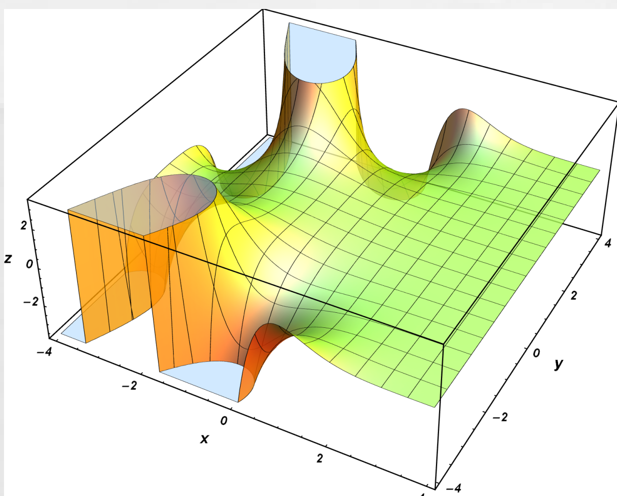
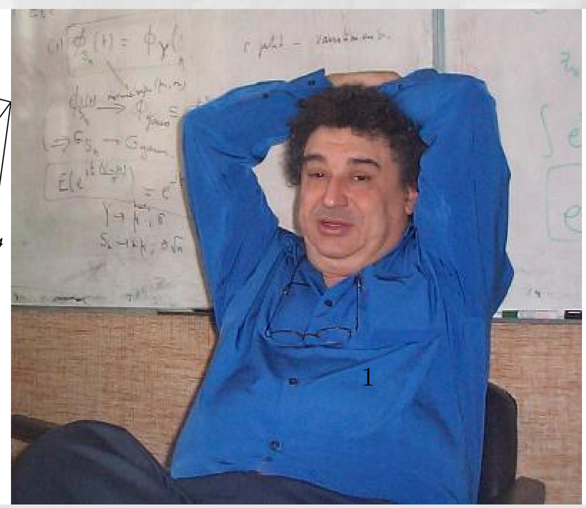
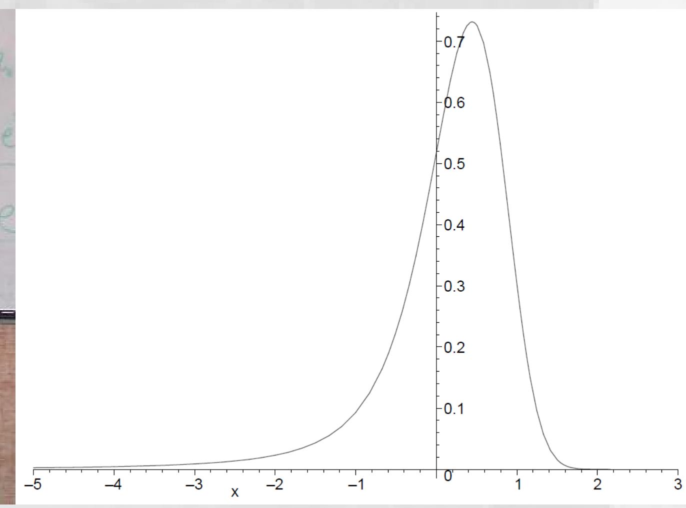
Philippe Flajolet's contributions on the Airy distributions

Cyril Banderier (U. Nord de Paris) & Guy Louchard (U. Libre de Bruxelles)

December 16, 2011







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6. Analytic variations on the Airy distribution
7. Hashing, trees, paths and graphs

2

Which one is Philippe's grandfather?

3

Which one is Philippe's grandfather?

Sir George Biddell Airy
(1801-1892)
Royal astronomer
Cambridge

Philippe Flajolet
(1885-1948)
Republican astronomer
Observatory of Lyon

4

The Airy function

Sir George Biddel Airy:
On the intensity of Light in the neighbourhood of a Caustic.
Trans. Camb. Phil. Soc. v. 6 (1838)

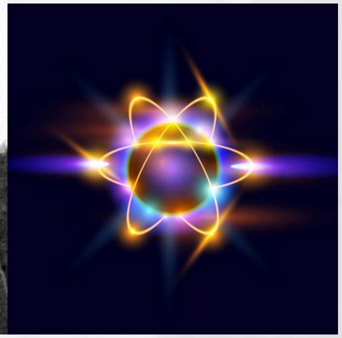
Airy Function $y'' - zy = 0$

$$\mathrm{Ai}(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i(zt+t^3/3)}\, dt$$

$$= \frac{1}{\pi 3^{2/3}} \sum_{n=0}^{\infty} \frac{\Gamma(\frac{n+1}{3})}{n!} \sin\left(\frac{2(n+1)\pi}{3}\right)\left(3^{1/3}x\right)^n$$

Airy $\approx$ hypergeometric series ${}_2F_0(z)$
$\approx$ integral representation $\approx$ Bessel functions $I_\nu(z), K_\nu(z)$ at $\nu = 1/3$
physics: optics, quantum mechanics, electromagnetics, radiative transfer
combinatorics : in some limit laws of discrete structures

The Airy function: a special function ubiquitous in physics

Schrödinger equation:

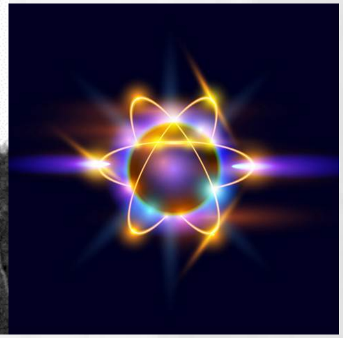
$$-\frac{\hbar}{2w}\psi''(x) + gx\psi(x) = E\psi(x)$$

[If you don't know about physics,
just see this as a differential equation
for a classical function ψ from $\mathbb{C}$ to $\mathbb{C}$.]

For which E do we have $\psi(0) = 0$?

Hint: Airy Function $y'' - zy = 0$

The Airy function: a special function ubiquitous in physics



Schrödinger equation:

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for a classical function ψ from $\mathbb{C}$ to $\mathbb{C}$.]

For which E do we have $\psi(0) = 0$?

Hint: Airy Function $y'' - zy = 0$

Answer: A change of variable gives that E has to be a zero of the Airy function!

Recently proven phenomenon in physics: Quantum states of neutrons in the Earth's gravitational field, at energy levels being nothing else than quotients of Airy zeroes! (up to 4 significative digits!)

[Thanks to the polish gang "Combinatorial Physics" for this nice example!]

**************** [Note added after the talk:]

We were asked some precisions about the previous example, as its sketchy presentation sounds to contradict the Cauchy–Lipschitz theorem (aka Picard–Lindelöf theorem).

It is true that for a given N, and some initial conditions, the equation $-a.y'' + b.x.y = N.y$ has a unique solution $y(z) = A.\,\mathrm{AiryAi}(blah(z)) + B.\,\mathrm{AiryBi}(blah(z))$ where $blah(z)$ is the appropriate linear change of variable.

If we stop here, there will always be a solution with $y(0) = 0$, but now, the quantum mechanics comes into the game: the physics of the Schrödinger equation implies that $y(\pm\infty) < \infty$ and thus $B = 0$.

The initial condition $y(0) = 0$ then implies
either $A = 0$ (but then $y(z) = 0$, which is not a physical solution)
either $blah(0)$ is a zero α_k of the Airy function, which in turn constraints N to belong to a discrete set of values: $N = -a^{1/3} b^{2/3} \alpha_k$.
****

The Airy function

George Biddel Airy:
On the intensity of Light in the neighbourhood of a Caustic.
Trans. Camb. Phil. Soc. v. 6 (1838)

Airy Function $y'' - zy = 0$

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Airy $\approx$ hypergeometric series ${}_2F_0(z)$
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physics: optics, quantum mechanics, electromagnetics, radiative transfer
COMBINATORICS: in some limit laws of discrete structures

Airy limit lawSSSSSSSS

There are 3 families of limit laws related to the Airy function:
- Tracy-Widom distribution (distribution of spectra of random matrices; zeroes of ζ; patience sorting)
- Area-Airy distribution (area under Brownian motion)
- Map-Airy distribution (largest connected component in a graph, stable law 3/2)

All of them can be obtained via analytic combinatorics

(c) [Flajolet–Sedgewick 2009].

10

THE FIRST CYCLES IN AN EVOLVING GRAPH

Philippe FLAJOLET,* Donald E. KNUTH,** and Boris PITTEL†

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Revised November 1988

If successive connections are added at random to an initially disconnected set of n points, the expected length of the first cycle that appears will be proportional to $n^{\frac{1}{6}}$, with a standard deviation proportional to $n^{\frac{1}{4}}$. The size of the component containing this cycle will be of order $n^{\frac{1}{2}}$, on the average, with standard deviation of order $n^{\frac{7}{12}}$. The average length of the kth cycle is proportional to $n^{\frac{1}{6}}(\log n)^{k-1}$. Furthermore, the probability is $\sqrt{\frac{2}{3}} + O(n^{-\frac{1}{3}})$ that the graph has no components with more than one cycle at the moment when the number of edges passes $\frac{1}{2}n$. These results can be proved with analytical methods based on combinatorial enumeration with multivariate generating functions, followed by contour integration to derive asymptotic formulas for the quantities of interest.

