^2 + 24z^3 + 176z^4 + 1456z^5$
$\mathcal{T}_2$ triangulations, $n+3$ vert	$T_2(z) = z + 3z^2 + 13z^3 + 68z^4 + 399z^5$
$\mathcal{T}_3$ irreducible triangulations, $n+3$ vert	$T_3(z) = z + z^3 + 3z^4 + 12z^5 + 52z^6 + 241z^7$

6.1. Map-related Composition Schemas

We start with a few definitions of classes of maps that have proved to be of interest in the combinatorial literature.

Families of Maps. A map is *loopless* if it does not contain any loop; *bridgeless* if it does not contain any bridge (a bridge, or isthmus, is an edge whose removal

TABLE 2 Generating functions, parameterizations and singular expansions for the families of Table 1

$\mathcal{M}$	ϕ	Ψ	$1/\rho$	Singular expansion ($Z = 1 - z/\rho$)
$\mathcal{M}_1$	$3(1+y)^2$	$\frac{2y-y^2}{3}$	12	$\frac{1}{3} - \frac{4}{3}Z + \frac{8}{3}Z^{3/2} + O(Z^2)$
$\mathcal{M}_2$	$3(1+\frac{y}{4})^4$	$\frac{y(y^2+3y-9)}{27}$	$\frac{256}{27}$	$\frac{5}{27} - \frac{16}{27}Z + \frac{32\sqrt{6}}{81}Z^{3/2} + O(Z^2)$
$\mathcal{M}_3$	$\frac{(y+3)^2}{3-y}$	$\frac{-y(y^2+3y-9)}{27}$	8	$\frac{5}{27} - \frac{32}{81}Z + \frac{256}{729}Z^{3/2} + O(Z^2)$
$\mathcal{M}_4$	$(1+y)^3$	$\frac{y(2+y-y^2)}{(1+y)^3}$	$\frac{27}{4}$	$\frac{1}{3} - \frac{4}{9}Z + \frac{8\sqrt{3}}{81}Z^{3/2} + O(Z^2)$
$\mathcal{M}_5$	$\frac{(y+1)^6}{(2y+1)^2}$	$\frac{y(-y^2+y+1)}{(y+1)^3}$	$\frac{729}{128}$	$\frac{5}{27} - \frac{32}{135}Z + \frac{2^8\sqrt{15}}{3^4 5^3}Z^{3/2} + O(Z^2)$
$\mathcal{M}_6$	$\frac{1}{1-y}$	$\frac{y^5(y^2+y-1)}{(1+y)^3(y^2-y-1)}$	4	$\frac{1}{540} - \frac{167}{8100}Z + \frac{32}{729}Z^{3/2} + O(Z^2)$
$\mathcal{B}_1$	$2(1+y)^2$	$\frac{y(2-y)}{4}$	8	$\frac{1}{4} - Z + 2Z^{3/2} + O(Z^2)$
$\mathcal{B}_2$	$\frac{8(1+y)^2}{4+2y-y^2}$	$\frac{y(2-y)}{4}$	$\frac{32}{5}$	$\frac{1}{4} - \frac{5}{9}Z + \frac{50\sqrt{5}}{243}Z^{3/2} + O(Z^2)$
$\mathcal{B}_3$	$\frac{(y+2)^6}{32(1+y)^2}$	$\frac{y^2(8-4y^2+4y-y^3)}{32(1+y)^2}$	$\frac{729}{128}$	$\frac{7}{128} - \frac{189}{640}Z + \frac{18\sqrt{15}}{125}Z^{3/2} + O(Z^2)$
$\mathcal{B}_4$	$\frac{32(1+y)^2}{(y^2-2y-4)^2}$	$\frac{y(2-y)}{4}$	$\frac{128}{25}$	$\frac{1}{4} - \frac{5}{13}Z + \frac{50}{2197}Z^{3/2} + O(Z^2)$
$\mathcal{B}_5$	$\frac{128(1+y)^2}{(4+2y-y^2)^3}$	$\frac{y(y-2)}{4}$	$\frac{512}{125}$	$\frac{1}{4} - \frac{5}{17}Z + \frac{50\sqrt{85}}{4931}Z^{3/2} + O(Z^2)$
$\mathcal{T}_1$	$2(1+y)^3$	$-\frac{y(y-1)}{2}$	$\frac{27}{2}$	$\frac{1}{8} - \frac{3}{8}Z + \frac{\sqrt{3}}{3}Z^{3/2} + O(Z^2)$
$\mathcal{T}_2$	$(1+y)^4$	$-y(y^2+y-1)$	$\frac{256}{27}$	$\frac{5}{27} - \frac{16}{27}Z + \frac{32\sqrt{6}}{81}Z^{3/2} + O(Z^2)$
$\mathcal{T}_3$	$\frac{1}{(y-1)^2}$	$\frac{y(y^2+y-1)}{(y-1)(1+y)^2}$	$\frac{27}{4}$	$\frac{5}{32} - \frac{27}{128}Z + \frac{9\sqrt{3}}{128}Z^{3/2} + O(Z^2)$

In this table, $\boldsymbol{M}(z) = \Psi(\boldsymbol{L}(z))$, where $\boldsymbol{L}(z) = z\phi(\boldsymbol{L}(z))$

"universal" phenomenon by providing the parameterizations, dominant singularity and singular expansion for the families of Table 1.

