Dr Flajolet's elixir
or
Mellin transform and asymptotics

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Mellin transform

$$f^*(s) = \int_0^{+\infty} f(x)\, x^{s-1}\, dx$$

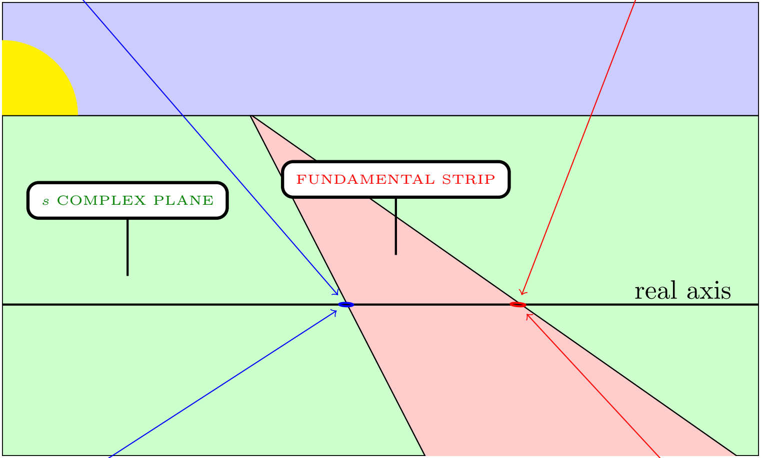
image
defined somewhere
or nowhere
in the complex
plane

original
defined on the
real positive
half-line

Fundamental strip

$f(x) \underset{x\to 0}{=} O(x^{\alpha})$

$f(x) \underset{x\to +\infty}{=} O(x^{-\beta})$



abscissa $-\alpha$

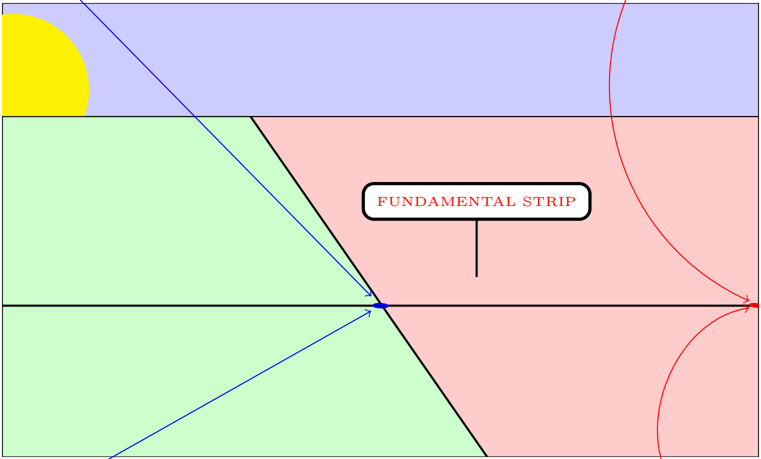
$$f^*(s) = \int_0^{+\infty} f(x) x^{s-1}\, dx$$

abscissa β

The most basic example: the Gamma function

$e^{-x} \underset{x \to 0}{=} O(x^{0})$

$e^{-x} \underset{x \to -\infty}{=} O(x^{-\infty})$

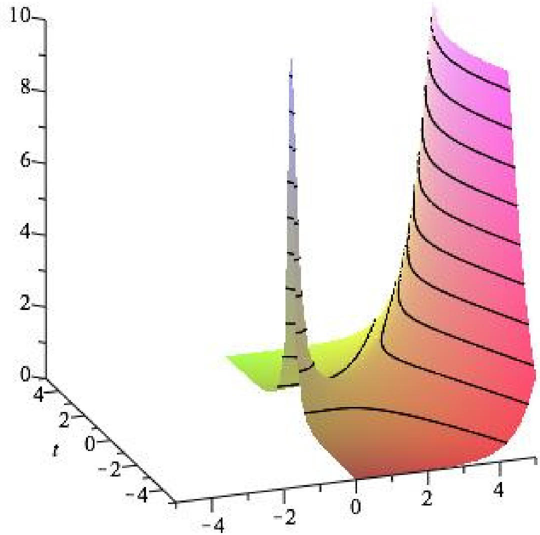


abscissa 0

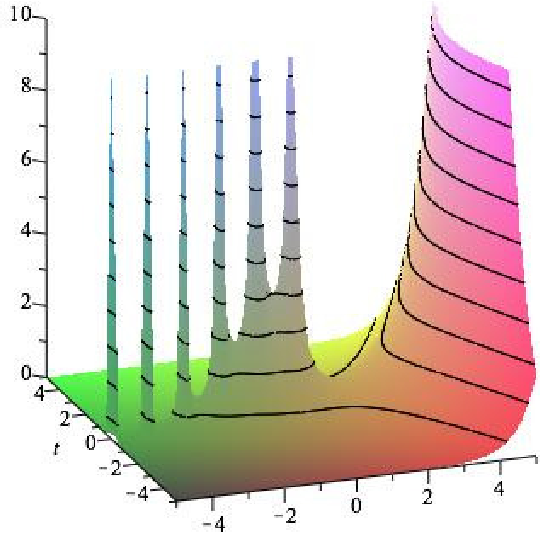
$$\Gamma(s) = \int_0^{+\infty} e^{-x} x^{s-1}\, dx$$

