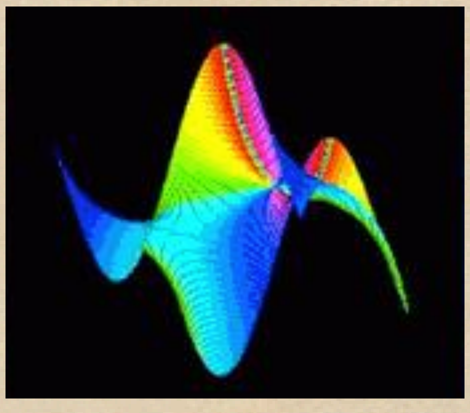
Philippe Flajolet and Symbolic Computation

Bruno Salvy
Algorithms Project - INRIA



December 15, 2011

My Claim:

Philippe did not publish much in SC,

but he was doing nothing else!

My Claim:

Philippe did not publish much in SC,
but he was doing nothing else!

1. PF's combinatorics is SC.
2. PF's complex analysis is SC!
3. PF's results in SC.

1. Combinatorics & Algorithms

Chronology

[31] PF & J.-M. Steyaert. A complexity calculus for classes of recursive search programs over tree structures. FOCS (1981).

[35]——. Journal version. Mathematical Systems Theory (1987).

[79]PF & BS & P. Zimmermann. Lambda-Upsilon-Omega: An assistant algorithms analyzer. AAECC (1988).

[80]——. Lambda-Upsilon-Omega: The 1989 Cookbook(RR).

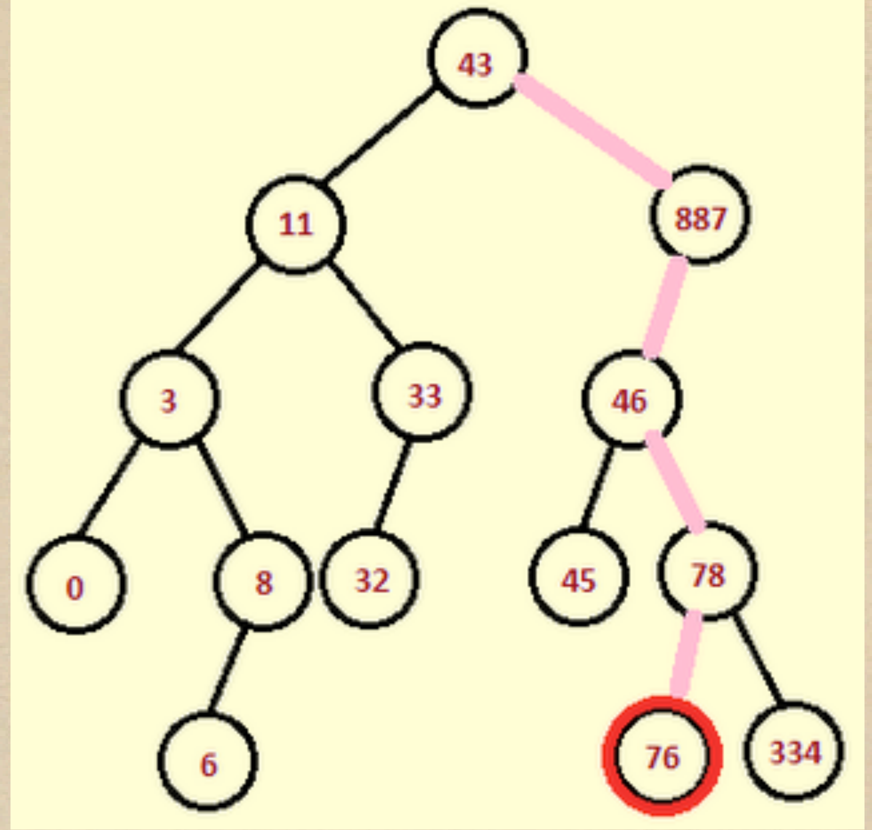
[94]——. Automatic average-case analysis of algorithms. TCS (1991).