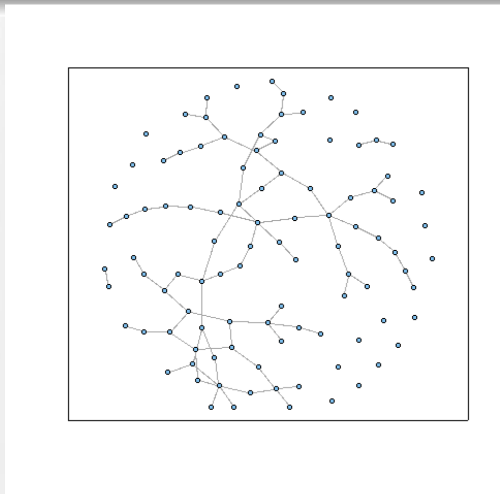
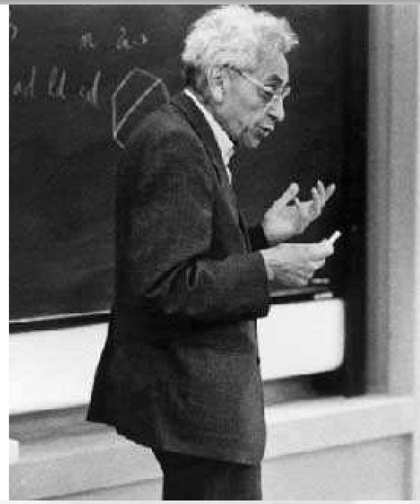
A classic paper by Erdős and Rényi [6] inaugurated the study of the random graph process, in which we begin with a totally disconnected graph and enrich it by successively adding edges. Algorithms that deal with graphs often mimic such a process, inputting a sequence of edges until some stopping criterion occurs, based on the configuration of edges seen so far. To analyze such algorithms, we wish to estimate relevant characteristics of the resulting graph. For example, we might stop when the graph first contains a particular kind of subgraph, and we might ask how large that subgraph is.

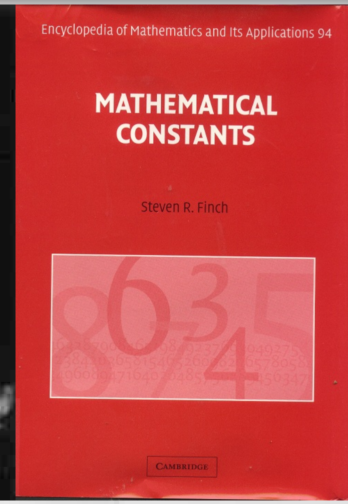
The purpose of this paper is to introduce analytical methods by which such questions can be answered systematically. In particular, we will apply the ideas to an interesting question posed by Paul Erdős and communicated by Edgar Palmer to the 1985 Seminar on Random Graphs in Posnań: "What is the expected length of the first cycle in an evolving graph?" The answer turns out to be rather surprising: The first cycle has length $Kn^{\frac{1}{6}} + O(n^{\frac{3}{22}})$ on the average, where

$$K = \frac{1}{\sqrt{8\pi}\,\mathrm{i}} \int_{-\infty}^{\infty} \int_{\Gamma} \mathrm{e}^{(\mu+2s)(\mu-s)^2/6} \frac{\mathrm{d}s}{s}\, \mathrm{d}\mu \approx 2.0337$$

for a certain contour Γ. The form of this result suggests that the expected

This research was supported in part by the National Science Foundation under grant CCR-86-10181, and by Office of Naval Research contract N00014-87-K-0502.



Erdős–Rényi model of random graphs (1959):

G(n,p): graphs with n vertices, and probability p to get an edge

Lot of transition phases when the proportion of edges is increasing.

The first cycle has length $\sim Kn^{1/6}$ where

$$K = \frac{1}{\sqrt{8\pi i}} \int_{-\infty}^{\infty} \int \exp\left((\mu + 2s) \frac{(\mu - s)^2}{6} \right) \frac{ds}{s} d\mu \approx 2.0337$$

PF: Give me an integral!

Further works: the Janson–Knuth–Łucwak–Pittel 1993 "giant paper of the giant component" ... story still goes on in the 2000's, e.g. Ravelomanana, Wormald, Noy, Drmota...

Airy Phenomena and Analytic Combinatorics of Connected Graphs

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Submitted: Oct 11, 2002 & Mar 12, 2004; Accepted: Apr 7, 2004; Published: May 27, 2004.
MR Subject Classifications: 05A15, 05A16, 05C30, 05C40, 05C80

Abstract

Until now, the enumeration of connected graphs has been dealt with by probabilistic methods, by special combinatorial decompositions or by somewhat indirect formal series manipulations. We show here that it is possible to make analytic sense of the divergent series that expresses the generating function of connected graphs. As a consequence, it becomes possible to derive analytically known enumeration results using only first principles of combinatorial analysis and straight asymptotic analysis—specifically, the saddle-point method. In this perspective, the enumeration of connected graphs by excess (of number of edges over number of vertices) derives from a simple saddle-point analysis. Furthermore, a refined analysis based on coalescent saddle points yields complete asymptotic expansions for the number of graphs of fixed excess, through an explicit connection with Airy functions.

Introduction

E. M. Wright, of Hardy and Wright fame, initiated the enumeration of labelled connected graphs by number of vertices and edges in a well-known series of articles [34, 35, 36]. In particular, he discovered that the generating functions of graphs with a fixed excess of number of edges over number of vertices has a rational expression in terms of the tree function $T(z)$. Wright's approach is based on the fact that deletion of an edge in a

Connected Graphs

Philippe Flajolet, Bruno Salvy, and Gilles Schaeffer.
Airy phenomena and analytic combinatorics of connected graphs.
Electronic Journal of Combinatorics, 2004.

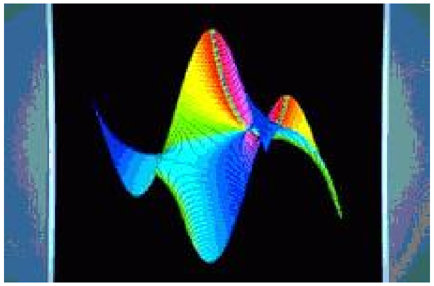
It is possible to make analytic sense of the divergent series that expresses the generating function of connected graphs $C(z) = \ln\left(\sum 2^{\binom{n}{2}} z^n\right)$.

The enumeration of connected graphs by excess (of number of edges over number of vertices) derives from a simple saddle-point analysis. Furthermore, a refined analysis based on coalescent saddle points yields complete asymptotic expansions for the number of graphs of fixed excess, through an explicit connection with Airy functions.

build on works by E.M. Wright 70's, Knuth/Flajolet/Pittel 1989, Janson/Knuth/Łuczak/Pittel 1993 "Giant paper on the giant component"

Algorithms Project's Logo

Our logo shows the behaviour in the complex plane of the generating function of connected graphs counted according to number of nodes and edges. In critical regions, two saddle points coalesce giving rise to a so-called "monkey saddle" (a saddle that you'd use if you had three legs!)



The fine analysis of this coalescence is crucial to the understanding of connectivity in random graphs. This problem has applications in the design of communication networks and it relates to a famous series of problems initiated by Erdös and Renyi in the late 1950's. See the paper [Janson, Knuth, Luczak, Pittel: The birth of the giant component. Random Structures Algorithms 4 (1993), no. 3, 231--358]. As said by Alan Frieze in his review [**MR**94h:05070]:

> *"This paper and its predecessor [**MR**90d:05184] mark the entry of generating functions into the general theory of random graphs in a significant way. Previously, their use had mainly been restricted to the study of random trees and mappings. Most of the major results in the area, starting with the pioneering papers of P. Erdös and A. Renyi [**MR**22#10924] have been proved without significant use of generating functions. However, at the early stages of the evolution of a random graph we find that it is usually not too far from being a forest, and this allows them an entry..."*

The icon was generated by Maple code like this:

```
f:=u^3/3-3*u+1;
plots[complexplot3d]([Re(f),argument(f)],
              u=-4-3*I..4+3*I,style=patchcontour,
                      contours=30,numpoints=50*50);
```

Algorithmica (1998) 22: 490–515

Algorithmica

On the Analysis of Linear Probing Hashing[1]

P. Flajolet,[2] P. Poblete,[3] and A. Viola[4]

Dedicated to Don Knuth on the occasion of the 35*th anniversary of his first analysis of an algorithm in* 1962–1963.

Abstract. This paper presents moment analyses and characterizations of limit distributions for the construction cost of hash tables under the linear probing strategy. Two models are considered, that of full tables and that of sparse tables with a fixed filling ratio strictly smaller than one. For full tables, the construction cost has expectation $O(n^{3/2})$, the standard deviation is of the same order, and a limit law of the Airy type holds. (The Airy distribution is a semiclassical distribution that is defined in terms of the usual Airy functions or equivalently in terms of Bessel functions of indices $-\frac{1}{3}, \frac{2}{3}$.) For sparse tables, the construction cost has expectation $O(n)$, standard deviation $O(\sqrt{n})$, and a limit law of the Gaussian type. Combinatorial relations with other problems leading to Airy phenomena (like graph connectivity, tree inversions, tree path length, or area under excursions) are also briefly discussed.

Key Words. Analysis of algorithms, Hashing, Linear probing, Parking problem, Airy functions.

Introduction. *Linear probing hashing*, defined below, is certainly the simplest "in place" hashing algorithm [14], [23], [38].

> A table of length m, $T[1\,..\,m]$ is set up, as well as a hash function h that maps keys from some domain to the interval $[1\,..\,m]$ of table addresses. A collection of n elements with $n \le m$ are entered sequentially into the table according to the following rule: Each element x is placed at the first unoccupied location starting from $h(x)$ in cyclic order, namely the first of $h(x), h(x)+1, \ldots, m, 1, 2, \ldots, h(x)-1$.