abscissa $+\infty$

Plot of the gamma function

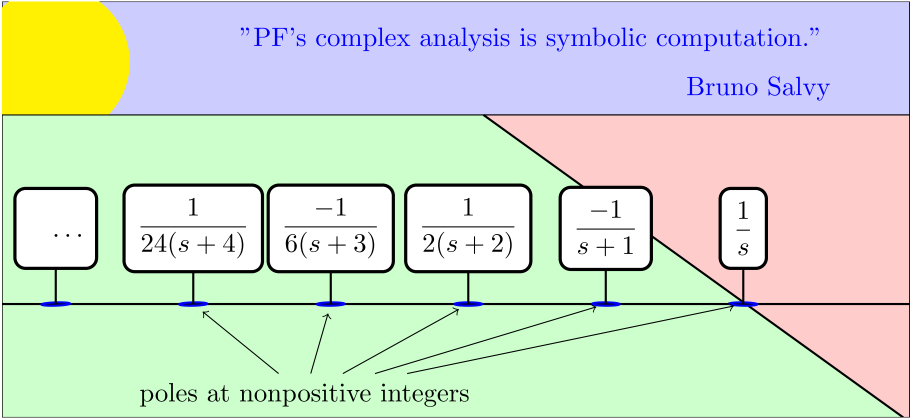


Plot of the gamma function



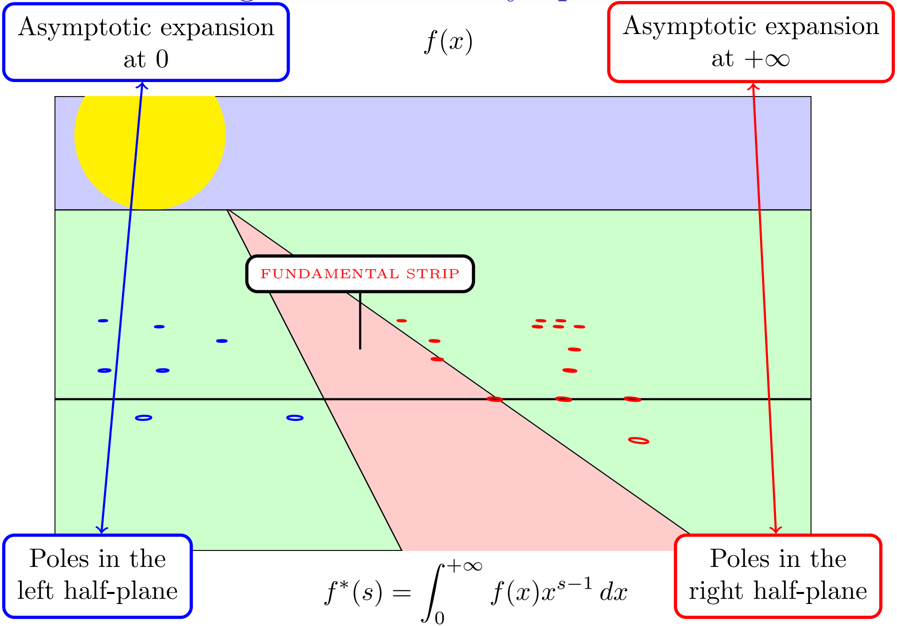
Analytic functions as data structures

$$f(x) = e^{-x}$$



$$f^*(s) = \Gamma(s) = \int_0^{+\infty} e^{-x} x^{s-1}\, dx$$

Singularities and asymptotics

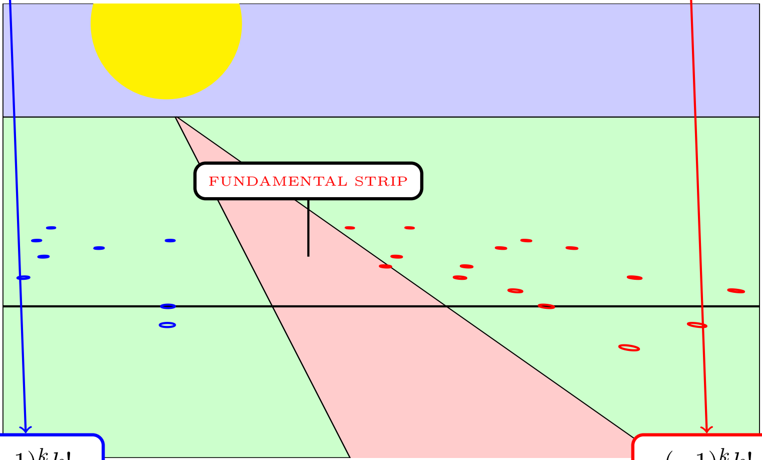


Singularities and asymptotics

$x^{\xi}\ln^{k}x$ $\quad x\to 0$

$f(x)$

$-x^{\xi}\ln^{k}x$ $\quad x\to+\infty$



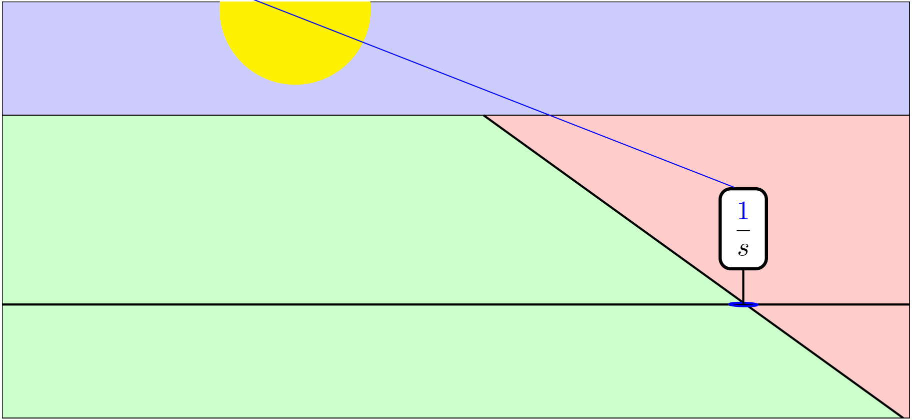
$$\frac{(-1)^{k}k!}{(s+\xi)^{k+1}}$$