ΛΥΩ

Input:

```
type btree = epsilon |
                 product(o, btree, btree);
    o = atom(1);

procedure pathlength(t: btree):
begin
    case t of
        ()          : nil;
        o(u,v)      : begin
                size(u); size(v);
                pathlength(u);
                pathlength(v);
        end;
    end;
end;
```



Output:

$$av\_tau\_pathlength\_n = Pi^{1/2} \, n^{(3/2)} + O(n)$$

average distance to the root ≈ √n

Combinatorial dictionary (≃81)

- Structures ➝ Gen. Fcns: [➥RS this morning]

$\mathcal{A} \cup \mathcal{B}$	$A(z) + B(z)$
$\mathcal{A} \times \mathcal{B}$	$A(z) \times B(z)$
$\textsc{Seq}(\mathcal{C})$	$\dfrac{1}{1 - C(z)}$
$\textsc{Cyc}(\mathcal{C})$	$\log \dfrac{1}{1 - C(z)}$
$\textsc{Set}(\mathcal{C})$	$\exp(C(z))$

(egf)

- Algorithms ➝ Gen. Fcns: [➥JMS a moment ago]

$R(a \in \mathcal{A})$: $P(a); Q(a);$	$\tau_R(z) = \tau_P(z) + \tau_Q(z)$
$P(a \in \mathcal{A}, b \in \mathcal{B})$: $Q(a);$	$\tau_P(z) = \tau_Q(z) \times B(z)$

Clearly symbolic computation

Then, Maple

The history of Maple and Inria were quite intertwined at the very beginning. This was due to my sabbatical year which I took in Inria Rocquencourt. I spent from Sep/84 to Feb/85 there. This was the time that I was working almost 100% on Maple, and of course I took Maple with me. [...] At the time we were analyzing the expected behaviour of some tries and we were using (and developing) Maple as we go. So Philippe was a user and developer of Maple from that time. He made all sorts of contributions.

Gaston Gonnet (sorry he couldn't come),
mail from this Monday.

```
    |\^/|
._|\|   |/|_.
\   MAPLE   /
<____ ____>
      |
```

1st public version (3.3):
March 85

June 1986: SC course



June 1986: SC course

- PF, Marc Giusti, Jean-Marc Steyaert

June 1986: SC course



- PF, Marc Giusti, Jean-Marc Steyaert
- I got hooked!

June 1986: SC course

- PF, Marc Giusti, Jean-Marc Steyaert
- I got hooked!
- ΛΥΩ started in 87-88 with P. Zimmermann.

June 1986: SC course

- PF, Marc Giusti, Jean-Marc Steyaert
- I got hooked!
- ΛΥΩ started in 87-88 with P. Zimmermann.

➥ Aug. 88: my 1st trip to Waterloo.

ΛΥΩ → combstruct today

```
> bintree:={B=Union(Epsilon,Prod(Z,B,B))}:
> PathLength:={pl(B)=Union(0,Prod(0,size(B)+pl(B),size(B)+pl(B)))}:
> sys:=combstruct[agfeqns](bintree,PathLength,unlabelled,z,[[u,pl]]);
```

$$sys := \left[B(z,u) = 1 + z\,B(z\,u,u)^2, Z(z,u) = z\,u\right]$$

```
> combstruct[agfmomentsolve](sys,0);
```

$$\left\{B(z) = -\frac{1}{2}\,\frac{-1+\sqrt{1-4\,z}}{z}, Z(z) = z\right\}$$

```
> equivalent(subs(%,B(z)),z,n);
```

$$\frac{\left(\frac{1}{n}\right)^{3/2}\left(e^{-n}\right)^{-2\ln(2)}}{\sqrt{\pi}} + O\left(\left(\frac{1}{n}\right)^{5/2}\left(e^{-n}\right)^{-2\ln(2)}\right)$$

```
> combstruct[agfmomentsolve](sys,1):
> equivalent(subs(%,B[2](z)),z,n);
```

$$\left(e^{-n}\right)^{-2\ln(2)} + O\left(\sqrt{\frac{1}{n}}\left(e^{-n}\right)^{-2\ln(2)}\right)$$

```
> asympt(%/%%%,n);
```