For each element x that gets placed at some location y, the circular distance between y and $h(x)$ (that is, $y - h(x)$ if $h(x) \le y$, and $m + h(x) - y$ otherwise) is called its *displacement*. Displacement is both a measure of the cost of inserting x and of the cost of searching x in the table. *Total displacement* corresponding to a sequence of hashed values is the sum of the individual displacements of elements. As it determines the *construction cost* of the table, we use both terms interchangeably.

We analyze here the total displacement $d_{m,n}$ of a table of length m (the number of table locations) and size n (the number of keys), under the assumption that all m^n hash

[1] The work of Philippe Flajolet was supported in part by the Long Term Research Project *Alcom-IT* (# 20244) of the European Union. The work of Patricio Poblete was supported in part by FONDECYT(Chile) under Grant 1960881. The work of Alfredo Viola was supported in part by proyecto BID-CONICYT 140/94 and proyecto CONICYT fondo Clemente Estable 2078/96.
[2] Algorithms Project, INRIA, Rocquencourt, 78150 Le Chesnay, France. Philippe.Flajolet@inria.fr.
[3] Department of Computer Science, University of Chile, Casilla 2777, Santiago, Chile. ppoblete@dcc.uchile.cl.
[4] Pedeciba Informatica, Casilla de Correo 16120, Distrito 6, Montevideo, Uruguay. viola@fing.edu.uy.

Received October 5, 1997; revised January 15, 1998. Communicated by H. Prodinger and W. Szpankowski.

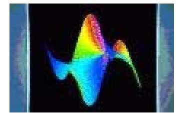
Linear Probing Hashing

Philippe Flajolet, Patricio Poblete, and Alfredo Viola.
On the analysis of linear probing hashing.
Algorithmica, 1998.
is solving an old problem, first studied by Knuth, 1962 [website]
Knuth then decided to create the field "analysis of algorithms" and to write *The Art of Computer Programming*

They get a limit law with moments given by Ai/Ai'.

Reversing the time: Linear probing hashing $\approx$ fragmentation process
(Bertoin, Pitman "additive coalescent", Rényi "parking functions", ...)

[more in Alfredo Viola's talk!]



Research News, November 1997

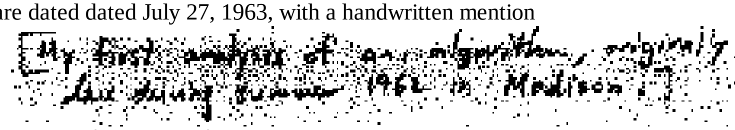
These pages are edited by Philippe.Flajolet@inria.fr.

Issue of November 1997.

[02/11/97] **Knuth's first analysis of an algorithm.** Thanks to the web, everybody can now have easy access to an original of Don Knuth's ***NOTES ON "OPEN" ADDRESSING*** . The note has 6 pages. It provided for the first time a precise analysis of **hashing with linear probing**. As Knuth repeatedly indicates, this problem has had a strong influence on his carreer.


NOTES ON "OPEN" ADDRESSING.

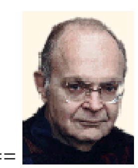
The notes are dated dated July 27, 1963, with a handwritten mention



"My first analysis of an algorithm, originally done during Summer 1962 in Madison."

There, you'll find for the first time the connection between the analysis of full tables and what was to become later known as the Knuth-Ramanujan function $Q(n)$. Of course, the analysis of sparse tables is also there.

- **Download Knuth's note**. Each file is about 30k bytes.
  [***Page1*** | ***Page2*** | ***Page3*** | ***Page4*** | ***Page5*** | ***Page6***]

 ============== 

[02/11/97] **The Ramanujan-Knuth Q-function.** Knuth's note of 1963 concludes with a conjecture about the asymptotics of the Knuth-Ramanujan function (which Knuth proved in 1965). We now know that this function together with a statement of its asymptotic behaviour was in fact already appearing in Ramanujan's first letter to Hardy dated Januray 16, 1913. Here is a version of a paper that provides a recent asymptotic analysis of this function (P. Flajolet, P. Grabner, P. Kirschenhofer, H. Prodinger. In *J. Computational and Applied Mathematics*, vol. **58** (1), 1995, pp. 103-116). Bruce Berndt has devoted a considerable part of his life to editing Ramanujan's works. See his home page for more about this fascinating adventure.

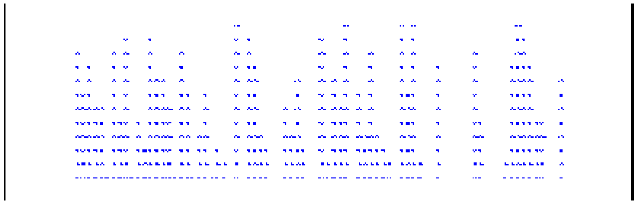
As we see from Knuth's note, Ramanujan's analysis entails that the total displacement of elements, *i.e.*, the construction cost, for a linear probing hashing table that is full is about *N*^(3/2).

[02/11/97] **Cost distribution in linear probing hashing.** In the July issue of *Research Notes*, we saw that Poblete-Viola and Knuth had independently solved the problem of the determining the variance of the construction cost of linear probing hashing. Knuth's analysis from July 1997 is available on the web here. In July 1997, Poblete and Viola merged forces with Flajolet. As a result, the variance analysis can be in fact extended to a full distributional analysis of linear probing hashing. The report is available on the web.

Here's what goes on. For sparse tables, that is to say tables whose filling ratio *a* is less than 1, then there are a large number of clusters that tend to be small. Accordingly, the distribution of costs is Gaussian in the asymptotic limit. For full tables, the distribution has a standard deviation that is of the same order as the mean and thus it is rather spread out. In the limit, what we have for this case of full tables is a so-called Airy distribution that is related to our pet icon and involves Airy functions.

For reasons explained in the above papers, the Airy distribution appears in a diversity of combinatorial problems like: (*i*) path length in trees, (*ii*) area below random walks and excursions, (*iii*) inversions in trees, (*iv*) connectivity in random graphs.

Here's the way clusters are formed in a linear probing table as it fills up (from top to bottom). See Sedgewick's new book for more of such displays (and also better ones!!)



[10/11/97] **Knuth's original note in TeX**. Here is the TeX version of Knuth's "*Notes on `Open' Addressing*". Helmut Prodinger has very kindly retyped them into TeX for everyone to see and enjoy. Versions: [TeX | dvi | postscript | ps.gz]

[30/11/97] **Odlyzko's monograph on "Asymptotic enumeration methods"**. In 1995, Elsevier published the *Handbook of Combinatorics* edited by R. L. Graham, M. Groetschel, and L. Lovasz. Andrew Odlyzko has there a survey of asymptotic enumeration methods (Volume 2 Chapter 22, pp. 1063-1229) that is really a thorough monograph of some healthy 166 pages! A preprint version of this inspiring text is on the web, as well as many other goodies that Andrew is offering on his Home Pages. I cannot resist listing the table of contents from this document that is a **must** for anyone interested in combinatorial asymptotics.

- Identities, indefinite summations. Basic estimates of factorials and binomial coefficients. Estimates of sums (alternating, Euler-Maclaurin and Poisson summation, bootstrapping, integrals). Generating functions (GF's). Formal power series. Elementary estimates for convergent GF's. Recurrences. Analytic generating functions. Small singularity of analytic functions (singularity analysis). Large singularities of analytic functions (saddle points). Multivariate GF's.

Mellin and other integral transforms. Functional equations. Algorithmic and automated asymptotics.

It is amusing to contrast a quotation of Odlyzko (1995)

- *"Analytic methods are extremely powerful and when they apply, they often yield estimates of unparalleled precision."*

and a quotation of John Riordan (1968):

- *"Combinatorialists use recurrence, generating functions, and such transformations as the Vandermonde convolution; others* ***to my horror****, use contour integrals, differential equations, and other resources of mathematical analysis."*

This was 29 years ago...

Under Odlyzko's page, you'll find a lot of relevant papers from the **guru** of analytic methods. :-) Here are some that are relative to general methods and are offered electronically:

- Analytic methods in asymptotic enumeration, A. M. Odlyzko, *Discrete Math.* 153 (1996), pp. 229-238.
- Asymptotic enumeration methods, A. M. Odlyzko, in *Handbook of Combinatorics*, vol. 2, R. L. Graham, M. Groetschel, and L. Lovasz, eds., Elsevier, 1995, pp. 1063-1229.
- Singularity analysis of generating functions, P. Flajolet and A. M. Odlyzko, *SIAM J. Discrete Math.*, 3 (1990) pp. 216-240.
- Asymptotic expansions for the coefficients of analytic generating function, A. M. Odlyzko and L. B. Richmond, *Aequationes Math.*, 28 (1985), pp. 50-63.
- Explicit Tauberian estimates for functions with positive coefficients, A. M. Odlyzko, *J. Computational Appl. Math.*, 41 (1992), pp. 187-197.

[My first analysis of an algorithm, originally done during Summer 1962 in Madison.]

NOTES ON "OPEN" ADDRESSING.

D. Knuth 7/22/63

1. Introduction and Definitions. Open addressing is a widely-used technique for keeping "symbol tables." The method was first used in 1954 by Samuel, Amdahl, and Boehme in an assembly program for the IBM 701. An extensive discussion of the method was given by Peterson in 1957 [1], and frequent references have been made to it ever since (e.g. Schay and Spruth [2], Iverson [3]). However, the timing characteristics have apparently never been exactly established, and indeed the author has heard reports of several reputable mathematicians who failed to find the solution after some trial. Therefore it is the purpose of this note to indicate one way by which the solution can be obtained.

