Pólya urns: THE analytic approach

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Philippe Flajolet and Analytic Combinatorics
Conference in the memory of Philippe Flajolet
Paris-Jussieu
December 15th 2011

Balanced Pólya urns

Replacement matrix $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$

Entries are nonrandom integers

Initial composition $\begin{pmatrix} a_0 \\ b_0 \end{pmatrix}$

- Balance hypothesis:
$\alpha + \beta = \gamma + \delta = s$.

- Tenability assumptions:
$\gamma, \beta \geq 0$;
if $\alpha \leq -1$, arithmetic conditions $\alpha | \gamma$ and $\alpha | a_0$; *idem* with δ.

Similarly, more than two colors...

Balanced Pólya urns

Question: what is the composition of the urn after n drawings?

More precisely, two main points of view
• Distribution of balls of different colors at time n when n is large (*slice* problem)
• After suitable normalization, limit distribution of the composition vector (*limit* problem)

Widely applicable model in physics, biology, medicine, theoretical computer science, *etc*.
Simple, elegant and fascinating object in mathematics.

[Extensions to random coefficients or unbalanced replacement matrices.]

Methods

• Combinatorial enumeration
[Jacob Bernoulli, Laplace, Pólya,... Mahmoud,...]

• Probabilistic methods
[Bagchi&Pal, Gouet, Athreya&Karlin, Aldous&Flannery&Palacios, Janson, (algebraic flavour)...]

• The approach by analytic combinatorics:

Philippe Flajolet

with co-authors
Philippe Dumas, Joaquim Gabarró, Thierry Huillet,
Basile Morcrette, Helmut Pekari, Vincent Puyhaubert

Books: Johnson&Kotz, Mahmoud

Coding by histories

Code the urn process by words $(W_n)_{n\geq 0}$ in the alphabet $\{b, w\}$.

Start with

$$W_0 = b^{a_0} w^{b_0} = \underbrace{bbb\dots b}_{a_0}\underbrace{www\dots w}_{b_0}.$$

If (say) the second ball is drawn,
get W_1 replacing the second letter (say b) by the word $b^{\alpha+1}w^{\beta}$:

$$W_1 = bb^{\alpha+1}w^{\beta}b\dots bwww\dots w.$$

And so on:
Black ball drawn $\rightsquigarrow$ the corresponding letter b is replaced by $b^{\alpha+1}w^{\beta}$;
White ball drawn $\rightsquigarrow$ the corresponding letter w is replaced by $b^{\gamma}w^{\delta+1}$.

Histories: generating functions

An *history* of length n is a sequence $(W_0, W_1, \ldots, W_n)$ of such words.

Notations: $H_n \begin{pmatrix} a_0 & a \\ b_0 & b \end{pmatrix}$ is the number of histories of length n that lead from the initial composition $\begin{pmatrix} a_0 \\ b_0 \end{pmatrix}$ to the final composition $\begin{pmatrix} a \\ b \end{pmatrix}$.

The hero: the trivariate GF

$$H\left(x, y, z \,\middle|\, \begin{matrix} a_0 \\ b_0 \end{matrix}\right) := \sum_{n,a,b \geq 0} H_n \begin{pmatrix} a_0 & a \\ b_0 & b \end{pmatrix} x^a y^b \frac{z^n}{n!}.$$

Two key properties.

Histories: convolution property

The urn composition is essentially described by the two elementary processes starting respectively from one black ball only and one white ball only:

$$H\left(x,y,z \,\middle|\, \begin{matrix} a_0 \\ b_0 \end{matrix}\right) = H\left(x,y,z \,\middle|\, \begin{matrix} 1 \\ 0 \end{matrix}\right)^{a_0} H\left(x,y,z \,\middle|\, \begin{matrix} 0 \\ 1 \end{matrix}\right)^{b_0}$$

Stochastic independence flavour,
as in continuous time branching processes

... endless passionate discussions with PF and Brigitte Chauvin.

Histories: differential system

The couple of functions $\left(H\left(x,y,z\,\middle|\,\begin{matrix}1\\0\end{matrix}\right), H\left(x,y,z\,\middle|\,\begin{matrix}0\\1\end{matrix}\right)\right)$

is THE solution to the (monomial, homogeneous) ODS

$$\begin{cases} f'(z) = f(z)^{\alpha+1} g(z)^{\beta} \\ g'(z) = f(z)^{\gamma} g(z)^{\delta+1} \end{cases}$$

with initial conditions $f(0) = x$ and $g(0) = y$.

PF: "basic isomorphism".
Proof: formal derivation of H functions mimicks the replacement rules.

[First version: PDE on $H(x, 1, z)$.]

And then...

Dream

• solve the fundamental differential system;

• get explicit expressions for GF, or explicit parametrizations;

• derive probabilistic results on the urn's distribution such as explicit laws or moments at finite time (slice problem), local limit laws, large deviations (limit problem).