$$\frac{\sqrt{\pi}}{\left(\frac{1}{n}\right)^{3/2}} + O(n)$$

$$B(z,u) = \sum_{b\in\mathcal{B}} z^{|b|}u^{\mathrm{pl}(b)}$$

$$\left.\frac{\partial}{\partial u}B(z,u)\right|_{u=1} = \sum_{b\in\mathcal{B}} \mathrm{pl}(b)z^{|b|}$$

higher moments accessible too

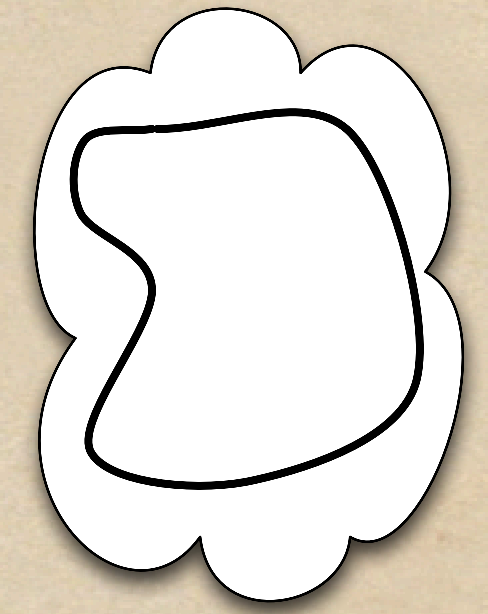
Mishna 2003. Attribute grammars and automatic complexity analysis.

II. PF's Complex Analysis is Symbolic Computation

Cauchy's residue theorem

Cauchy's residue theorem

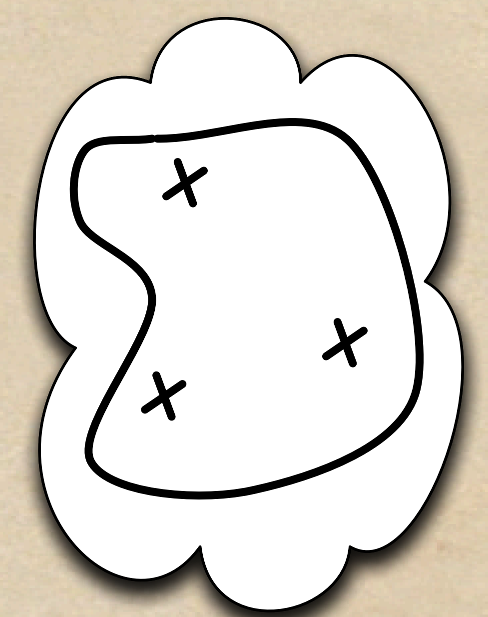
An integral is evaluated by:



Cauchy's residue theorem

An integral is evaluated by:

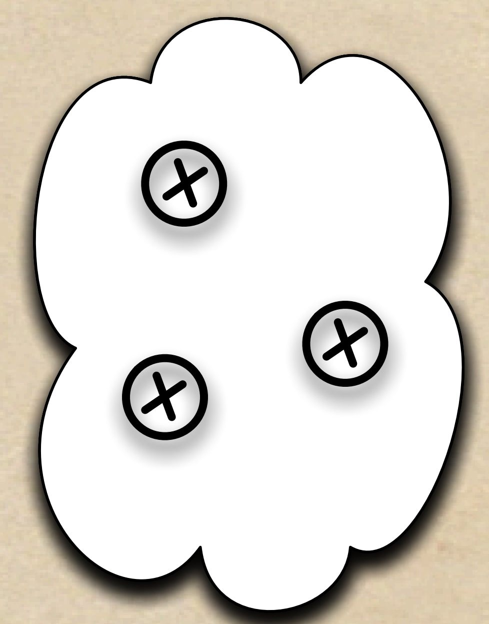
1. locating singularities;



Cauchy's residue theorem

An integral is evaluated by:

