THE UBIQUITOUS GAUSSIAN LAW IN ANALYTIC COMBINATORICS

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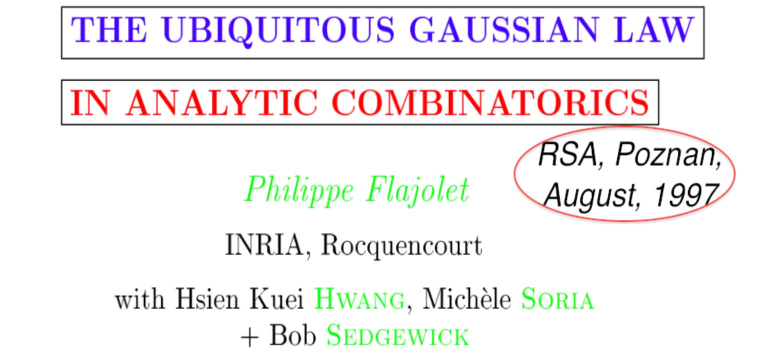
December 15, 2011



MOTIVATION: INRIA RAPPORT DE RECHERCHE

INRIA

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET AUTOMATIQUE

The Average Case Analysis of Algorithms: Multivariate Asymptotics and Limit Distributions

Philippe FLAJOLET, Robert SEDGEWICK

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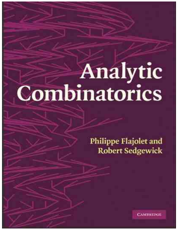
PART C OF ANALYTIC COMBINATORICS (2009)



UNPUBLISHED MANUSCRIPT

The ubiquitous Gaussian law in analytic combinatorics

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Working draft

Q: Why and how Gaussian law is so common?
(from an analytic viewpoint)

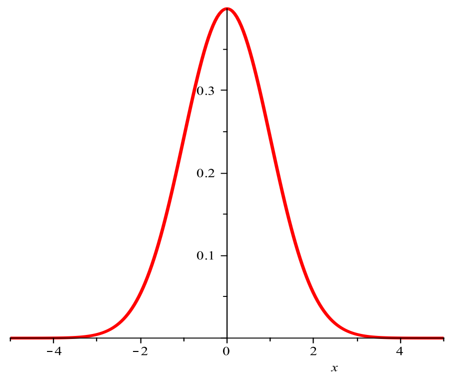
AIM HERE (FOR GAUSSIAN LAW IN AC)

Survey major techniques, focusing on PF-papers

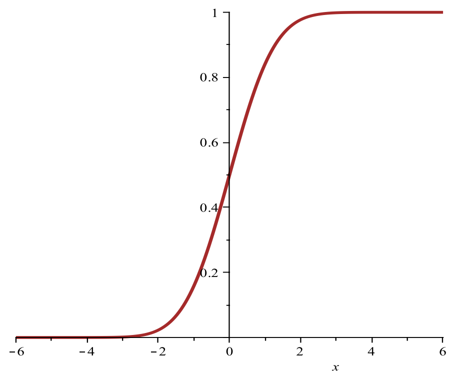
A simple classification will be given

GAUSSIAN (NORMAL) DISTRIBUTION

$$\phi(x) := \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$



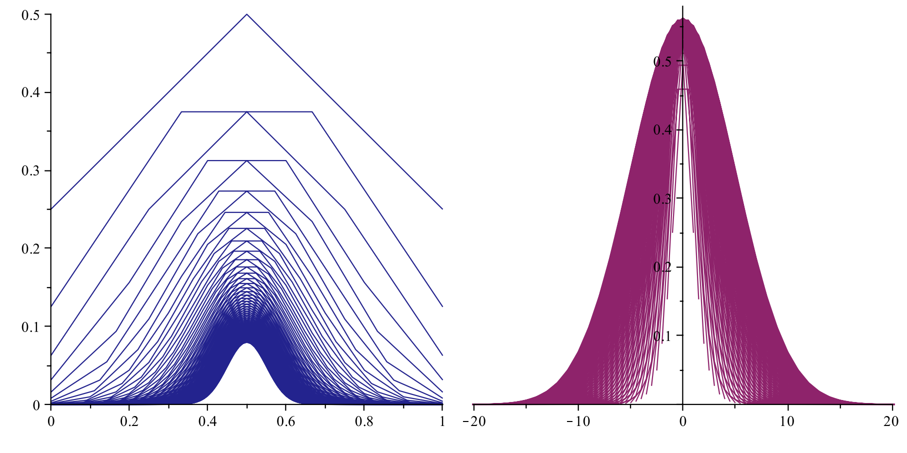
$$\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} \mathrm{d}t$$



$$\mathbb{E}(e^{\mathscr{N} it}) = e^{-t^2/2}$$

BINOMIAL TO GAUSSIAN: LOCAL LIMIT THEOREM

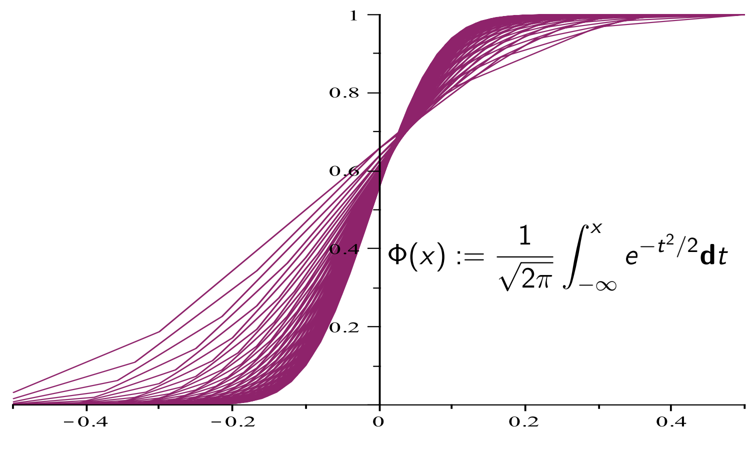
Toss n fair coins; how many heads?