$$f^{*}(s)=\int_{0}^{+\infty}f(x)x^{s-1}\,dx$$

$$\frac{(-1)^{k}k!}{(s+\xi)^{k+1}}$$

Again, the most basic example

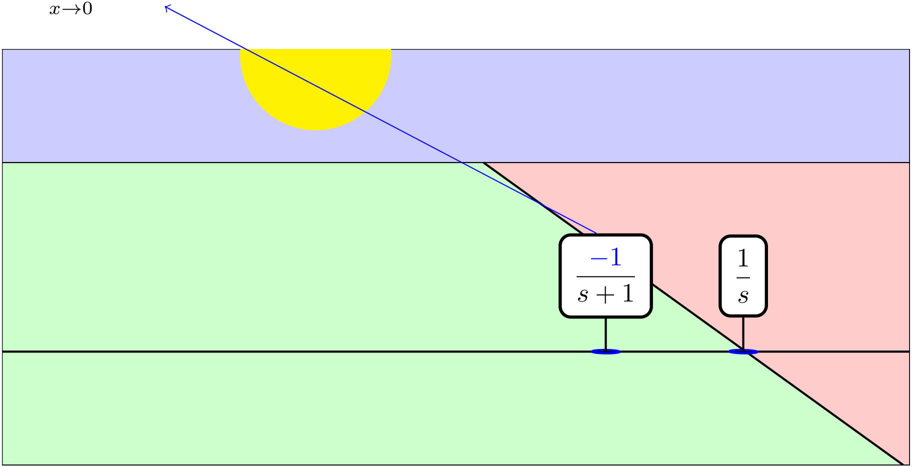
$$e^{-x} \underset{x\to 0}{=} 1$$



$$f^*(s) = \Gamma(s) = \int_0^{+\infty} e^{-x} x^{s-1}\, dx$$

Again, the most basic example

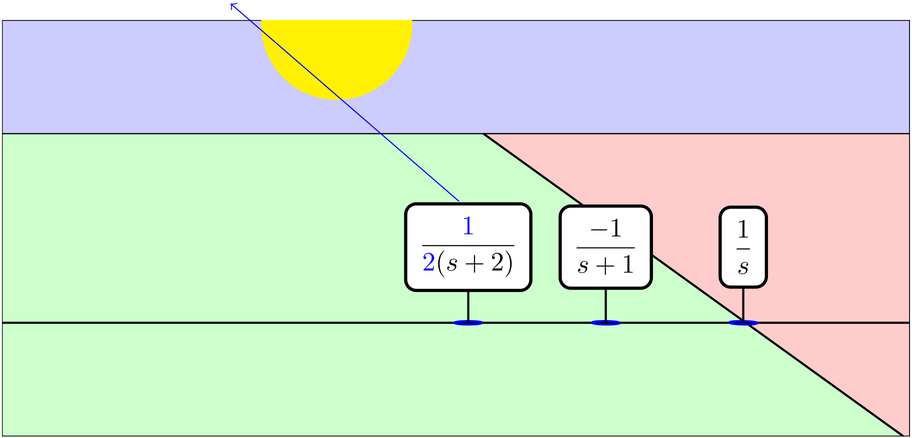
$$e^{-x} \underset{x\to 0}{=} 1 - x$$



$$f^*(s) = \Gamma(s) = \int_0^{+\infty} e^{-x} x^{s-1}\, dx$$

Again, the most basic example

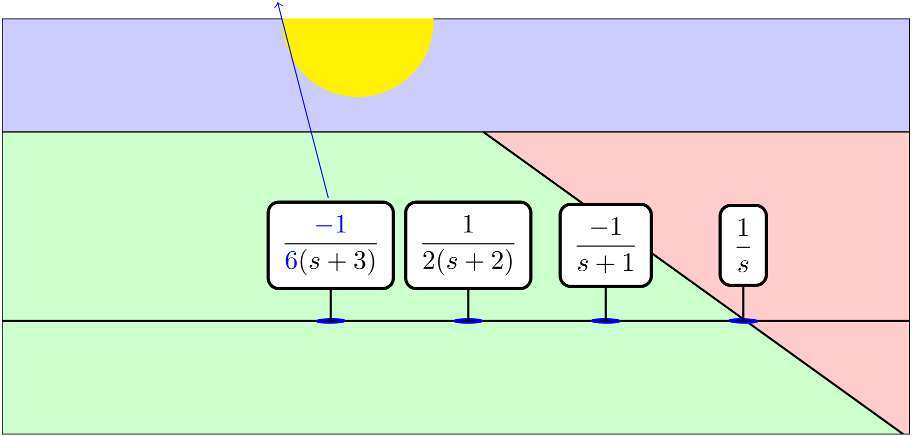
$$e^{-x} \underset{x\to 0}{=} 1 - x + \frac{1}{2}x^2$$