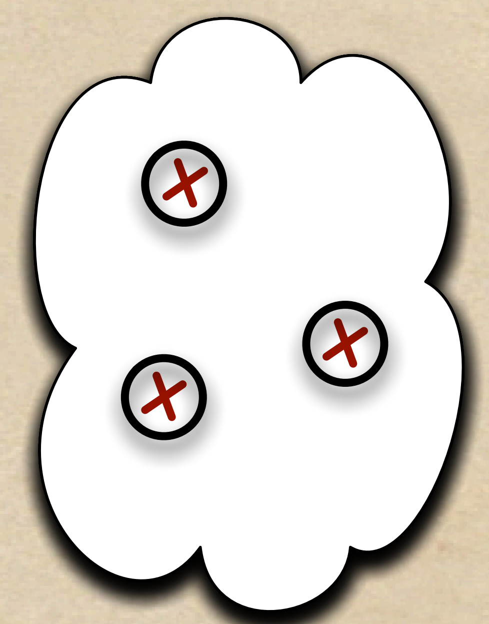
1. locating singularities;
2. moving the contour;



Cauchy's residue theorem

An integral is evaluated by:

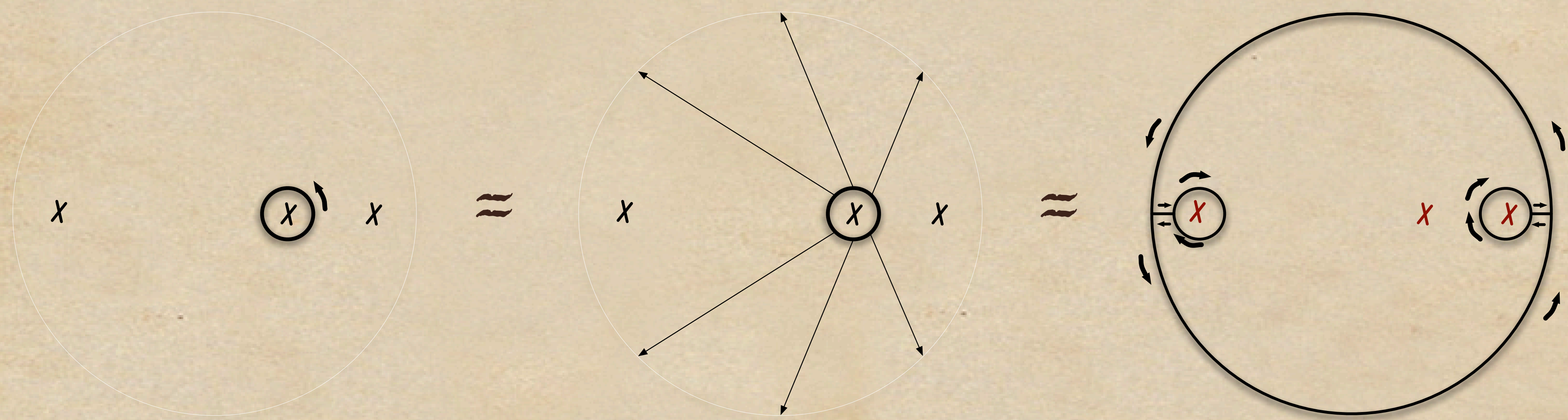
1. locating singularities;
2. moving the contour;
3. local analysis
   (= Symbolic Computation).