de Moivre (1738): $\binom{n}{k}2^{-n} \sim \sqrt{\frac{2}{\pi n}}\, e^{-\frac{(k-n/2)^2}{n/2}}$

BINOMIAL TO GAUSSIAN: CENTRAL LIMIT THEOREM

$$\sup_x \left| \sum_{j < \frac{n}{2} + x\frac{\sqrt{n}}{2}} \binom{n}{j} 2^{-n} - \Phi(x) \right| \to 0$$



$$\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} \mathbf{d}t$$

CENTRAL LIMIT THEOREM (CLT)

or Asymptotically normal, Asymptotically Gaussian , . . .

Write

$$X_n \sim \mathcal{N}(\mu_n, \sigma_n)$$

if

$$\lim_{n\to\infty} \sup_x \left| \mathbb{P}\left(\frac{X_n - \mu_n}{\sigma_n} < x \right) - \Phi(x) \right| = 0$$

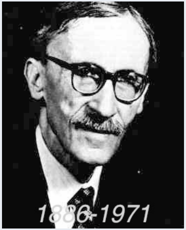
Lévy's continuity theorem

If

$$\mathbb{E}\left(e^{\frac{X_n - \mu_n}{\sigma} it} \right) \to e^{-t^2/2},$$

then

$$X_n \sim \mathcal{N}(\mu_n, \sigma_n).$$



CENTRAL LIMIT THEOREM (CLT): FIRST USE

CLT first appears in 1920 in the title

Über den zentralen Grenzwertsatz der Wahrscheinlichkeitsrechnung und das Momentenproblem.

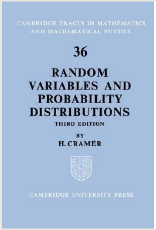
Von

Georg Pólya in Zürich.


(1887-1985)

MATH. Z. 8 (1920) 171-181,
DOI: 10.1007/BF01206525

CLT first appears in English in 1937 in Cramér's book



CHAPTER VI

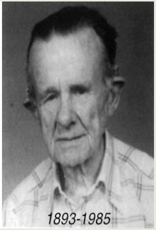
THE NORMAL DISTRIBUTION AND THE CENTRAL LIMIT THEOREM

1. The *normal distribution function*[1] $\Phi(x)$ is defined by the relation

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt.$$

The corresponding *normal frequency function* is

$$\Phi'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

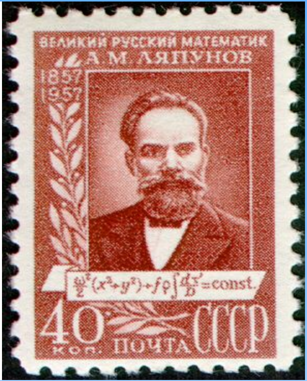


CENTRAL LIMIT THEOREM: EARLY DEVELOPMENTS



Abraham de Moivre (1667–1754)	**Pierre-Simon Laplace (1749–1827)**	**Carl Friedrich Gauss (1777–1855)**

CENTRAL LIMIT THEOREM: EARLY DEVELOPMENTS

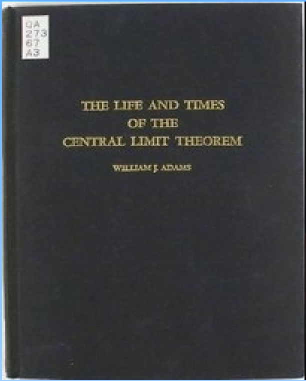


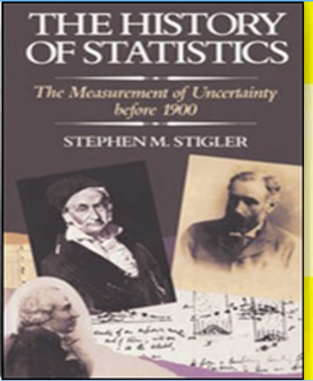
Pafnuty L. Chebyshev (1821–1894)

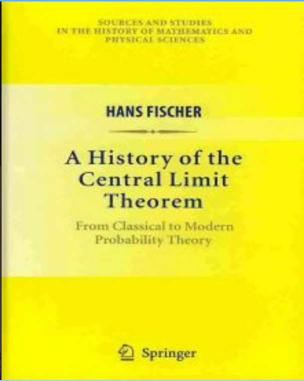
Andrey A. Markov (1856–1922)

Aleksandr M. Lyapunov (1857–1918)

BOOKS ON HISTORY OF CENTRAL LIMIT THEOREM







William J. Admas
1974

S. M. Stigler
1990

Hans Fischer
2011

GAUSSIAN LIMIT LAW: AN ANALYTIC PATTERN

The most classical approach

Sums of RVs	$\longrightarrow$	**Char. fun.**	$\longrightarrow$	**Gaussian law**
$S_n := \sum_{1\le j\le n} X_j$	$\longrightarrow$	$\varphi(t)^n$	$\longrightarrow$	$e^{-t^2/2}$

$\Longrightarrow S_n \sim \mathscr{N}(\mu n, \sigma^2 n)$ **(convergence in distribution)**

$$\mathbb{E}\left(e^{\frac{X_n-\mu n}{\sigma\sqrt{n}}\, it}\right) = e^{-\frac{\mu}{\sigma}\sqrt{n}it}\varphi\left(\frac{t}{\sigma\sqrt{n}}\right)^n$$
$$= e^{-\frac{\mu}{\sigma}\sqrt{n}it}\left(e^{\frac{\mu}{\sigma\sqrt{n}}\, it-\frac{t^2}{2n}+\cdots}\right)^n$$
$$\to e^{-\frac{t^2}{2}}.$$