We will use the following abstract model to describe the method: N is a positive integer, and we have an array of N variables $x_1, x_2, \ldots, x_N$. At the beginning, $x_i = 0$, for $1 \leqq i \leqq N$.

To "enter the k-th item in the table," we mean that an integer a_k is calculated, $1 \leqq a_k \leqq N$, depending only on the item, and the following process is carried out:

1. Set $j = a_k$.
2. "The comparison step." If $x_j = 0$, set $x_j = 1$ and stop; we say "the k-th item has fallen into position x_j."
3. If $j = N$, go to step 5.
4. Increase j by 1 and return to step 2.
5. "The overflow step." If this step is entered twice, the table is full, i.e. $x_i = 1$ for $1 \leqq i \leqq N$. Otherwise set j to 1 and return to step 2.

Observe the cyclic character of this algorithm.

We are concerned with the statistics of this method, with respect to the number of times the comparison step must be executed. More precisely, we consider all of the N^k possible sequences $a_1, a_2, \ldots a_k$ to be equally probable, and we ask, "What is the probability that the comparison step is used precisely m times when the k-th item is placed?"

2. Non-overflow (self-contained) sequences.

Let $\left[{n \atop k}\right]$ denote the number of sequences $a_1, a_2, \ldots a_k$ $(1 \leqq a_i \leqq n)$ in which no overflow step occurs during the entire process of placing k items, if the algorithm is used for $N = n$. (By convention, we set

$$\left[{n \atop 0}\right] = 1.)$$

Lemma 1: If $0 \leq k \leq n+1$, then $$\left[{n \atop k}\right] = (n+1)^k - k(n+1)^{k-1}.$$

Proof: This proof is based on the fact that $\left[{n \atop k}\right]$ is precisely the number of sequences $b_1, b_2, \ldots b_k$ $(1 \leqq b_i \leqq n+1)$ in which, if the algorithm is carried out for $N = n+1$, then $x_{n+1} = 0$ at the end of the operation. This follows because every sequence of the former type is one of the latter, and conversely the condition implies in particular that $1 \leqq b_i \leqq n$, and that no overflow step occurs.

But sequences of the latter type are easily enumerated, because the algorithm has circular symmetry; of the $(n+1)^k$ possible sequences $b_1, b_2, \ldots b_k$, exactly $k/(n+1)$ of these leave $x_{n+1} \neq 0$. This shows that

$$\left[{n \atop k}\right] = (n+1)^k \left(1 - \frac{k}{n+1}\right).$$

Random Maps, Coalescing Saddles, Singularity Analysis, and Airy Phenomena

Cyril Banderier,[1] Philippe Flajolet,[1] Gilles Schaeffer,[2] Michèle Soria[3]
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ABSTRACT: A considerable number of asymptotic distributions arising in random combinatorics and analysis of algorithms are of the exponential-quadratic type, that is, Gaussian. We exhibit a class of "universal" phenomena that are of the exponential-cubic type, corresponding to distributions that involve the Airy function. In this article, such Airy phenomena are related to the coalescence of saddle points and the confluence of singularities of generating functions. For about a dozen types of random planar maps, a common Airy distribution (equivalently, a stable law of exponent $\frac{3}{2}$) describes the sizes of cores and of largest (multi)connected components. Consequences include the analysis and fine optimization of random generation algorithms for a multiple connected planar graphs. Based on an extension of the singularity analysis framework suggested by the Airy case, the article also presents a general classification of compositional schemas in analytic combinatorics. © 2001 John Wiley & Sons, Inc. Random Struct. Alg., 19, 194–246, 2001

Key Words: Airy function; analytic combinatorics; coalescing saddle points; multiconnectivity; planar map; random graph; random generation; singularity analysis; stable law

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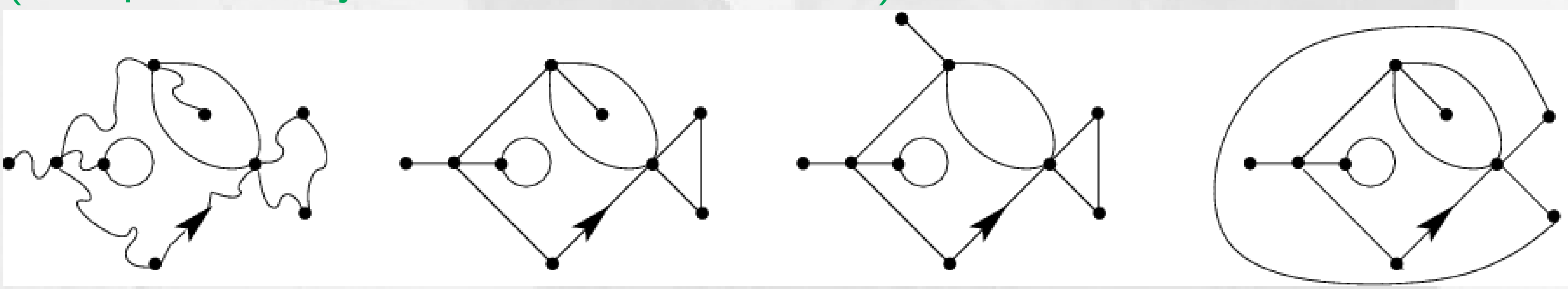
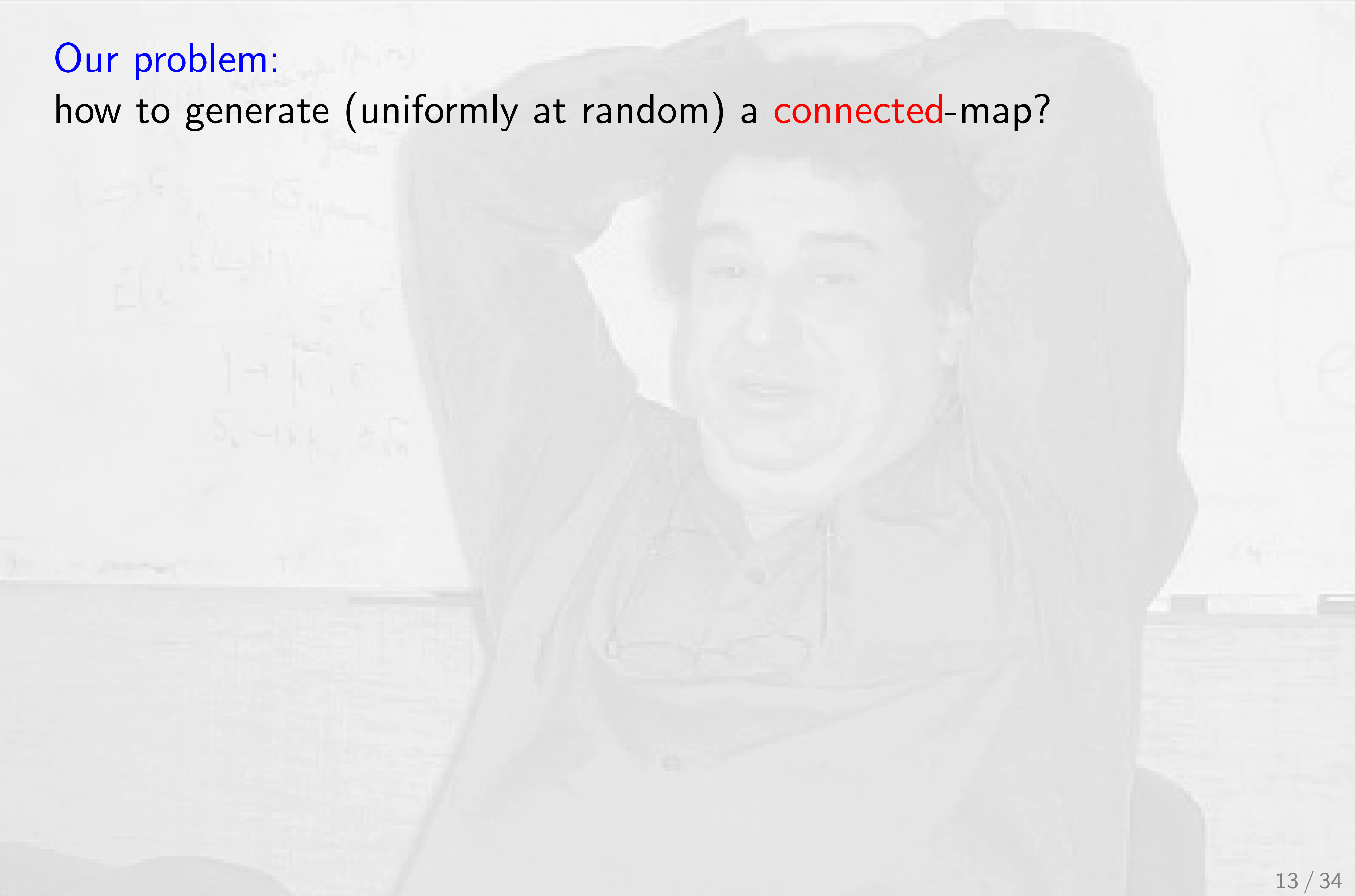
Random maps, coalescing saddles, and Airy

Cyril Banderier, Philippe Flajolet, Gilles Schaeffer, and Michèle Soria.
Random maps, coalescing saddles, singularity analysis, and Airy phenomena.
Random Structures & Algorithms, 2001.
(and preliminary version at ICALP'2000).



map = planar graph on the sphere.
Tutte wanted to refute the 4 color theorem, not succeeded but found a way to enumerate maps (∼1960).

Random maps



Our problem:
how to generate (uniformly at random) a connected-map?