- Deep analysis
- Wide range of applicability

- Deep analysis
- Wide range of applicability

Giménez, N., Rué
The largest 2-connected (and 3-connected) component in random planar graphs is Airy distributed

The largest 2-connected component has size $0.96n$ a.a.s.

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Discrete Mathematics 204 (1999) 203–229

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Analytic combinatorics of non-crossing configurations

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Received 3 July 1997; revised 16 February 1998; accepted 3 August 1998

Abstract

This paper describes a systematic approach to the enumeration of 'non-crossing' geometric configurations built on vertices of a convex n-gon in the plane. It relies on generating functions, symbolic methods, singularity analysis, and singularity perturbation. Consequences are both exact and asymptotic counting results for trees, forests, graphs, connected graphs, dissections, and partitions. Limit laws of the Gaussian type are also established in this framework; they concern a variety of parameters like number of leaves in trees, number of components or edges in graphs, etc.

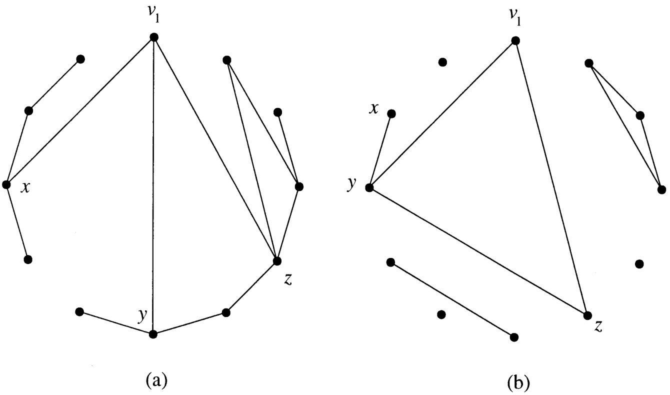

Fig. 2. (a) A connected graph; (b) an arbitrary graph.

(ii) *The number of graphs of size $n \geqslant 3$ is expressible in terms of Schröder numbers,*

$$G_n = 2^n c_{n-1}, \qquad c_n := \sum_{0 \leqslant v \leqslant (n/2)} (-1)^v \frac{1\cdot 3 \cdots (2n-2v-3)}{v!\,(n-2v)!} 3^{n-2v} 2^{-v-2}, \tag{14}$$

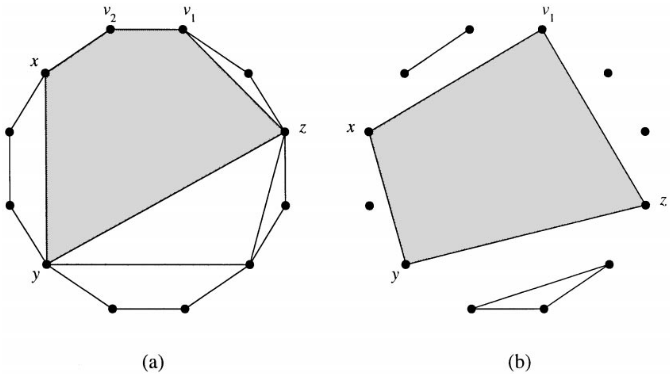

Fig. 3. (a) A dissection; (b) a non-crossing partition.

There is an alternative way to expand the GF, not to be found in Comtet's book [7]. Set $D = zy$. Then y satisfies an equation similar to (24),

$$y = z + \frac{y^2}{1-y} \quad \text{or} \quad z = y\frac{1-2y}{1-y}.$$

This equation is of the Lagrange type and it can be subjected to inversion

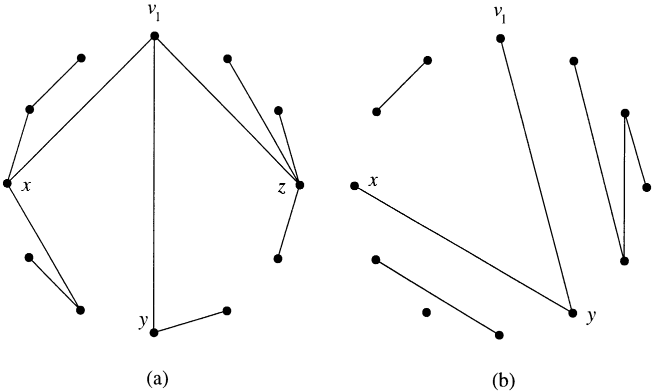

Fig. 1. (a) A tree; (b) a forest.