$$f^*(s) = \Gamma(s) = \int_0^{+\infty} e^{-x} x^{s-1}\, dx$$

Again, the most basic example

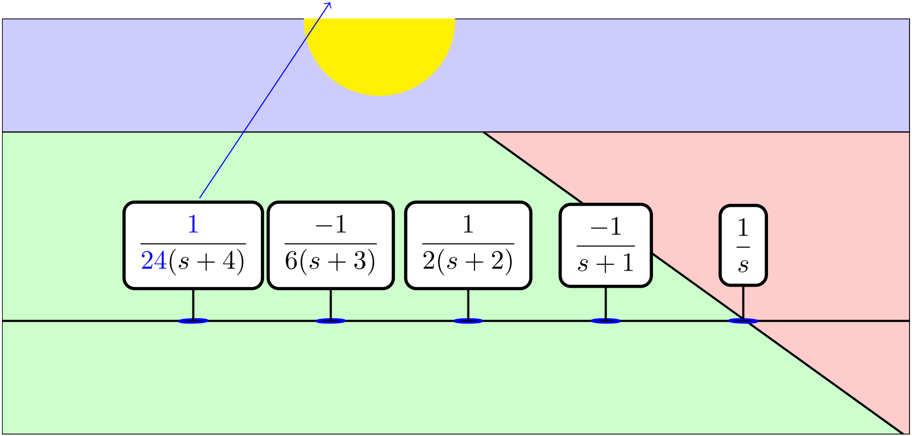
$$e^{-x} \underset{x\to 0}{=} 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3$$



$$f^*(s) = \Gamma(s) = \int_0^{+\infty} e^{-x} x^{s-1}\, dx$$

Again, the most basic example

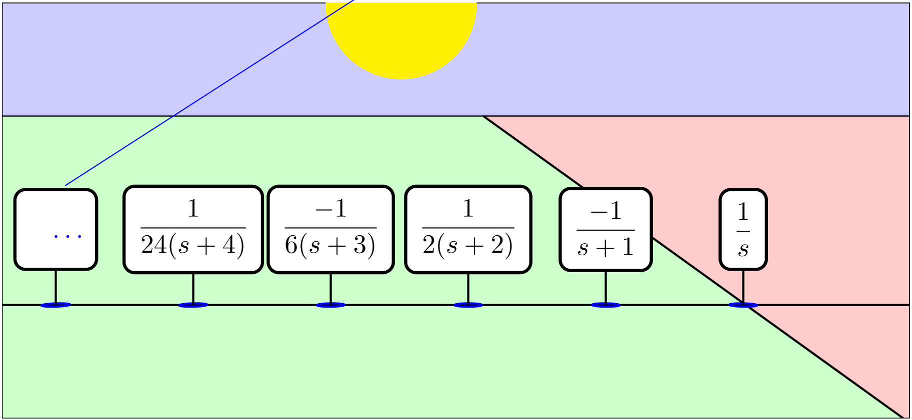
$$e^{-x} \underset{x\to 0}{=} 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4$$



$$f^*(s) = \Gamma(s) = \int_0^{+\infty} e^{-x} x^{s-1}\, dx$$

Again, the most basic example

$$e^{-x} \underset{x\to 0}{=} 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 + \dots$$



$$f^*(s) = \Gamma(s) = \int_0^{+\infty} e^{-x} x^{s-1}\, dx$$

Harmonic sums

base function

$f(x)$

Mellin transform

$f^*(s)$

Harmonic sums

base function

harmonic sum

$f(x)$ → harmonic sum → $F(x) = \sum_k \lambda_k f(\mu_k x)$

Mellin transform ↓

$f^*(s)$

Harmonic sums

base function | harmonic sum

$f(x)$ — harmonic sum → $F(x) = \sum_k \lambda_k f(\mu_k x)$

Mellin transform ↓

$f^*(s)$ → $F^*(s) = f^*(s)\Lambda(s)$

$$\Lambda(s) = \sum_k \frac{\lambda_k}{{\mu_k}^s}$$

generalized Dirichlet series

Zigzag method

$f_1(x) = \exp(-x)$ $\longrightarrow$ $f_1^*(s) = \Gamma(s) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!} \frac{1}{s+k}$

Zigzag method

$$f_1(x) = \exp(-x)$$

$$f_1^*(s) = \Gamma(s) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!} \frac{1}{s+k}$$

$$f_2(x) = \frac{x}{\exp(x) - 1}$$

$$= \sum_{n=1}^{+\infty} x \exp(-nx)$$

$$f_2^*(s) = \Gamma(s+1)\zeta(s+1)$$

$$= \frac{1}{s} + \sum_{j=0}^{+\infty} \frac{(-1)^j \zeta(-j)}{j!} \frac{1}{s+j+1}$$