GAUSSIAN LIMIT LAW: AN ANALYTIC PATTERN

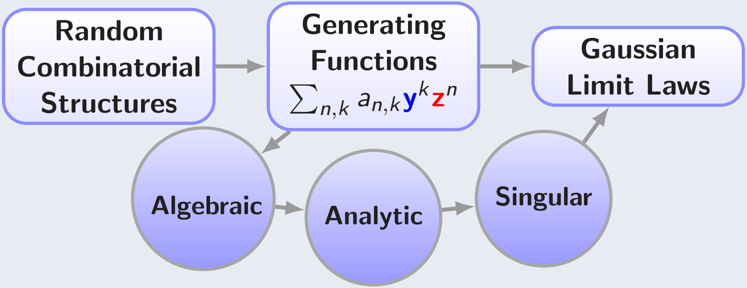
The general question

From a double-indexed nonnegative sequence $\{a_{n,k}\}$, define

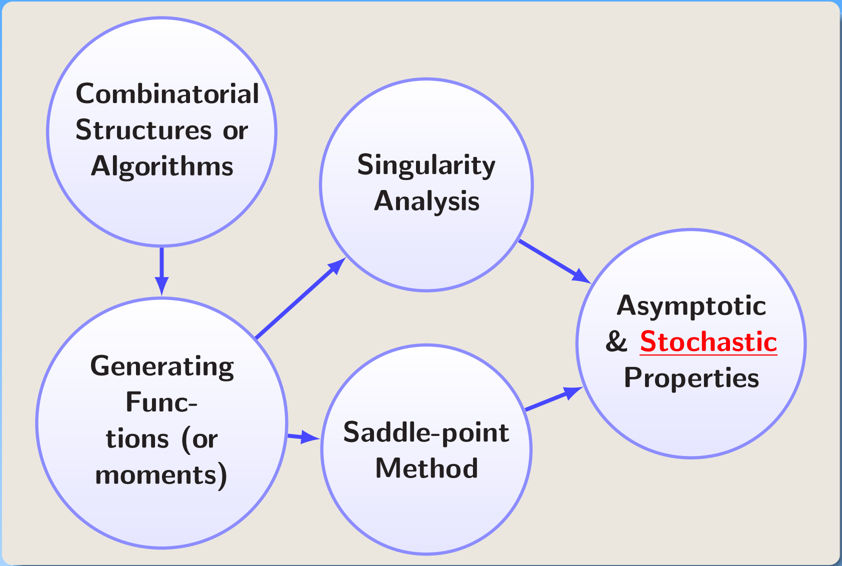
$$\mathbb{P}(X_n = k) := \frac{a_{n,k}}{\sum_j a_{n,j}}.$$

Q: Limit distribution of X_n and finer properties?

The Flajolet pattern

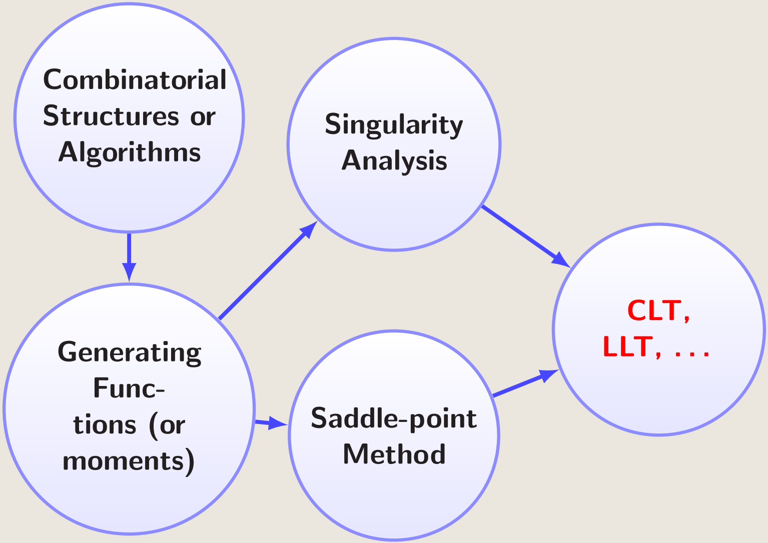


ANALYTIC COMBINATORICS



Focus on limit law = Gaussian

GAUSSIAN LAW IN ANALYTIC COMBINATORICS



SINGULARITY ANALYSIS

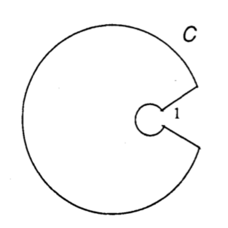
Singularity analysis: $f(z) = \sum_n a_n z^n$

WLOG, assume the (finite) singularity $z = e^{-\zeta} = 1$

$$f\left(e^{-\zeta}\right) \sim \sum_{j\geq 0} c_j \zeta^{j-\alpha} \qquad (\zeta \sim 0);$$

then (under suitable conditions)

$$a_n \sim \sum_{j\geq 0} \frac{c_j}{\Gamma(\alpha - j)} n^{\alpha-j-1}$$



Asymptotic expansion of f near the dominant singularity

$\Updownarrow$

Asymptotic expansion of a_n at infinity

Suitable for finite singularity and polynomial growth

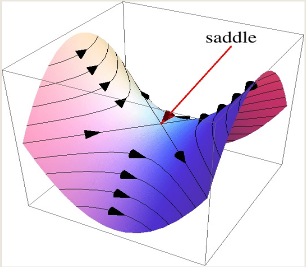
SADDLE-POINT METHOD

Saddle-point method: $f(z) = \sum_n a_n z^n$

$$a_n = \frac{1}{2\pi i} \oint_{|z|=r} z^{-n-1} f(z) \mathbf{d}z \le r^{-n} f(r) \quad (0 < r < \rho \le \infty).$$