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Solution: Use a rejection algorithm to reach size n

1. generate a map of size $f(n)$
2. extract largest connected component C
3. if its size $\neq n$, then goto 1
4. output C.

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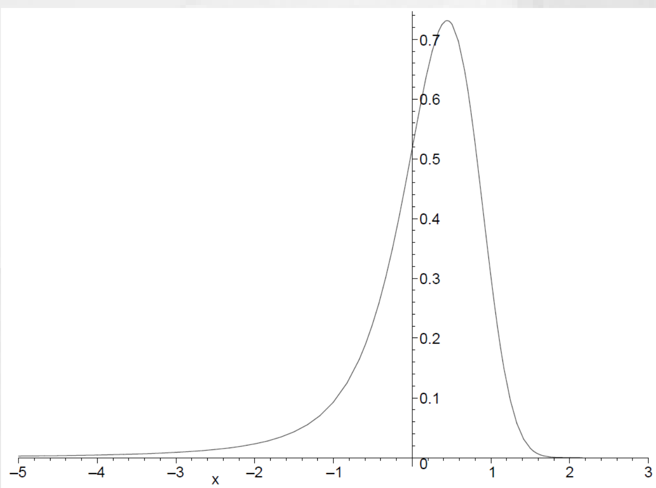

Theorem The fastest algorithm consists in choosing $f(n) = 3n - (3n)^{2/3} x_0$.

$x_0 := \max(\mathcal{A}(x))$

= peak of the Map-Airy distribution:

$\mathcal{A}(x) = 2 \exp\left(-\frac{2}{3}x^3\right)\left(x\mathrm{Ai}(x^2) - \mathrm{Ai}'(x^2)\right)$

$\int_{-\infty}^{+\infty} \mathcal{A}(x) dx = 1$

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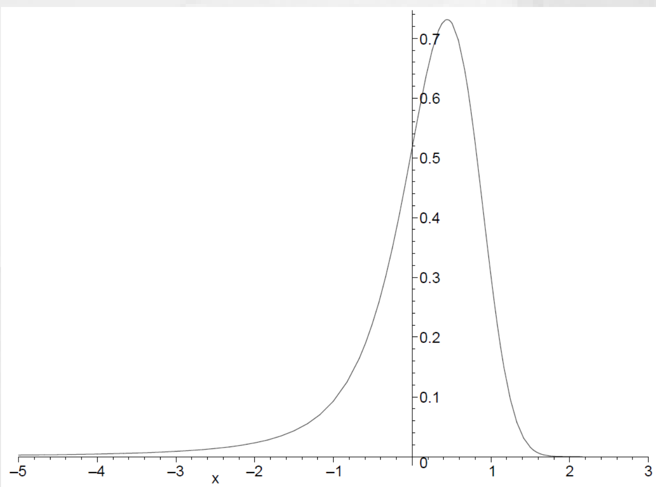

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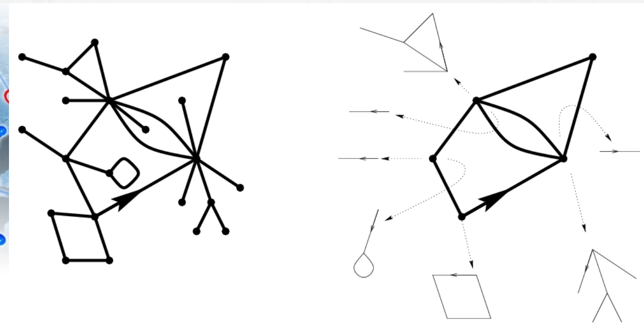
Rejection with optimization of the "parameter" to reach size $\approx n$:
Flajolet's "Boltzmann method" [M. Soria's talk]

Proof of the rejection algorithms

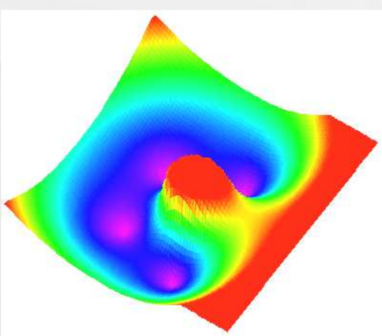



largest 2-connected component

functional equation (Tutte) via the "rootface decomposition"

$Mn, k := \#$ maps with n edges and face of degree k.

$$M(z,u) = \sum M_{n,k} z^n u^k = 1 + u^2 z M(z,u)^2 + uz \frac{M(z,1) - uM(z,u)}{1-u}$$



prob that a map of size n has a kernel of size k =

$$Pr(X_n = k) = \frac{M_{n,k}}{M_n} = C_k \frac{1}{2i\pi} \int_\gamma F(z) G(z)^k H(z)^n dz$$

$$\int_\Gamma \exp(n(a_0 + a_1(z-\tau) + a_2(z-\tau)^2 + a_3(z-\tau)^3 + \dots))dz$$

double saddle $\Leftrightarrow a_1 = 0$ et $a_2 = 0$, so one gets Airy.

Bonus: universal stable law for critical compositions [Marc Noy's talk]

Algorithmica (2001) 31: 361–377
DOI: 10.1007/s00453-001-0056-0

Algorithmica

Analytic Variations on the Airy Distribution[1]

P. Flajolet[2] and G. Louchard[3]

Abstract. The Airy distribution (of the "area" type) occurs as a limit distribution of cumulative parameters in a number of combinatorial structures, like path length in trees, area below walks, displacement in parking sequences, and it is also related to basic graph and polyomino enumeration. We obtain curious explicit evaluations for certain moments of the Airy distribution, including moments of orders -1, -3, -5, etc., as well as $+\frac{1}{3}$, $-\frac{5}{3}$, $-\frac{11}{3}$, etc. and $-\frac{7}{3}$, $-\frac{13}{3}$, $-\frac{19}{3}$, etc. Our proofs are based on integral transforms of the Laplace and Mellin type and they rely essentially on "non-probabilistic" arguments like analytic continuation. A by-product of this approach is the existence of relations between moments of the Airy distribution, the asymptotic expansion of the Airy function $\mathrm{Ai}(z)$ at $+\infty$, and power symmetric functions of the zeros $-\alpha_k$ of $\mathrm{Ai}(z)$.

Key Words. Brownian excursion area, Airy function, Parking problem, Linear probing hashing.

Introduction. For probabilists, the *Airy distribution* considered here is nothing but the distribution of the *area* under the Brownian excursion. The name is derived from the connection between Brownian motion and the Airy function, a fact discovered around 1980 by several authors; see [16] and [20]. For combinatorialists and theoretical computer scientists, this Airy distribution (of the "area type") arises in a surprising diversity of contexts like parking allocations, hashing tables, trees, discrete random walks, merge-sorting, etc.

The most straightforward description of the Airy distribution is by its moments themselves defined by a simple nonlinear recurrence. We follow here the notations and the normalization of [11].

DEFINITION 1. The Airy distribution (of the "area" type) is the distribution of a random variable $\mathcal{A}$ whose moments are

$$\mu_r \equiv E(\mathcal{A}^r) = \frac{-\Gamma(-\frac{1}{2})}{\Gamma((3r-1)/2)}\Omega_r, \qquad r \geq 1, \tag{1}$$

where the "Airy constants" Ω_r are determined by the quadratic recurrence

$$\Omega_0 = -1, \qquad 2\Omega_r = (3r-4)r\Omega_{r-1} + \sum_{j=1}^{r-1}\binom{r}{j}\Omega_j\Omega_{r-j} \qquad (r \geq 1). \tag{2}$$

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[1] This research was partially supported by the IST Programme of the EU under Contract Number IST-1999-14186 (ALCOM-FT).

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Online publication August 28, 2001.

The Airy distribution of the Brownian excursion area

The Airy distribution is the distribution of a random variable $\mathcal{A}$ whose moments are

$$\mu_r \equiv E(\mathcal{A}^r) = \frac{-\Gamma(-1/2)}{\Gamma((3r-1)/2)}\Omega_r, \qquad r \geq 1,$$

where the "Airy constants" Ω_r satisfy the quadratic recurrence (more or less distant echo of some simple combinatorial tree decomposition):

$$\Omega_0 = -1, \qquad 2\Omega_r = (3r-4)r\Omega_{r-1} + \sum_{j=1}^{r-1} \binom{r}{j} \Omega_j \Omega_{r-j} \;\; (r \geq 1).$$

The normalized random variable $\frac{\mathcal{A}}{\sqrt{8}}$ is called Brownian excursion area.

r	0	1	2	3	4	5	6	7
Ω_r	-1	$\frac{1}{2}$	$\frac{5}{4}$	$\frac{45}{4}$	$\frac{3315}{16}$	$\frac{25425}{4}$	$\frac{18635625}{64}$	$\frac{18592875}{1}$
μ_r	1	$\sqrt{\pi}$	$\frac{10}{3}$	$\frac{15}{4}\sqrt{\pi}$	$\frac{884}{63}$	$\frac{565}{32}\sqrt{\pi}$	$\frac{662600}{9009}$	$\frac{19675}{192}\sqrt{\pi}$

Table: A table of the Airy constants Ω_r and of the Airy moments μ_r.

We shall see later that the Airy distribution is uniquely determined by its moments and that it admits a continuous density $w(x)$.
Let us notice that the first analysis of the Airy distribution was done following a question by P. Flajolet concerning the asymptotic distribution of the area under a Dyck path.