Zigzag method

$$f_1(x) = \exp(-x)$$

$$f_1^*(s) = \Gamma(s) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!} \frac{1}{s+k}$$

$$f_2(x) = \frac{x}{\exp(x) - 1}$$

$$= \sum_{n=1}^{+\infty} x \exp(-nx)$$

$$= 1 - \frac{1}{2}x + \sum_{k=1}^{+\infty} B_{2k} \frac{x^{2k}}{(2k)!}$$

$$B_3 = B_5 = \ldots = 0$$

$$f_2^*(s) = \Gamma(s+1)\zeta(s+1)$$

$$= \frac{1}{s} + \sum_{j=0}^{+\infty} \frac{(-1)^j \zeta(-j)}{j!} \frac{1}{s+j+1}$$

Zigzag method

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$$= \frac{1}{s} - \frac{1}{2}\frac{1}{s+1} + \sum_{k=1}^{+\infty} \frac{B_{2k}}{(2k)!} \frac{1}{s+2k}$$

$$B_3 = B_5 = \ldots = 0$$

$$\zeta(-2) = \zeta(-4) = \ldots = 0$$

Zigzag method

$$f_1(x) = \exp(-x)$$

$$f_1^*(s) = \Gamma(s) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!} \frac{1}{s+k}$$

$$f_2(x) = \frac{x}{\exp(x)-1}$$

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$$B_3 = B_5 = \ldots = 0$$

$$\zeta(-2) = \zeta(-4) = \ldots = 0$$

$$f_3(x) = \sum_{k=1}^{+\infty} \exp(-k^2x^2)$$

$$f_3^*(s) = \frac{1}{2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

Zigzag method

$$f_1(x) = \exp(-x)$$

$$f_2(x) = \frac{x}{\exp(x) - 1}$$

$$= \sum_{n=1}^{+\infty} x \exp(-nx)$$

$$= 1 - \frac{1}{2}x + \sum_{k=1}^{+\infty} B_{2k} \frac{x^{2k}}{(2k)!}$$

$$B_3 = B_5 = \ldots = 0$$

$$f_3(x) = \sum_{k=1}^{+\infty} \exp(-k^2 x^2)$$

$$= \frac{\sqrt{\pi}}{2} \frac{1}{x} - \frac{1}{2}$$

$$f_1^*(s) = \Gamma(s) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!} \frac{1}{s+k}$$

$$f_2^*(s) = \Gamma(s+1)\zeta(s+1)$$

$$= \frac{1}{s} + \sum_{j=0}^{+\infty} \frac{(-1)^j \zeta(-j)}{j!} \frac{1}{s+j+1}$$

$$= \frac{1}{s} - \frac{1}{2} \frac{1}{s+1} + \sum_{k=1}^{+\infty} \frac{B_{2k}}{(2k)!} \frac{1}{s+2k}$$

$$\zeta(-2) = \zeta(-4) = \ldots = 0$$

$$f_3^*(s) = \frac{1}{2} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

$$= \frac{\sqrt{\pi}}{2} \frac{1}{s-1} - \frac{1}{2} \frac{1}{s}$$

Zigzag method

$$f_1(x) = \exp(-x)$$

$$f_2(x) = \frac{x}{\exp(x) - 1}$$

$$= \sum_{n=1}^{+\infty} x \exp(-nx)$$

$$= 1 - \frac{1}{2}x + \sum_{k=1}^{+\infty} B_{2k} \frac{x^{2k}}{(2k)!}$$

$$B_3 = B_5 = \ldots = 0$$

$$f_3(x) = \sum_{k=1}^{+\infty} \exp(-k^2 x^2)$$

$$\underset{x \to 0}{=} \frac{\sqrt{\pi}}{2} \frac{1}{x} - \frac{1}{2} + O(x^{+\infty})$$

$$f_1^*(s) = \Gamma(s) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!} \frac{1}{s+k}$$

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$$= \frac{\sqrt{\pi}}{2} \frac{1}{s-1} - \frac{1}{2}\frac{1}{s} + \ \ldots$$

Zigzag method

$$f_3(x) = \sum_{k=1}^{+\infty} \exp(-k^2 x^2)$$

$$\underset{x \to 0}{=} \frac{\sqrt{\pi}}{2}\frac{1}{x} - \frac{1}{2} + O(x^{+\infty})$$

$$f_3^*(s) = \frac{1}{2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

$$= \frac{\sqrt{\pi}}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s} + \ \ldots$$

$$f_4(x) = \sum_{k=1}^{+\infty} d(k) \exp(-k^2 x^2)$$

$$\underset{x \to 0}{=} -\frac{\sqrt{\pi}}{2}\frac{\ln x}{x} + \frac{C}{x} + \frac{1}{4} + O(x^{+\infty})$$

$$f_4^*(s) = \frac{1}{2}\Gamma\left(\frac{s}{2}\right)\zeta(s)^2$$

$$= \frac{\sqrt{\pi}}{2}\frac{1}{(s-1)^2} + \frac{C}{s-1} + \frac{1}{s} + \ \ldots$$

$$C = \frac{\sqrt{\pi}}{4}(3\gamma - \ln 4)$$

Average case analysis of algorithms and harmonic sums

De Bruijn, Knuth, Rice, 1972	average height of planar trees	$\sum_{k\geq 1} k^b d(k) \exp(-k^2/n)$
Knuth, 1973	radix exchange sorting	$\sum_{j\geq 1} 2^j (\exp(-n/2^j) - 1 + n/2^j)$
Sedgewick, 1978	odd-even merge	$\sum_{k\geq 1} \nabla\Delta F(k) \exp(-k^2/j)$
Kemp, 1979	register allocation	$\sum_{k\geq 1} k^a v_2(2k) \exp(-16k^2/n)$