So $\boxed{a_n \le \min_{0<r<\rho} r^{-n} f(r).}$
If $f(z)$ blows up fast enough ($\gg$ polynomial) near ρ, then

$$a_n \sim \frac{r^{-n} f(r)}{\sqrt{2\pi\sigma^2(r)}},$$



where

$$\frac{rf'(r)}{f(r)} = n, \quad \sigma^2(r) := \frac{rf'(r)}{f(r)} + \frac{r^2 f''(r)}{f(r)} - \left(\frac{rf'(r)}{f(r)}\right)^2.$$

Suitable for fast-growing functions

A CLASSIFICATION BY APPROACH

CLT by singularity analysis

- $\mathscr{N}(cn, c'n)$: **moving singularity**
- $\mathscr{N}(c \log n, c' \log n)$: **moving exponent**
- $\mathscr{N}(\mu_n, \sigma_n^2)$: **quasi-power framework**

CLT by saddle-point method

- $\mathscr{N}(cn, c'n)$: **high powers of GFs** $f(\mathbf{z}, \mathbf{y})^{bn}$ **or** $g(\mathbf{z}, \mathbf{y}) f(\mathbf{z})^{bn}$
- $\mathscr{N}(\mu_n, \sigma_n^2)$: **de-Poissonization (Jacquet-Szpankowski)**
- $\mathscr{N}(\mu_n, \sigma_n^2)$: **Sachkov's framework**

CLT by other approaches

- **Method of moments (analytic, probabilistic, ...)**
- **Sum of RVs, U-statistics, etc.**
- **Martingale, m-dependent, branching processes, etc.**
- **Stein's method, . . .**

CLT BY SINGULARITY ANALYSIS

Two prototypes: $\begin{cases} \text{linear} & \Rightarrow \text{moving singularity} \\ \text{logarithmic} & \Rightarrow \text{moving exponent} \end{cases}$

$$f(z, y) := \sum_{n\geq 0} \mathbb{E}\left(y^{X_n}\right) z^n \quad (y \sim 1)$$

#($\nearrow$ runs) in permutations $\longrightarrow \dfrac{y(1-y)}{e^{z(1-y)} - y} \longrightarrow \mathscr{N}\left(\dfrac{n}{2}, \dfrac{n}{12}\right)$

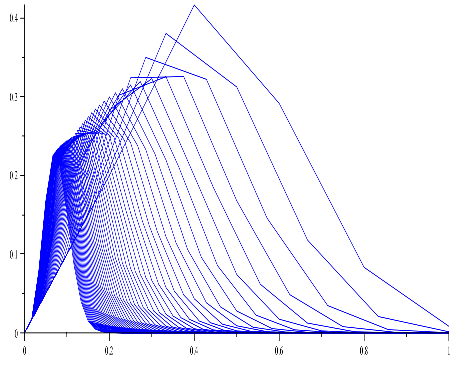
$$f(z, y) = \frac{1}{\frac{\log y}{y-1} - z} + \cdots \Longrightarrow \mathbb{E}\left(y^{X_n}\right) \sim \left(\frac{y-1}{\log y}\right)^{n+1}$$

#(cycles) in permutations $\longrightarrow \exp\left(y \log \dfrac{1}{1-z}\right) \longrightarrow \mathscr{N}(\log n, \log n)$

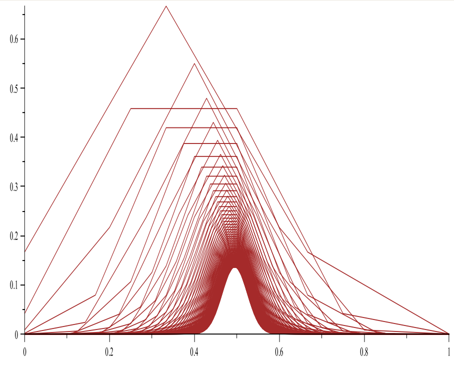
$$f(z, y) = (1-z)^{-y} \Longrightarrow \mathbb{E}\left(y^{X_n}\right) \sim \frac{n^{y-1}}{\Gamma(y)} = \frac{e^{(y-1)\log n}}{\Gamma(y)}$$

THE HISTOGRAMS

Stirling # 1st kind



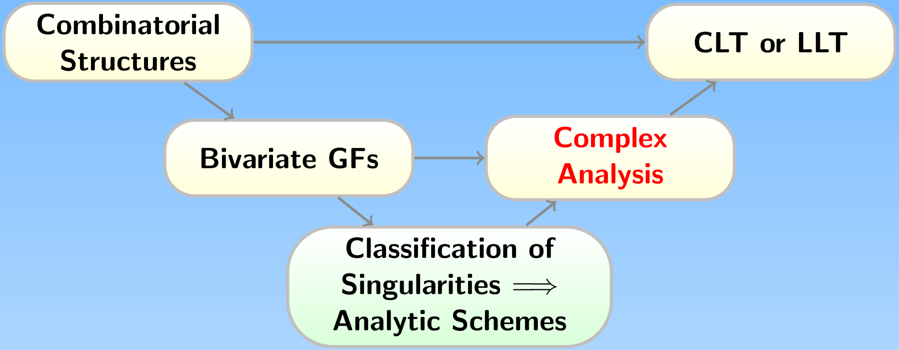
Eulerian numbers



CLT BY SINGULARITY ANALYSIS: ANALYTIC SCHEMES

Two typical cases

exp(log)	$(1 - z)^{-\alpha(y)}$	$\longrightarrow$	$\mathscr{N}(c \log n, c' \log n)$
alg-log	$(1 - \rho(y)z)^{-\alpha}$	$\longrightarrow$	$\mathscr{N}(cn, c'n)$



ANALYTIC SCHEMES

Integer partitions: long history

Hardy-Ramanujan (1917), Statistical physicists in 1930's, Haselgrov and Temperley (1954), Lee (1993), Richmond (1994), Mutafchiev (2011), etc.