The Airy distribution: generating functions

The Airy constants Ω_r are characterized by any of the following expansions:

$$\frac{\mathrm{Ai}'(z)}{\mathrm{Ai}(z)} \underset{z\to+\infty}{\sim} \sum_{r=0}^{\infty} \frac{\Omega_r}{2^r} \frac{(-1)^r z^{-(3r-1)/2}}{r!}$$

We also have a linear recurrence on the Airy coefficients Ω_r:

$$18^r\Omega_r = \frac{12r}{6r-1}\frac{\Gamma(3r+1/2)}{\Gamma(r+1/2)} - \sum_{k=1}^{r-1}\binom{r}{k}\frac{\Gamma(3k+1/2)}{\Gamma(k+1/2)}18^{r-k}\Omega_{r-k}. \qquad (1)$$

The Airy distribution: Laplace transform and density

The relation between Airy coefficients and moments of the Airy distribution involves products and quotients of factorials that, at generating functions level, are well known to correspond to direct and inverse Laplace transforms. Let $w(x)$ be the density function of the Airy distribution:

$$w(x) = \frac{d}{dx}\mathbb{P}\{\mathcal{A} \leq x\},$$

the corresponding moment generating function is

$$E\left[e^{-y\mathcal{A}}\right] = \sum_{r\geq 0} \mu_r \frac{(-y)^r}{r!} = \int_0^\infty e^{-yt}\, w(t)dt.$$

We know that the Airy distribution function satisfies the double Laplace transform relation:

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty \left(e^{-zy} - 1\right) E\left[e^{-y^{2/3}\frac{\mathcal{A}}{\sqrt{8}}}\right] \frac{dy}{y^{3/2}} = 2^{1/3}\left(\frac{\mathrm{Ai}'(2^{1/3}z)}{\mathrm{Ai}(2^{1/3}z)} - \frac{\mathrm{Ai}'(0)}{\mathrm{Ai}(0)}\right).$$

The moment generating function and the density of the Airy distribution are given by

$$E\left[e^{-y\frac{\mathcal{A}}{\sqrt{8}}}\right] = \sqrt{2\pi}y\sum_{k=0}^{\infty}\exp\left(-\alpha_k y 2^{-1/3}\right)$$

$$w(x) = \frac{8\sqrt{3}}{x^2}\sum_{k=1}^{\infty}e^{-v_k}\,v_k^{2/3}\,U\left(\frac{-5}{6},\frac{4}{3};v_k\right) \qquad v_k = \frac{16\alpha_k^3}{27x^2}.$$

There, the quantities $-\alpha_k$ are the zeros of the Airy function $\mathrm{Ai}(z)$ and $U(a, b; z)$ is the confluent hypergeometric function.

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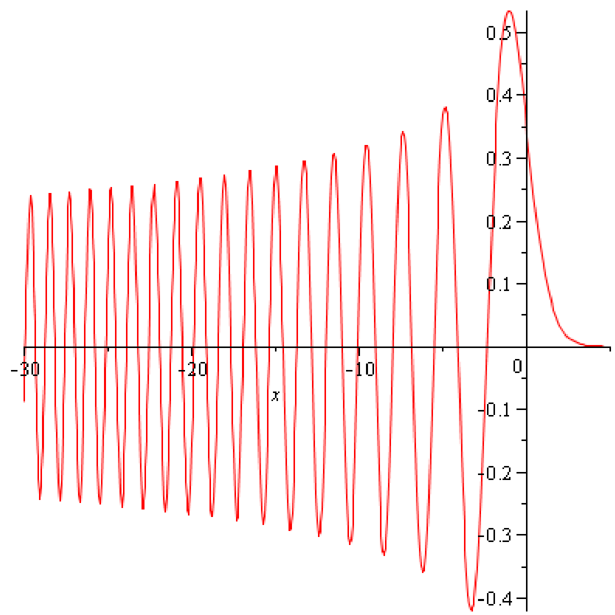
> plot(AiryAi(x),x=-30..5);
```




```
> evalf(seq(AiryAiZeros(k),k=1..10));
```

$-2.338107410, -4.087949444, -5.520559828, -6.786708090, -7.944133587, -9.022650853, -10.04017434, -11.00852430, -11.93601556, -12.82877675$ **(1)**