(iv) *The GF of forests, the BGF of trees and leaves, and the BGF of forests and components, are algebraic functions given by* (10), (6) *and* (11).

Trees were first enumerated by Dulucq and Penaud [12], and their result is summarized in part (i) of the theorem; the enumeration of forests by GF in (10) below is

Classical subject: polygon triangulations, dissections, non-crossing partitions...

Systematic analysis

- For all objects under consideration, asymptotic enumeration of the form
$$f_n \sim \frac{\gamma}{\sqrt{\pi}} n^{-3/2} \omega^n$$
- For all parameters X_n under consideration (number of edges, number of components...) asymptotic normal law with
$$\mathbf{E}(X_n) \sim \kappa n \qquad \sigma^2(X_n) \sim \lambda n$$

Table 4
The constants appearing in the statement of Theorem 4

	Class	ω	Num. value	γ
(T)	Trees	$\frac{27}{4}$	6.75000	$\frac{\sqrt{3}}{27}$
(F)	Forests	$\frac{1}{\xi}$	8.22469	0.07465
(C)	Connected graphs	$6\sqrt{3}$	10.39230	$\frac{\sqrt{6}}{9} - \frac{\sqrt{2}}{6}$
(G)	Graphs	$6+4\sqrt{2}$	11.65685	$\frac{1}{4}\sqrt{-140+99\sqrt{2}}$
(D)	Dissections	$3+2\sqrt{2}$	5.82842	$\frac{1}{4}\sqrt{-140+99\sqrt{2}}$
(P)	Partitions	4	4.00000	1

Note: ξ denotes the root of the polynomial $4-32x-8x^2+5x^3$ that is near 0.121, and 0.07465 represents the explicit algebraic number of degree 6 equal to $\beta/2$, with β given in the text.

near the dominant singularity ρ being

$$f(z) \sim c_0 + c_1\sqrt{1-z/\rho}. \tag{26}$$

Then singularity analysis [16] is used to achieve the transfer of (26) to coefficients

Table 5
The constants appearing in the statement of Theorem 5

Class, parameter	κ (mean)		λ (variance)	
Trees, leaves	$\frac{4}{9}$	0.444	$\frac{28}{243}$	0.115
Forests, components	$\frac{8}{37} - \frac{13}{37}\xi + \frac{15}{74}\xi^2$	0.176	$\frac{192}{1369} + \frac{5}{2738}\xi - \frac{47}{2738}\xi^2$	0.140
Connected graphs, edges	$\frac{1}{2} + \frac{\sqrt{3}}{2}$	1.366	$\frac{1}{4}$	0.250
Graphs, edges	$\frac{1}{2} + \frac{\sqrt{2}}{2}$	1.207	$\frac{1}{4} + \frac{\sqrt{2}}{8}$	0.426
Graphs, components	$\frac{5}{7} - \frac{3}{7}\sqrt{2}$	0.108	$\frac{50}{2401} + \frac{255}{4802}\sqrt{2}$	0.095
Dissections, regions	$\frac{\sqrt{2}}{2}$	0.707	$\frac{\sqrt{2}}{8}$	0.176
Partitions, blocks	$\frac{1}{2}$	0.500	$\frac{1}{8}$	0.125

Note: ξ denotes the root near 0.121 of the polynomial $4 - 32z - 8z^2 + 5z^3$.

Theorem 5. *Consider the following parameters: number of leaves in trees, components in forests, edges in connected graphs, components in graphs, edges in graphs, regions*

June 1995 after FPSAC in Paris

```
Dear Dr. Noy,

I have read your interesting note on non-crossing
trees. I think I can solve the open problem you
mention on the number of leaves, thus confirming your
conjecture.
The analysis is based on specifications of
combinatorial objects. You can specify a non-crossing
tree as...
```

June 1995 after FPSAC in Paris

Dear Dr. Noy,

I have read your interesting note on non-crossing trees. I think I can solve the open problem you mention on the number of leaves, thus confirming your conjecture.
The analysis is based on specifications of combinatorial objects. You can specify a non-crossing tree as...

A few weeks later

Dear Marc,

I am coming next month to Barcelona for a number theory conference, maybe we could meet and discuss about

...

Cheers,
Philippe

Philippe Flajolet, maître à penser