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Done

• in many particular cases, including famous problems (see Basile M.);

PF's amazing inventiveness, sense of metaphore

• for families of two-color urns (see below).

Analytic urns [183, FlaGabPek]

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But

theoretical obstruction due to the generic non solvability of the differentiable system in dimension ≥ 3.

So

investigations to get results without explicitly solving the ODS (à suivre...).

Complex analysis

Article [183, FlaGabPek].
$\alpha, \delta \leq -1$.
$h = s - \alpha - \delta$.

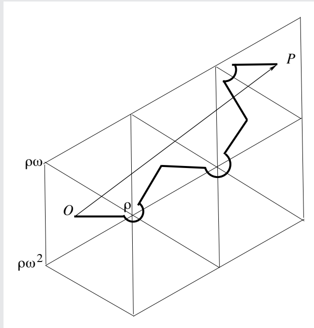
The functions H are parametrized by the analytic inverse of the Abelian integral

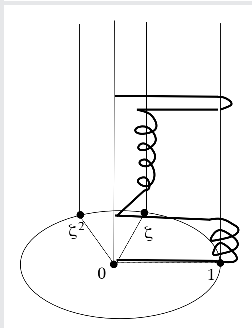
$$\int_{\{u^h+v^h=1\}} u^{-\alpha-1} v^{\alpha+\delta} du.$$

Conformal mapping theory, underlying Riemann surface, elementary kite that generate a fundamental polygonal domain, beautiful figures (*FP's enthusiasm and pride*).

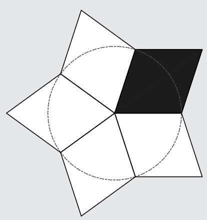
Classification of "elliptic" urns, parametrized by Weierstrass $\wp$ functions.

PF's pictures or slides



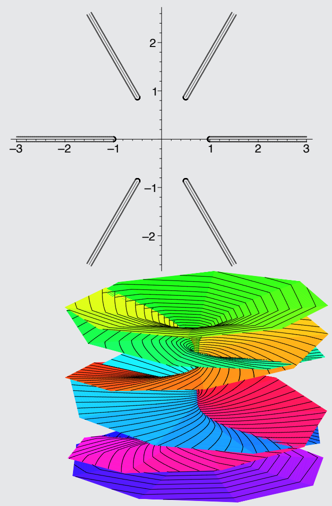


Definition 1. *The* fundamental polygon *of an urn model is the (closure of) the union of h regularly rotated versions of the elementary kite about the origin.*



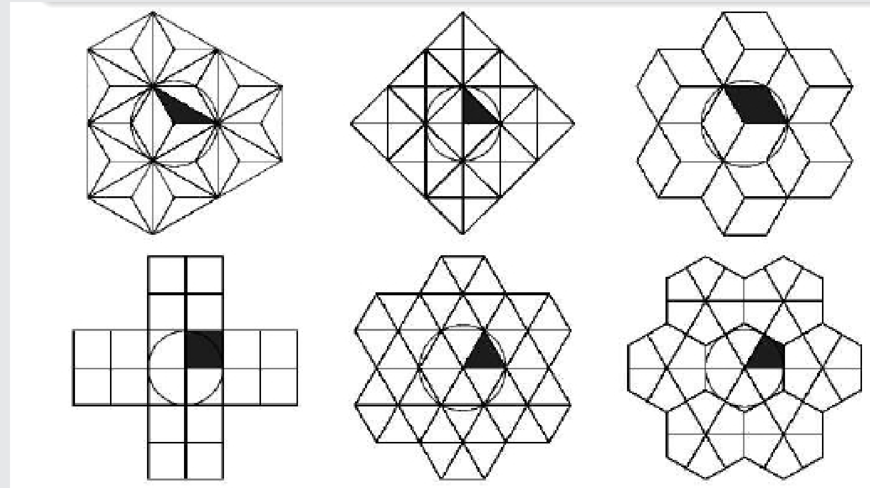
The elementary kite and the fundamental polygon of the urn

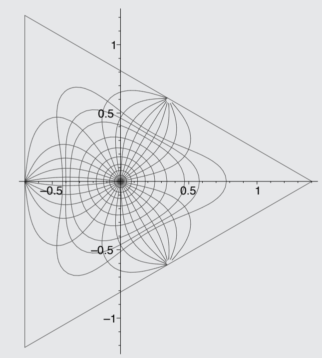
$$\begin{pmatrix} -1 & 4 \\ 4 & -1 \end{pmatrix}$$



The region R_0 (left) and a rendering of the six-sheeted Riemann surface $\mathfrak{R}$ of $\delta(u) \equiv (1 - u^6)^{1/6}$ for u near 1 (right).

Because of double parameterization, taking u in a half-plane suffices.





Another view of the image of $(R_0 \cap H)$ by $I(u)$ giving the fundamental triangle T: a representation of the images of rays emenating from 0 and of circles centred at 0