```
>
```

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The left tail of the Airy distribution is described as follows. Let $-\alpha_1 \doteq -2.33810\,74104$ be the first Airy zero. As $x \to 0^+$, one has

$$2^{3/2} w(2^{3/2} x) \quad \sim \quad e^{-2\alpha_1^3/(27x^2)} \left(\frac{8}{35} \frac{\alpha_1^{9/2}}{x^5} - \frac{48}{35} \frac{\alpha_1^{3/2}}{x^3} + \cdots \right).$$

When x tends to 0, the first term in the density corresponding to α_1 dominates exponentially all the other ones.

The right tail was less precisely quantified at the moment P. Flajolet was working on the analytic properties of the Airy distribution. As he said " it would be of interest to be able to get access to the right tails by purely analytic methods".

The Airy zeros admit an asymptotic expansion of the form:

$$\alpha_k \sim \rho k^{2/3} \left(1 + \sum_{j=1}^{\infty} \frac{a_j}{k^j}\right) \qquad \text{with} \quad \rho = \left(\frac{3\pi}{2}\right)^{\frac{2}{3}}$$

and the expansion starts as

$$\alpha_k \sim \rho k^{2/3} \left(1 - \frac{1}{6k} - \frac{\rho^3 - 15}{144 \rho^3 k^2} - \frac{\rho^3 - 45}{1296 \rho^3 k^3} - \cdots \right).$$

The moments

In what follows, an essential rôle is played by what may be called the "root zeta function" of the Airy function. This function is defined by

$$\Lambda(s) := \sum_{k=1}^{\infty} (\alpha_k)^{-s} \qquad (\Re(s) > \frac{3}{2}), \tag{2}$$

where the sum is *a priori* defined and analytic for $\Re(s) > 3/2$, given the growth of the α_k. Furthermore, since α_k admits a complete asymptotic expansion, $\Lambda(s)$ can be in fact continued as a meromorphic function in the whole of the complex plane. The continuation property results from a classical process of inserting "convergence terms" in a divergent series. This was first started by P. Flajolet in some " Reflexions ferroviaires".

For instance, the identity

$$\Lambda(s) = \sum_{k\geq 1} (3\pi k/2)^{-2s/3} + \sum_{k\geq 1} \left((\alpha_k)^{-s} - (3\pi k/2)^{-2s/3} \right) \tag{3}$$

initially valid for $\Re(s) > 3/2$ extends by analytic continuation to $\Re(s) > 0$, where the first term in (3) is $(3\pi/2)^{-2s/3}\zeta(s)$ with $\zeta(s)$ the Riemann zeta function and the second term has summands that are $O(k^{-2/3s-1})$ because of the Airy zeros asymptotic expansion.

This construction generalizes.

Using Mellin transforms, one derive that the moments of the Airy distribution exist for any $s \in \mathbb{C}$ and satisfy

$$\mathbf{E}\left[\left(\frac{\mathcal{A}}{\sqrt{8}}\right)^s\right] = 3\sqrt{\pi}2^{-s/2}\frac{\Gamma(\frac{3}{2}(1-s))}{\Gamma(-s)}\Lambda\left(\frac{3}{2}(1-s)\right),$$

where $\Lambda(\frac{3}{2}(1-s))$ is to be taken either as the power sum symmetric function of the α_k by (2) for $\Re(s) < 0$, or as one of the analytic continuation forms.

The moments of negative order of the Airy distribution are expressible in terms of the root zeta function Λ in its region of convergence:

$$E\left[\left(\frac{\mathcal{A}}{\sqrt{8}}\right)^{-m}\right] = 3\sqrt{\pi}2^{m/2}\frac{\Gamma(\frac{3}{2}+\frac{3}{2}m)}{\Gamma(m)}\Lambda(\frac{3}{2}+\frac{3}{2}m).$$

So, the moments of odd negative order $-m = -1-2p$ (and some negative fractional moments) are rationally expressible in terms of $\sqrt{\pi}$, $\Gamma(1/3)$, and algebraic radicals:

$$E\left[\left(\frac{\mathcal{A}}{\sqrt{8}}\right)^{-1-2p}\right] = -3\sqrt{2\pi}2^{p}\frac{\Gamma(3+3p)}{\Gamma(1+2p)}(-1)^{3p}[z^{2+3p}]\frac{\mathrm{Ai}'(z)}{\mathrm{Ai}(z)},$$

$$E\left[\left(\frac{\mathcal{A}}{\sqrt{8}}\right)^{-3}\right] = \sqrt{2\pi}\left(18-480\frac{\pi^3\sqrt{3}}{\Gamma(1/3)^6}+7680\frac{\pi^6}{\Gamma(1/3)^{12}}\right),$$

$$E\left[\left(\frac{\mathcal{A}}{\sqrt{8}}\right)^{-5/3}\right] = \frac{9\sqrt{\pi}2^{5/6}}{\Gamma(1/3)^7}\left(3^{1/3}\Gamma(1/3)^6 - 8\,3^{5/6}\pi^3\right).$$

The moments of positive order are by definition given by the asymptotic behaviour of the Airy function $\mathrm{Ai}(z)$ at $+\infty$. Here, we show that such moments are also indirectly determined by the behaviour of $\mathrm{Ai}(z)$ towards $-\infty$, where the function oscillates rapidly, this via the coefficients in the asymptotic expansion of the zeros $-\alpha_k$.
Consider the function

$$h(x) := \frac{\mathrm{Ai}'(x)}{\mathrm{Ai}(x)} - \frac{\mathrm{Ai}'(0)}{\mathrm{Ai}(0)}.$$

As $x \to \infty$, one has

$$h(x) \sim -x^{1/2} - \frac{\mathrm{Ai}'(0)}{\mathrm{Ai}(0)} - \frac{1}{4}x^{-1} + \frac{5}{32}x^{-5/2} - \frac{15}{64}x^{-4} + \frac{1105}{2048}x^{-11/2} - \dots$$

The Mellin transform of $h(x)$ exists *a priori* in the fundamental strip $-1 < \Re(s) < -\frac{1}{2}$ where it equals

$$h^*(s) = \frac{\pi}{\sin \pi s}\Lambda(1-s), \tag{4}$$

and $\Lambda(s)$ is known already to be meromorphically continuable (with simple poles at most) to the whole of $\mathbb{C}$. Finally, the mapping property of Mellin transforms implies that the singular expansion of $h^*(s)$ is the image of the asymptotic expansion of $h(x)$ at $+\infty$:

$$h^*(s) \asymp \left(\frac{\mathrm{Ai}'(0)}{\mathrm{Ai}(0)}\frac{1}{s}\right)_{s=0} - \sum_{r\geq 0}\left((-1)^r\frac{\Omega_r}{2^r r!}\frac{1}{s-(3r-1)/2}\right)_{s=(3r-1)/2}. \tag{5}$$

The moment of order $\frac{1}{3}$ of the Airy distribution is expressible in terms of $\Gamma(1/3)$ as well as in terms of Airy zeros as ($\rho = (3\pi/2)^{2/3}$)

$$E\left[\left(\frac{\mathcal{A}}{\sqrt{8}}\right)^{1/3}\right] = 2^{-1/6}3^{1/3}\frac{\sqrt{\pi}}{\Gamma(1/3)}$$

$$= -\frac{2^{5/6}}{3^{1/2}\sqrt{\pi}}\Gamma(1/3)\left[\frac{\zeta(2/3)}{\rho} + \sum_k \left(\frac{1}{\alpha_k} - \frac{1}{\rho k^{2/3}}\right)\right].$$

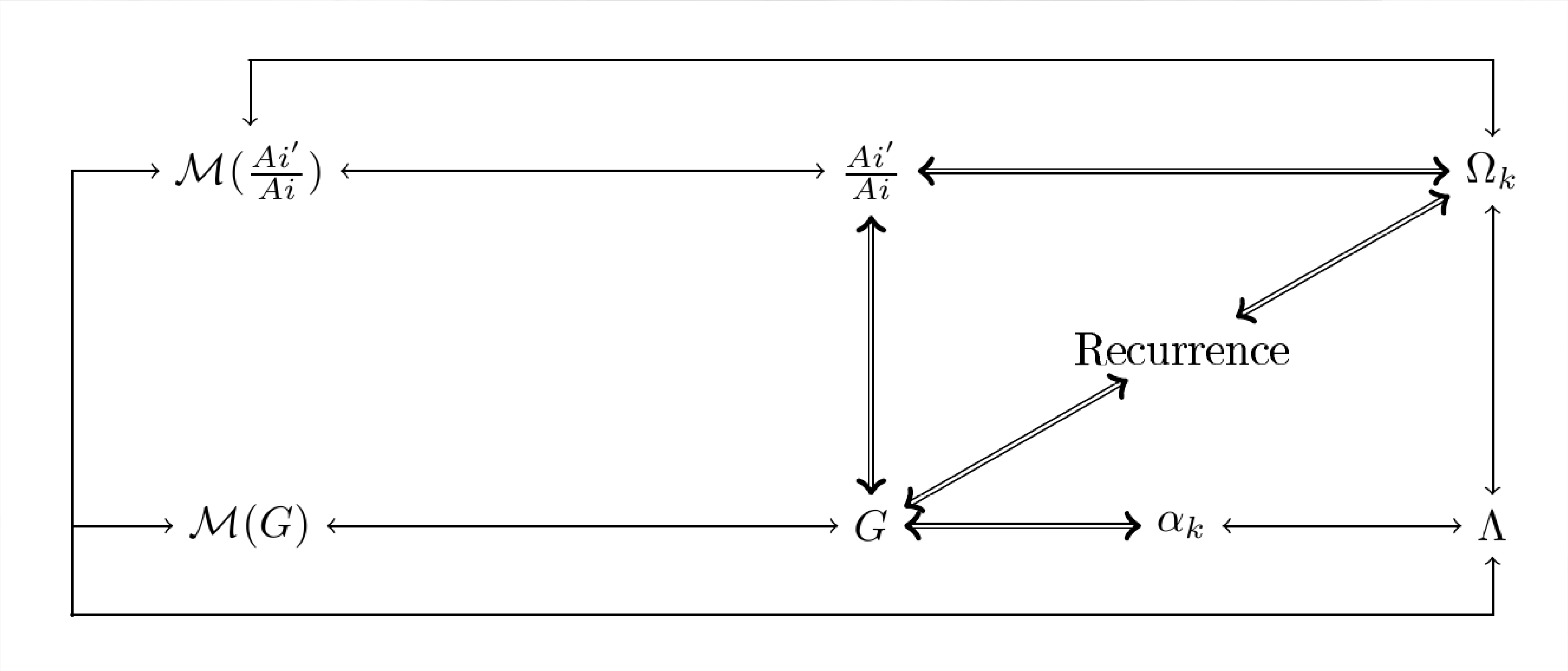
The moments of positive order are described as follows. Let m be any real number satisfying $m > 0$ and $m \neq 2, 4, 6, \ldots$. The moment of order m of the Airy distribution is expressible in terms of sums over Airy zeros:

$$\mathbf{E}\left[\left(\frac{\mathcal{A}}{\sqrt{8}}\right)^m\right] = 3\sqrt{\pi}2^{-m/2}\frac{\Gamma(\frac{3}{2}(1-m))}{\Gamma(-m)}\Lambda\left(\frac{3}{2}(1-m)\right).$$

The moments of even order $m = 2, 4, 6, \ldots$ are expressible as finite polynomial forms in the coefficients $\{a_i\}$ in the expansion of the zeros α_k.

Philippe's contributions to the analytic properties of Airy

[$\Leftrightarrow$ shows probabilistic analysis, while $\leftrightarrow$ shows analytic analysis]



MATHÉMATIQUES ET INFORMATIQUE

Hachage, arbres, chemins & graphes

Philippe Chassaing[†] et Philippe Flajolet[‡]

Mathématiques discrètes et continues se rencontrent et se complètent volontiers harmonieusement. C'est cette thèse que nous voudrions illustrer en discutant un problème classique aux ramifications nombreuses— l'analyse du hachage avec essais linéaires. L'exemple est issu de l'analyse d'algorithmes, domaine fondé par Knuth et qui se situe lui-même « à cheval » entre l'informatique, l'analyse combinatoire, et la théorie des probabilités. Lors de son traitement se croisent au fil du temps des approches très diverses, et l'on rencontrera des questions posées par Ramanujan à Hardy en 1913, un travail d'été de Knuth datant de 1962 et qui est à l'origine de l'analyse d'algorithmes en informatique, des recherches en analyse combinatoire du statisticien Kreweras, diverses rencontres avec les modèles de graphes aléatoires au sens d'Erdös et Rényi, un peu d'analyse complexe et d'analyse asymptotique, des arbres qu'on peut voir comme issus de processus de Galton-Watson particuliers, et, pour finir, un peu de processus, dont l'ineffable mouvement Brownien ! Tout ceci contribuant *in fine* à une compréhension très précise d'un modèle simple d'aléa discret.

1. Hachage

Nous ferons commencer l'histoire par Knuth en 1962 ; Knuth a alors 24 ans et hésite entre une carrière en mathématiques discrètes et une passion pour l'informatique très concrète. L'un des tous premiers problèmes quantitatifs de la science informatique naissante consiste à comprendre comment se comporte une certaine méthode d'accès à des données qui apparaît comme présentant, au vu des simulations, de très bonnes caractéristiques de complexité. L'École de Feller est aussi « sur le coup » (dixit Knuth). Knuth apportera très vite une solution[1] à ces questions (à partir de 1962) et, comme il le dit lui-même, c'est ce succès scientifique initial qui déterminera très largement la suite de sa carrière. Pour

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`Philippe.Chassaing@iecn.u-nancy.fr`
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`Philippe.Flajolet@inria.fr`