Average case analysis of algorithms and harmonic sums

A very non-exhaustive list

Flajolet, Odlyzko, 1981/1982	average height of simple trees	$\sum_{n\geq 1} \sigma_r(n) \exp(-nu)$
Flajolet, Puech, 1983/1986	retrieval of multi-dimensional data	$\sum_{\ell}^{k-1} \varepsilon_\ell \sum_{j\geq 0} 2^{j(k-s)} \times$ $(1 - \exp(-x\alpha_{i,j}) - \beta_{j,\ell} \exp(-x\alpha_{j,\ell}))$
Fayolle, Flajolet, Hofri, 1986	multi-access brodcast channel	$\sum_{\sigma\in H} \exp(-\mu)\,(\,(1+K\mu)(\exp(-rx)-1-rx)$ $+\,Krx(\exp(-rx)-1)\,)$ with $H = \{\sigma_1, \sigma_2\}^*$, $\sigma_1(z) = \lambda + pz$, $\sigma_2(z) = \lambda + qz$, $\sigma(z) = \mu + rz$

Average case analysis of algorithms and harmonic sums

Flajolet, Richmond, 1992	generalized digital trees	$\displaystyle\sum_{k=0}^{+\infty} 2^k \frac{(1+2^k t)^{b-1}}{Q(2^k t)^b}$ with $Q(u) = \displaystyle\prod_{j=0}^{+\infty}\left(1+\frac{u}{2^j}\right)$
Mahmoud, Flajolet, Jacquet, Régnier, 2000	bucket selection and sorting	$z\displaystyle\sum_{k=0}^{+\infty}(1-\exp(-z/B^k))$ and relatives

Average case analysis of algorithms and harmonic sums

Flajolet, Fusy, Gandouet, Meunier, 2007	cardinality estimation	$\displaystyle\sum_{k=1}^{+\infty}(\exp(-x/2^k) - \exp(-2x/2^k))\exp(-xu/2^k)$
Broutin, Flajolet, 2010	height in non-plane binary trees	$\displaystyle\sum_{h\geq 1} h^r \frac{\exp(-ht)}{(1-\exp(-ht))^2}$

Why so many exponentials?

$$(1-a)^n \underset{n\to+\infty}{=} \exp(-na)\left(1+O(na^2)\right) \quad \text{with} \quad na \underset{n\to+\infty}{=} n^{\varepsilon}, \qquad \varepsilon < 1/2$$

$$\sum_{k=0}^{+\infty}\left(1-\left(1-\frac{1}{2^k}\right)^n\right) \underset{n\to+\infty}{=} F(n)+O\left(\frac{1}{\sqrt{n}}\right) \qquad F(x) = \sum_{k=0}^{+\infty}(1-\exp(-x/2^k))$$

$$\frac{\binom{2n}{n-k}}{\binom{2n}{n}} \underset{n\to+\infty}{=} \exp(-w^2)\left(1+O\left(\frac{1}{n}\right)\right) \quad \text{with} \quad k = w\sqrt{n}, \quad k \underset{n\to+\infty}{=} o(n^{3/4})$$

$$\sum_{k=1}^{n} d(k)\frac{\binom{2n}{n-k}}{\binom{2n}{n}} \underset{n\to+\infty}{=} G\left(\frac{1}{\sqrt{n}}\right)+o(1) \qquad G(x) = \sum_{k=1}^{+\infty} d(k)\exp(-k^2x^2)$$

Register allocation for binary trees

local behaviour of $E(z)$ at $z = 1/4$

$$E(z) = \frac{1-u^2}{u} \sum_{k \geq 1} v_2(k) u^k$$

$$z = \frac{u}{(1+u)^2}, \qquad u = \frac{1-r}{1+r}, \qquad r = \sqrt{1-4z}$$

change of variable $u = \exp(-t)$

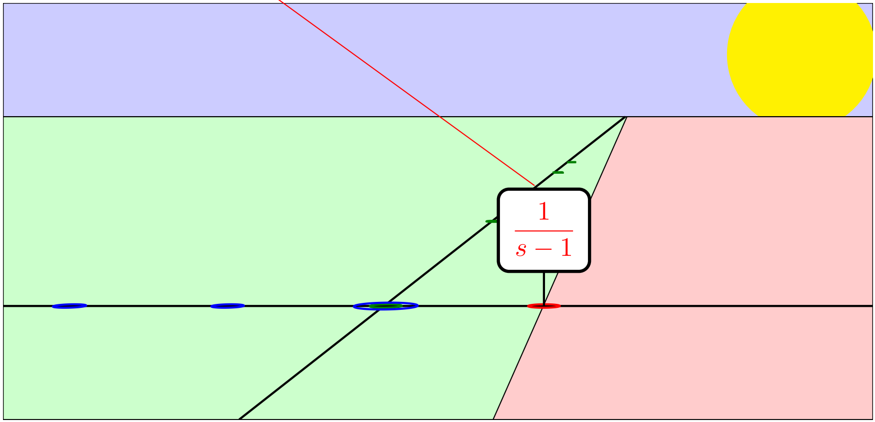
$$z = \frac{1}{4} \leftrightsquigarrow u = 1 \leftrightsquigarrow t = 0$$

$$V(t) = \sum_{k \geq 1} v_2(k) \exp(-kt) \qquad \longrightarrow \qquad V^*(s) = \frac{\zeta(s)}{2^s - 1} \times \Gamma(s)$$