Asymptotic analysis

Hayman (1956) (the classical paper, admissible functions for saddle-point method), many follow-up papers

Analytic Combinatorics (early developments)

Bender (1973) (the pioneering paper), Knopfmacher (1975), Charalambides (1976), Canfield (1977), Bender and Richmond (1983), Bender, Richmond and Williamson (1983), Flajolet and Odlyzko (1984), Charalambides and Kyriakoussis (1984), Kyriakoussis (1984), Flajolet and Soria (1990, 1993), Mutafchiev (1992), Gao and Richmond (1992), H. (1994, 1996, 1998), etc.

A VERY SIMPLE SCHEME

Frobenius (1910)-Harper (1967)

If $\mathbb{E}\left(\mathbf{y}^{X_n}\right)$ has only real zeros and $\mathbb{V}(X_n) \to \infty$, then $X_n \sim \mathscr{N}$.

Examples: binomial, Stirling numbers of both kind, Eulerian numbers, matching polynomials, occupancy problem, hypergeometric, etc.

holds also when all zeros lie in the left half-plane

All roots real $\Longrightarrow$ difficult to prove

THREE ANALYTIC SCHEMES: $\mathcal{N}(c \log n, c' \log n)$

Flajolet and Soria (1990) [88] exp(log): $(1 - z/\rho)^{-y} g(z, y)$

Cycles in permutations, rounds in children's yards, components in 2-regular graphs, cycles in random mappings, cycles in random mapping patterns, irreducibles in polynomials (over finite fields), arithmetical semigroups, etc.

Bergeron, Flajolet and Salvy (1992) [96]

Depth in increasing trees (binary, recursive, plane-oriented, ...)

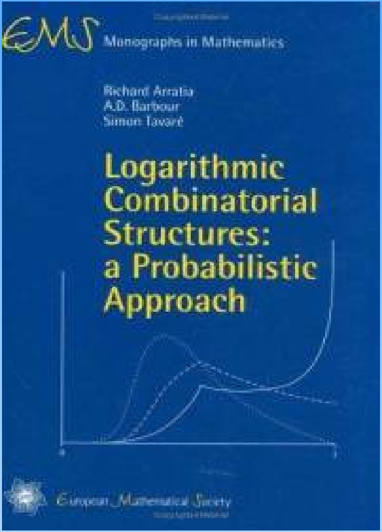
$$g'(\mathbf{z})^{\mathbf{y}} \int_0^{\mathbf{z}} g'(t)^{1-\mathbf{y}} \mathbf{d}t$$

Flajolet and Lafforgue (1994) [117]

Depth in quadtrees

$$\sum_{0 \le j \le r} \frac{c_j(\mathbf{z}, \mathbf{y})}{(1 - \mathbf{z})^j} \cdot \frac{\partial^{r-j}}{\partial \mathbf{z}^{r-j}} f(\mathbf{z}, \mathbf{y}) = 0 \Longrightarrow \textbf{linear systems}$$

A BOOK BY ARRATIA, BARBOUR AND TAVARÉ



CLT BY SINGULARITY ANALYSIS: LINEAR CASE

PF-papers with $\mathscr{N}(cn, c'n)$ (explicit)

- **Flajolet (1985) [47]: # wagons in random trains**

$$\frac{h(\mathbf{z})}{1-\mathbf{y}g(\mathbf{z})}$$

- **Flajolet and Soria (1993) [112]:**

$$\left(\frac{1}{1-\mathbf{y}C(\mathbf{z})}\right)^{\alpha}\left(\log\frac{1}{1-\mathbf{y}C(\mathbf{z})}\right)^{k},$$

(extending Bender's meromorphic scheme), many examples (ordered set partitions, integer compositions, etc.)

CLT BY SINGULARITY ANALYSIS: LINEAR CASE

PF-papers with $\mathscr{N}(cn, c'n)$ (explicit)

- **Flajolet, Gourdon and Salvy (1993) [107]: coefficients of polynomials**

$$\frac{\log \frac{1}{1-z}}{z\left(1 - y \log \frac{1}{1-z}\right)}$$

- **Banderier and Flajolet (2002) [168]: final altitude of meander**

$$\frac{g(z, y)}{1 - zP(y)}$$

- **Nicodème, Flajolet, and Salvy (2002): pattern occurrences**

$$\frac{g(z, y)}{\det(I_{m\times m} - z\mathbf{M}(y))}$$

CLT BY SINGULARITY ANALYSIS: LINEAR CASE

PF-papers with $\mathcal{N}(cn, c'n)$ (implicit)

- **Bergeron, Flajolet and Salvy (1992) [96]: leaves in increasing trees**

$$\int_0^{f(\mathbf{z},\mathbf{y})} \frac{\mathbf{d}t}{(\mathbf{y}-1)\phi_0+\phi(t)} = \mathbf{z}$$

- **Flajolet, Gourdon and Martínez (1997) [135]: patterns in random binary search trees**

$$\frac{\partial}{\partial \mathbf{z}} f(\mathbf{z},\mathbf{y}) = f(\mathbf{z},\mathbf{y})^2 + cm(\mathbf{y}-1)\mathbf{z}^{m-1}$$