[1]Le manuscrit est disponible en `http://algo.inria.fr/AofA/Research/11-97.html`.

(*ii*) *Des fonctions de parking aux partitions.* Une fonction parking f de n voitures sur $n+1$ places est également décrite par une partition ordonnée de l'ensemble $\{1, 2, \ldots, n\}$ en n morceaux, notée $(B_k(f))_{1 \le k \le n}$, avec

$$B_k(f) = f^{-1}(k) \ ;$$

$B_k(f)$ est l'ensemble des voitures, de cardinal noté $x_k(f)$, dont la place k est la première tentative. Définissons $y_k(f)$ de manière analogue à (11) :

(13)
$$y_k(f) = x_1(f) + x_2(f) + \cdots + x_k(f) - k + 1;$$

$y_k(f)$ représente alors le nombre de voitures ayant tenté, avec ou sans succès, de se garer sur la $k^{\text{ème}}$ place. Le déplacement total coïncidant avec le nombre total de tentatives infructueuses, on a

$$y_1(f) + y_2(f) + \cdots + y_n(f) - n = D_{n,n+1}(f),$$

et, naturellement, le moment factoriel du déplacement total s'exprime par

(14)
$$E\left[\binom{D_{n,n+1}}{k}\right] = \frac{1}{(n+1)^{n-1}} \sum_f \binom{y_1(f) + y_2(f) + \cdots + y_n(f) - n}{k},$$

où f parcourt l'ensemble des fonctions de parking. (Rappel : on a vu à la section 2 que $(n+1)^{n-1}$ dénombre les fonctions de parking.)

(*iii*) *L'équivalence.* Le point clé est que les ensembles de partitions « admissibles », d'une part l'ensemble des partitions $(A_k)_{1 \le k \le n}$ issues du parcours d'un arbre (ou plus généralement du parcours d'un graphe connexe) et d'autre part l'ensemble des partitions $(B_k)_{1 \le k \le n}$ issues d'une fonction parking, sont confondus. De fait, la condition pour qu'une partition soit admissible dans l'un ou l'autre des sens du terme, est que

(15)
$$y_k \ge 1, \qquad 1 \le k \le n.$$

En effet, pour un parcours d'arbre ou de graphe, y_k représente la longueur de la file d'attente après le $k^{\text{ème}}$ pas, et l'inégalité (15) traduit la contrainte de connexité sur le graphe ou l'arbre ; pour une fonction parking, l'inégalité (15) traduit que la seule place vide est la place $n+1$ (ou 0, indifféremment).

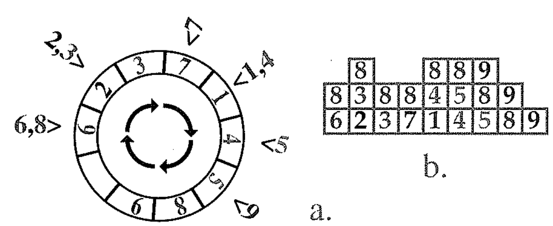

FIG. 3. Le parking induit par la partition ordonnée $(\{6,8\}, \{2,3\}, \varnothing, \{7\}, \{1,4\}, \{5\}, \{9\}, \varnothing, \varnothing)$, apparaissant déjà Figure 2 pour l'arbre γ (a), et les piles des voitures ayant essayé de se garer sur les places 1, 2, etc .., dans l'ordre chronologique (b).

ALGORITHME DU PARCOURS. On part du sommet 0. À tout instant, un sommet peut être dans deux états : l'état « inconnu » ou l'état « exploré ». Initialement, tout sommet (sauf la source) est « inconnu ». L'algorithme opère avec une file d'attente qui à chaque instant contient une liste de sommets déjà visités. Soit $\mathcal{F}_t$ l'état de la file à l'instant t. Initialement $t = 0$ et $\mathcal{F}_0 := \{0\}$. La « boucle » principale de l'algorithme consiste alors à répéter jusqu'à épuisement l'opération suivante :

> Au temps t ($t = 1, 2, \ldots$), choisir un sommet $s_t \in \mathcal{F}_{t-1}$; l'enlever de $\mathcal{F}_{t-1}$. Soit A_t l'ensemble des voisins de s_t (selon l'adjacence du graphe) qui sont encore « inconnus » ; on remplace alors dans $\mathcal{F}_{t-1}$ l'élément s_t par les éléments de A_t, de sorte que
>
> $$\mathcal{F}_t = (\mathcal{F}_{t-1} \setminus \{s_t\}) \cup A_t.$$
>
> Au moment où les nouveaux sommets sont insérés dans $\mathcal{F}_t$, leur état passe du statut « inconnu » au statut « exploré ».

Clairement, ce schéma permet de parcourir tous les sommets d'un graphe connexe en leur rendant visite une fois et une seule. De fait, ce schéma associe à un graphe connexe γ un *arbre couvrant* τ, l'arbre dont les arêtes relient s_t aux éléments de A_t, les arêtes de s_t vers les autres voisins étant inutilisées.

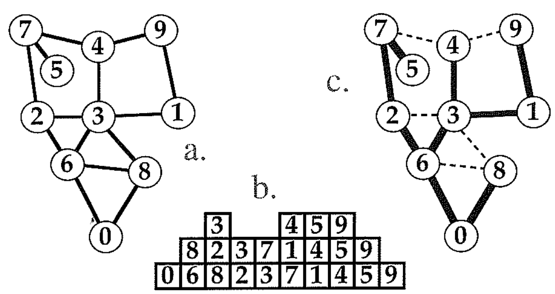

FIG. 2. Un graphe γ (a), les états $(\mathcal{F}_t)_{0 \le t \le 9}$ de la file (b) et l'arbre couvrant (c) associés au parcours en largeur de γ.

L'algorithme est complètement spécifié dès qu'une politique $\mathbf{\Pi}$ fixe le principe de sélection de s_t ainsi que l'ordre d'insertion de ses successeurs $s' \in A_t$. Les principes « LIFO » (*Last-In-First-Out* : on choisit le $s \in \mathcal{F}$ le plus récent, ce qui se gère par une pile) ou « FIFO » (*First-In-First-Out* : on sert le plus ancien dans la file) sont par exemple des politiques correspondant aux parcours appelés « en profondeur d'abord » et « en largeur d'abord ». Le principe par « priorité » (choix du plus petit ou plus grand numéro d'abord) est une autre politique particulièrement intéressante du point de vue combinatoire.

la suite des tailles et positions des blocs, complétée par une suite infinie de couples $(0,0)$. On considère les suites renormalisées

$$Z_\beta^{(m)}(t) = \frac{y_{\lceil mt \rceil}^{m,\lceil \beta\sqrt{m}\rceil}}{\sqrt{m}}, \qquad \Theta^{(m)}(\beta) = \frac{B^{m,\lceil \beta\sqrt{m}\rceil}}{m},$$

$$Z^{(m)} = \left(Z_\beta^{(m)}(t)\right)_{0\leq\beta,\,0\leq t\leq 1}, \qquad \Theta^{(m)} = \left(\Theta^{(m)}(\beta)\right)_{\beta\geq 0}.$$

Ces suites constituent respectivement les « profils de visite » et « profils de placement » du parking.

Soit par ailleurs, comme précédemment, $(e(t))$ l'excursion Brownienne. On définit alors « l'excursion élaguée » construite selon la règle

$$\begin{aligned} Z_\beta(t) &= e(t) - \beta t + \sup_{t-1\leqslant s\leqslant t} (\beta s - e(s)), \\ Z &= (Z_\beta(t))_{0\leq\beta,\,0\leq t\leq 1}, \end{aligned}$$

où β est un paramètre de contrôle. On note alors $\Theta(\beta)$ la suite des largeurs et positions des excursions du processus $t \longrightarrow Z_\beta(t)$.

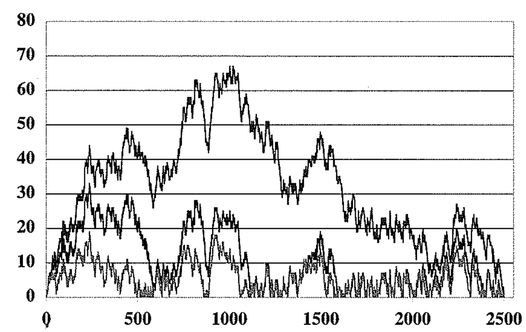

FIG. 5. $t \longrightarrow Z_\beta^{(m)}(t)$ pour $m = 2500$ places
et $\beta = 0, 2, 4$ successivement (1 puis 100 et 200 places vides)

Avec ces notations, en utilisant un couplage *parking–processus empirique*, on peut construire, sur un espace de probabilité approprié, des copies de $Z^{(m)}$ et de Z, telles que soit vérifié le principe suivant (qui généralise (16)) :

Convergence du profil. *Dans la région critique, le profil des visites du parking converge vers l'excursion Brownienne élaguée,*

$$\Pr\left(Z^{(m)} \to Z\right) = 1,$$

où $Z^{(m)}$ converge vers Z au sens de la convergence uniforme sur tout compact de $[0, +\infty) \times [0, 1]$. *En conséquence, les distributions finies-dimensionnelles du profil des placements du parking,* $\Theta^{(m)}$,

1. In probability theory, $\mathcal{B} = \mathcal{A}/\sqrt{8}$ is the distribution of the area $\int_0^1 e(t)\,dt$ where $e(t)$ is the classical Brownian excursion. The Airy distribution occurs in related problems relative to the analysis of dynamic data structures (like stacks under a Markovian model and priority queues under Knuth's model), and to the area of various classes of polyominoes
2. Accordingly, $\mathcal{A}$ is, up to normalization, the limit distribution of the area under discrete excursions, like the Bernoulli excursion (the probabilities of ± 1 jumps are $1/2$) or the Poisson excursion (each jump is $-1 + Y$ where Y is Poisson(1) distributed), where the latter was introduced by Spencer in order to model traversals of random connected graphs.
3. Path length in trees (that is, the sum of distances from the root to all the nodes) is asymptotically Airy distributed in Catalan trees as well as in other combinatorial families of trees that are defined by a finite set of allowed node degrees. This applies also to trees determined by branching processes and conditioned by the size of the total progeny.

4. The number of inversions in a random Cayley tree of large size is Airy distributed in the limit.
5. The total displacement of a random parking sequence or equivalently the construction cost of a hashing table under the linear probing strategy is Airy distributed in the limit.

6. The Airy constant Ω_r appears in the asymptotic enumeration of labelled connected graphs with n vertices and $n + r - 1$ edges, an intriguing fact that points to deep connections between random graphs and random walks
7. Breadth first search traversal of random trees has a cumulated cost that is asymptotically Airy distributed.
8. For a dozen of types of random planar graphs, an Airy distribution describes the sizes of cores and of largest (multi)connected components. Based on an extension of the singularity analysis framework suggested by the Airy case, a general classification of compositional schemas in analytic combinatorics is obtained.

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