Mellin transform

Register allocation for binary trees

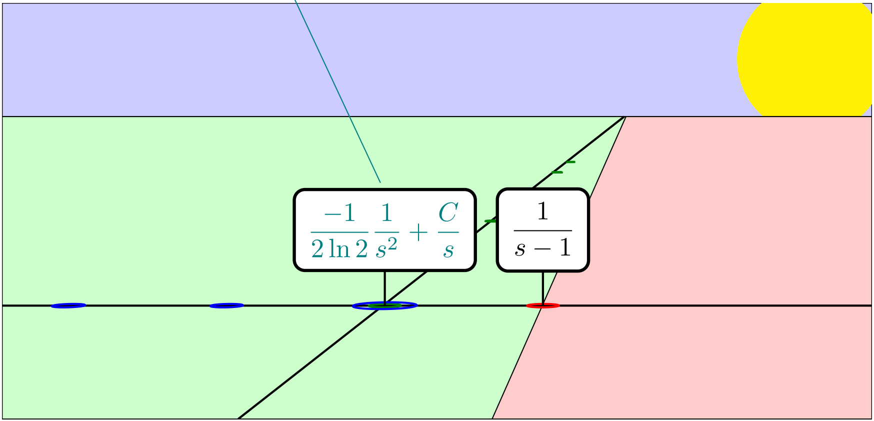
$$V(t) = \frac{1}{t} \qquad\qquad V^*(s) = \frac{\zeta(s)}{2^s - 1} \times \Gamma(s)$$



Register allocation for binary trees

$$V(t) = \frac{1}{t} + \frac{+1}{2\ln 2}\ln(t) + C \qquad V^*(s) = \frac{\zeta(s)}{2^s - 1} \times \Gamma(s)$$

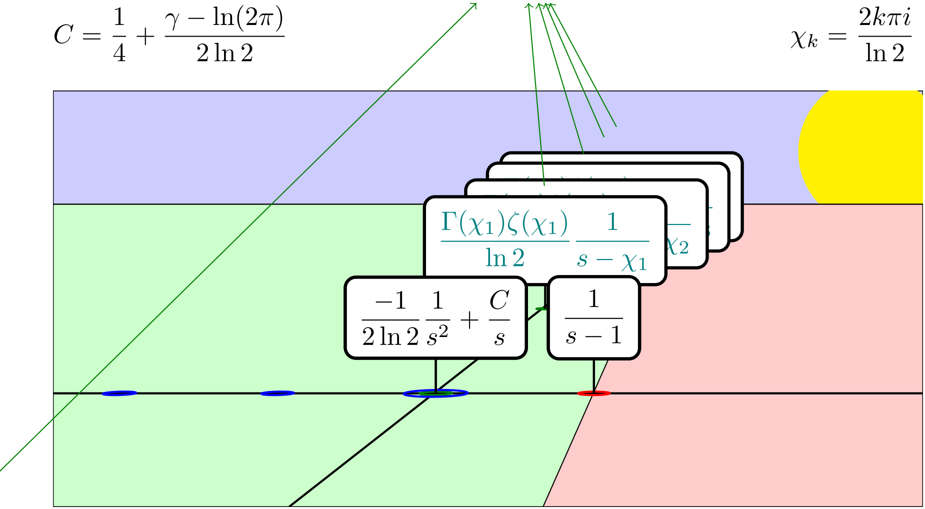
$$C = \frac{1}{4} + \frac{\gamma - \ln(2\pi)}{2\ln 2}$$



Register allocation for binary trees

$$V(t) = \frac{1}{t} + \frac{+1}{2\ln 2}\ln(t) + C + \sum_{k \neq 0} \frac{\Gamma(\chi_k)\zeta(\chi_k)}{\ln 2} t^{-\chi_k} \qquad V^*(s) = \frac{\zeta(s)}{2^s - 1} \times \Gamma(s)$$

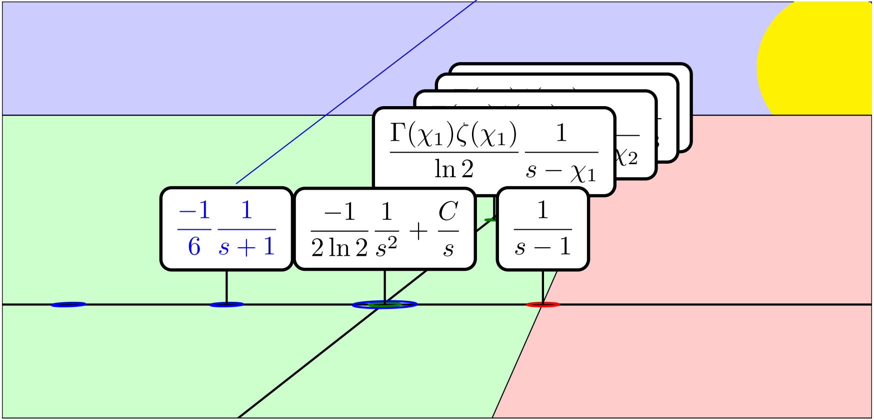
$$C = \frac{1}{4} + \frac{\gamma - \ln(2\pi)}{2\ln 2} \qquad \chi_k = \frac{2k\pi i}{\ln 2}$$