- **Flajolet and Noy (1999) [144]: non-crossing geometric configurations**

$$\sum_{0\le j\le r} c_j(\mathbf{z},\mathbf{y}) f^j(\mathbf{z},\mathbf{y}) = 0$$

CLT BY SINGULARITY ANALYSIS: LINEAR CASE

PF-papers with $\mathcal{N}(cn, c'n)$ (implicit)

- **Flajolet, Gabarró and Pekari (2006) [183] (also [186]): balls in urns**

$$\left(1 - s\mathbf{y}^{b+s}\mathbf{z}\right)\frac{\partial f}{\partial \mathbf{z}}\left(\mathbf{y}^{b+s+1} - \mathbf{y}^{1-a}\right)\frac{\partial f}{\partial \mathbf{y}} - t_0\mathbf{y}^{b+s}f = 0$$

- **Bóna and Flajolet (2009) [198]: symmetry in evolutional trees**

$$f(\mathbf{z},\mathbf{y}) = \tfrac{1}{2}f^2(\mathbf{z},\mathbf{y}) + \mathbf{z} + \left(\mathbf{y} - \tfrac{1}{2}\right)f(\mathbf{z}^2,\mathbf{y}^2)$$

Proving $(1-\rho(\mathbf{y})\mathbf{z})^{-\alpha}$ may be very challenging

QUASI-POWER FRAMEWORK

The prototype

$$\mathbb{E}\left(e^{(Y_1+\cdots+Y_n)it}\right) = \varphi(t)^n$$

$$\Longrightarrow \mathbb{E}\left(e^{(Y_1+\cdots+Y_n-\mu n)it/(\sigma\sqrt{n})}\right) \to e^{-t^2/2}$$

$$\Longrightarrow Y_1+\cdots+Y_n \sim \mathscr{N}(\mu n, \sigma^2 n)$$

First extensions

Bender and Richmond (1983): multivariate CLT and LLT

$$\mathbb{E}(\mathbf{y}^{X_n}) \sim h(\mathbf{y})g(\mathbf{y})^n \qquad (\mathbf{y} \sim 1)$$

see also Kyriakoussis (1984) (more restricted version)

my PhD Thesis (1994); "Quasi-Power" due to PF

$$\mathbb{E}(e^{X_n s}) = h(s)e^{\lambda_n g(s)}\left(1 + O\left(\varepsilon_n\right)\right) \Longrightarrow \mathscr{N}(c\lambda_n, c'\lambda_n)$$

TWO CLTs BY QUASI-POWER FRAMEWORK

Flajolet and Vallée (1998) [144]: $\mathscr{N}(cn, c'n)$

Continuants in continued fraction algorithms

$$\mathbb{E}(Q_n(x)^s) \sim \mathscr{N}_{2-s}^n \left[\frac{1}{1+u}\right](0) \sim h(s)g(s)^n$$

(Dirichlet series, transfer operators, spectral properties)

Flajolet, Szpankowski and Vallée (2006) [164]: $\mathscr{N}(cn, c'n)$

Occurrences of pattern (as a subsequence) in random texts with all gaps finite

$$\mathbb{E}(\mathbf{y}^{X_n}) \sim h(\mathbf{y})g(\mathbf{y})^{n-\delta} \qquad (\delta = \textbf{sum of gaps})$$

(finite-state automaton, matrix analysis, Perron-Frobenius)

CLT BY SADDLE-POINT METHOD

Large powers: $g(z,y)f(z,y)^{bn}$, exact or asymptotic

- **Flajolet and Odlyzko (1984) [45]:**

$$f_{k+1}(z) = \sum_{0\le j\le d} c_j(z) f^j(z) \Longrightarrow f_k(z) \sim h(z) e^{d^k g(z)}$$

- **Flajolet, Poblete and Viola (1998) [142]: # moves in sparse hashing table**
- **Mahmoud, Flajolet, Jacquet and Regnier (2000) [159]: distributive sort**
- **Banderier, Flajolet, Schaeffer and Soria (2001) [152] [160]: composition scheme**

CLT BY SADDLE-POINT METHOD

Functional equations via analytic de-Poissonization

Mahmoud, Flajolet, Jacquet and Regnier (2000) [159]: Radix selection

$$f(\mathbf{z},\mathbf{y}) = be^{(b-1)\mathbf{yz}/b} f\left(\frac{\mathbf{yz}}{b},\mathbf{y}\right) + \mathbf{z}(1-\mathbf{y})$$

(can be solved explicitly and then apply Rice's integral)

Radix sort

$$f(\mathbf{z},\mathbf{y}) = f^b\left(\frac{\mathbf{yz}}{b},\mathbf{y}\right) + \mathbf{z}(1-\mathbf{y})$$

Scahkov's extended Quasi-Power framework (1997)

If $\mathbb{E}(\mathbf{y}^{X_n}) \sim \mathbf{y}^{c_n} e^{g_n(\mathbf{y})}$**, where** $g_n \in C^3[1-\delta, 1+\delta]$ **and** $\frac{h_n'''(\mathbf{y})}{(g_n'(1)+g_n''(1))^{3/2}} \to 0$**,** $\mathbf{y} \in [1-\delta, 1+\delta]$**, then**

$$X_n \sim \mathscr{N}(g_n'(1) + c_n, g_n'(1) + g_n''(1))$$

DE-POISSONIZATION: IDEAS

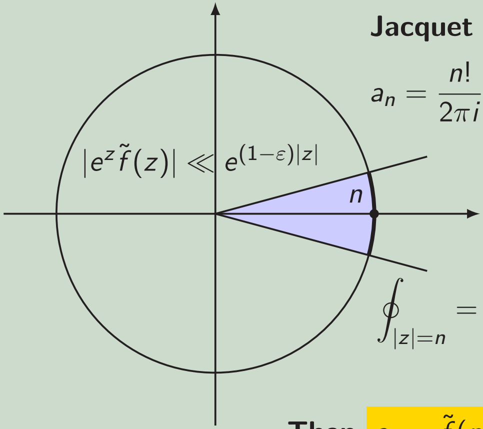
Poisson heuristic: $\tilde{f}(z) := e^{-z} \sum_{n \geq 0} a_n z^n / n!$

If a_n doesn't grow too fast, then $a_n \approx \tilde{f}(n)$.