Register allocation for binary trees

$$V(t) = \frac{1}{t} + \frac{1}{2\ln 2}\ln(t) + C + \sum_{k\neq 0} \frac{\Gamma(\chi_k)\zeta(\chi_k)}{\ln 2} t^{-\chi_k} + O(t)$$

$$C = \frac{1}{4} + \frac{\gamma - \ln(2\pi)}{2\ln 2} \qquad \chi_k = \frac{2k\pi i}{\ln 2}$$



Register allocation for binary trees

$$V(t) = \frac{1}{t} + \frac{1}{2\ln 2}\ln(t) + C + P(\log_2 t) + O(t)$$

$$P(\log_2 t) = \sum_{k\neq 0} \frac{\Gamma(\chi_k)\zeta(\chi_k)}{\ln 2} t^{-\chi_k} = \sum_{k\neq 0} \frac{\Gamma(\chi_k)\zeta(\chi_k)}{\ln 2} \exp(-2k\pi i \log_2(t))$$

next

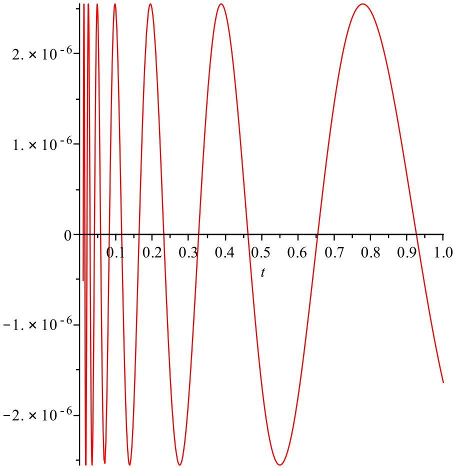
$$t = -\ln(u) = -\ln\frac{1-r}{1+r} \underset{r\to 0}{=} 2r + O(r^3) \qquad r = \sqrt{1-4z}$$

$$E(z) \underset{z\to 1/4}{=} 2 + 2r\log_2 r + (2C+1)r + 4rP(\log_2 r) + O(r^2)$$

and so on

Register allocation for binary trees

$$P(\log_2 t) = \sum_{k\neq 0} \frac{\Gamma(\chi_k)\zeta(\chi_k)}{\ln 2} t^{-\chi_k} = \sum_{k\neq 0} \frac{\Gamma(\chi_k)\zeta(\chi_k)}{\ln 2} \exp(-2\pi i \log_2(t))$$



$$|\Gamma(\chi_1)| \simeq 5.5\ 10^{-7}$$

$$|\Gamma(\chi_2)| \simeq 2.5\ 10^{-13}$$

$$|\Gamma(\chi_3)| \simeq 1.4\ 10^{-19}$$

$$C \simeq -0.66$$

Other topics

– Rice formula

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k f_k = \frac{(-1)^n}{2\pi i} \int_{\mathcal{C}} f(s) \frac{n!}{s(s-1)\cdots(s-n)}\, ds$$

– Poisson-Mellin-Newton cycle

$$f_n$$

Poisson

Rice

$$\hat{f}(t) = \sum_{n=0}^{\infty} f_n \frac{e^{-t} t^n}{n!}$$

Mellin

$$\hat{f}^*(s) = \Gamma(s) \sum_{n=0}^{\infty} f_n \frac{s(s+1)\cdots(s+n-1)}{n!}$$

Other topics

P. Flajolet and R. Sedgewick, *Mellin transforms and asymptotics: Finite differences and Rice's integrals*, Research Report 2231, Institut National de Recherche en Informatique et en Automatique (INRIA), 1994.

21 pages. For published version, see [2].

———, *Mellin transforms and asymptotics: Finite differences and Rice's integrals*, Theoretical Computer Science, 144 (1995), pp. 101–124.

For preliminary version, see [1].

Other topics

- magic duality

$$F(z) \underset{z\to 0}{=} \sum_{n\geq 1} \phi(n)(-z)^n, \qquad F(z) \underset{z\to +\infty}{=} -\sum_{n\geq 1} \phi(-n)(-z)^{-n} + \cdots$$

(see *Analytic combinatorics*, p. 238, for references)

- Mellin-Perron formula

$$\sum_{1\leq k<n} u_k + \frac{1}{2} u_k = \ldots$$

Bibliographie

P. Flajolet, X. Gourdon, and P. Dumas, *Mellin transforms and asymptotics: Harmonic sums*, Research Report 2369, Institut National de Recherche en Informatique et en Automatique (INRIA), 1994.

55 pages. For published version, see [2].

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For preliminary version, see [1].

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