Analytic justification: Cauchy + saddle-point method

Jacquet & Szpankowski (1998)

$$a_n = \frac{n!}{2\pi i} \oint_{|z|=n} z^{-n-1} e^z \tilde{f}(z) dz$$



$|\tilde{f}(z)| \ll e^{|z|^{\frac{1}{2}-\varepsilon}}$

$$\oint_{|z|=n} = \int_{\substack{|z|=n \\ |\theta| \leq \varepsilon}} + \int_{\substack{|z|=n \\ \varepsilon \leq |\theta| \leq \pi}}$$

Then $a_n \sim \tilde{f}(n)$.

CLT BY SADDLE-POINT METHOD

Flajolet and Noy (2000) [155]

 # crossings in random chord diagrams

$$\mathbb{E}(e^{X_n s})\frac{2^n n!}{(2n)!} = \sum_{-n\le k\le n}\binom{2n}{n+k}(-1)^k\frac{e^{\binom{k}{2}s}}{(1-e^s)^n}$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-x^2/2}\left(\frac{\sinh^2(\frac{x\sqrt{s}}{2}-\frac{s}{4})}{e^{\frac{s}{2}}\sinh(\frac{s}{2})}\right)^n \mathrm{d}x$$

Then $\mathscr{N}(\frac{n^2}{6},\frac{n^3}{45})$ by saddle-point method

DIFFERENT (LESS ANALYTIC) APPROACHES TO CLT

Flajolet and Golin (1994) [105][115]

Cost of top-down mergesort

$$X_n = X_{\lfloor n/2 \rfloor} + X^*_{\lceil n/2 \rceil} + Y_n$$
$$\Longrightarrow X_n \sim \mathscr{N}(n \log_2 n + P_1(\log_2 n)n, P_2(\log_2 n)n)$$

sum of independent RVs, **Lyapunov's condition ($(2+\delta)$-th absolute moment).**

Flajolet, Szpankowski and Vallée (2006) [164] [191]

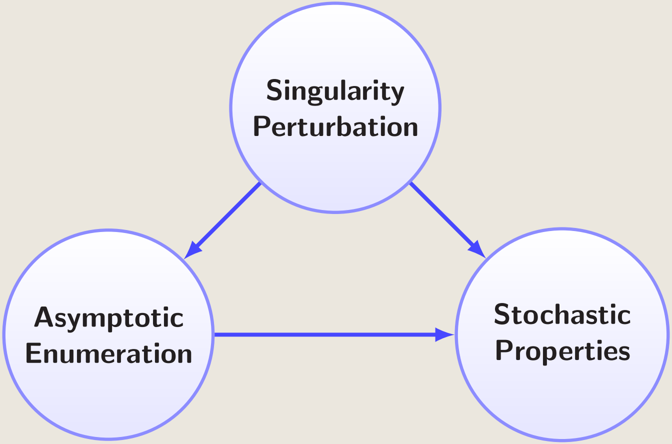
Occurrences of hidden patterns in random texts (with b unconstrained blocks)

$$X_n \sim \mathscr{N}(cn^b, c' n^{2b-1})$$

Method of moments, **combinatorial arguments, and analytic tools**

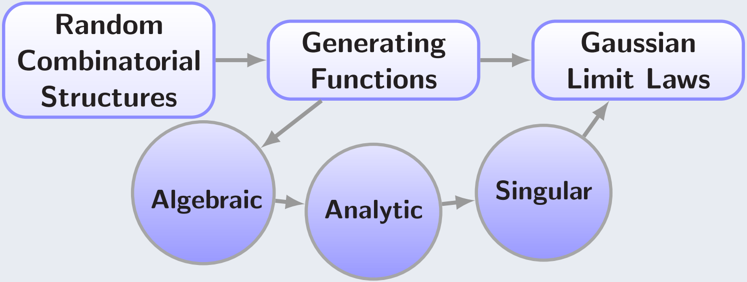
PF'S CONTRIBUTION TO LIMIT LAWS

Clarify the deep connection between



PF'S CONTRIBUTION TO LIMIT LAWS

Systemize (schematize) the use of



PF'S CONTRIBUTION TO LIMIT LAWS

Modernize classical tools

Research becomes easier after him

PF'S CONTRIBUTION TO LIMIT LAWS

LLT and Large deviations

Prove LLT for

- **wagons in trains [47]**
- **leaves of BSTs [135]**
- **composition scheme [152], [160]**
- **motif statistics [151], [174]**
- **urns [183]**
- **symmetry in trees [198],**

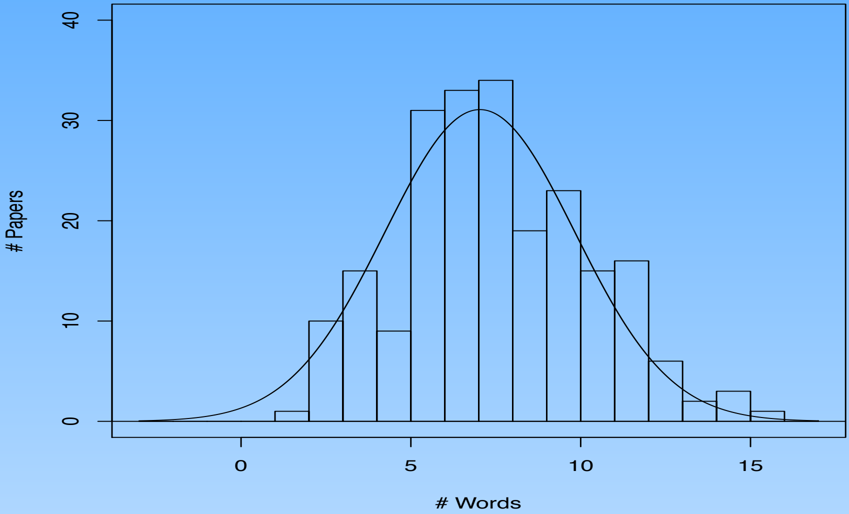
$$\sup_x \left| \sigma_n \mathbb{P}\left(X_n = \lfloor \mu_n + x\sigma_n \rfloor\right) - \frac{e^{-x^2/2}}{\sqrt{2\pi}} \right| = o(1)$$

some enhanced with explicit large deviations estimates.

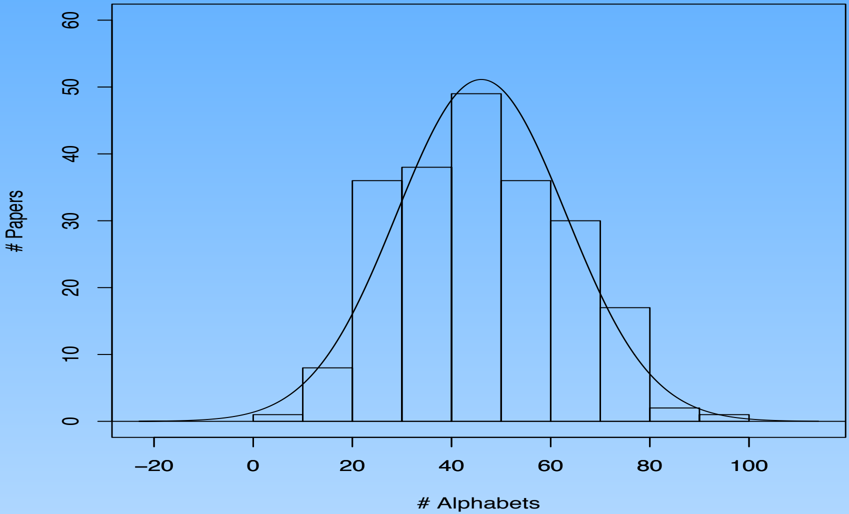
A. M. Odlyzko (1995)

"Analytic methods are extremely powerful, and when they apply, they often yield estimates of unparalleled precision."

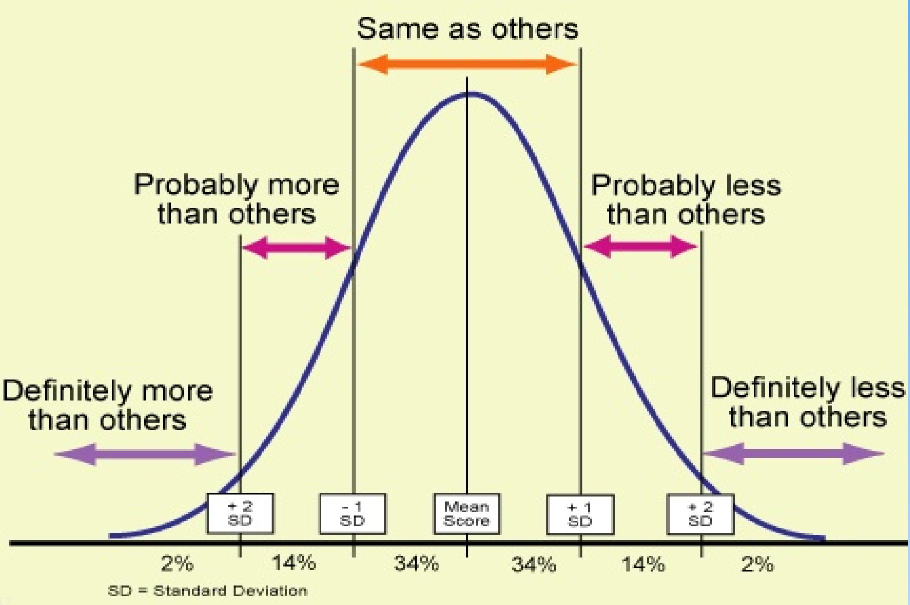
HISTOGRAM OF THE #(TITLE-WORDS) IN ALL PF-PAPERS



HISTOGRAM OF THE #(TITLE-ALPHABETS) IN ALL PF-PAPERS



GAUSSIAN LAW EVERYWHERE



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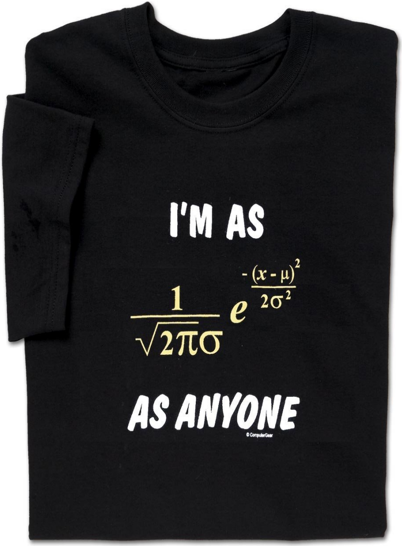

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