Under technical conditions

Example 1: singularity analysis

$$[z^n]f(z) = \frac{1}{2\pi i}\oint f(z)\,\frac{dz}{z^{n+1}}$$

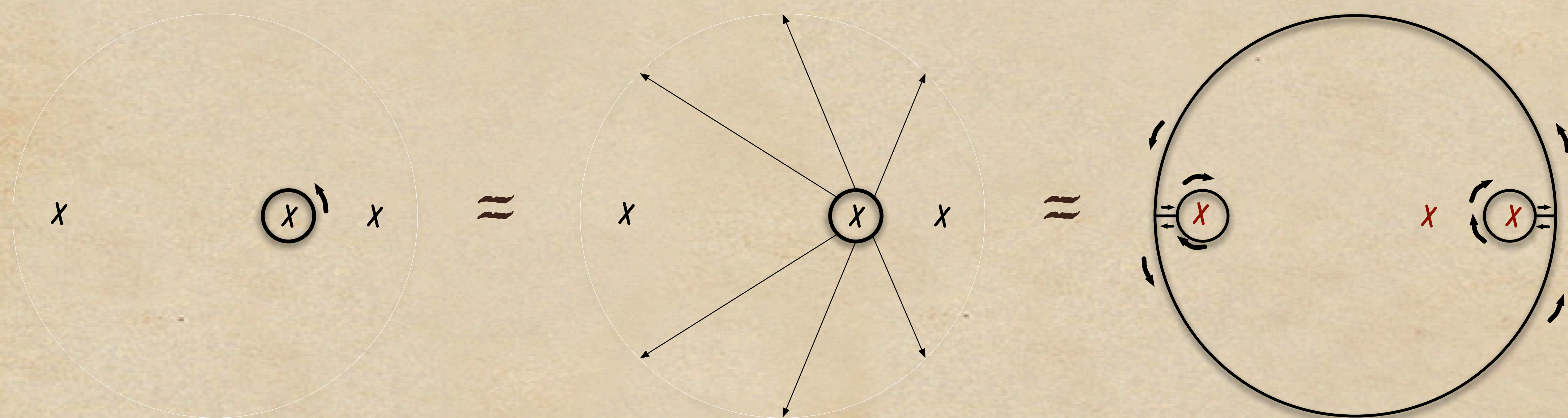


[➥AO tomorrow]

[86] PF & A. Odlyzko (1990).

Example 1: singularity analysis

$$[z^n]f(z) = \frac{1}{2\pi i} \oint f(z) \, \frac{dz}{z^{n+1}}$$



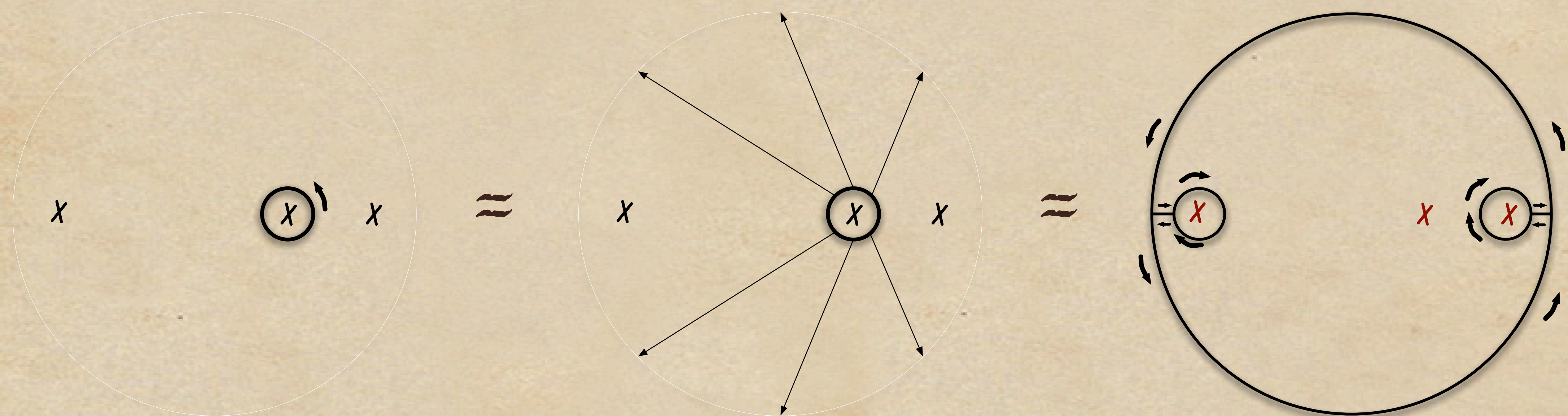
$$B(z) = \frac{1 - \sqrt{1 - 4z}}{2z}$$

[➥AO tomorrow]

[86] PF & A. Odlyzko (1990).

Example 1: singularity analysis

$$[z^n]f(z) = \frac{1}{2\pi i}\oint f(z)\,\frac{dz}{z^{n+1}}$$



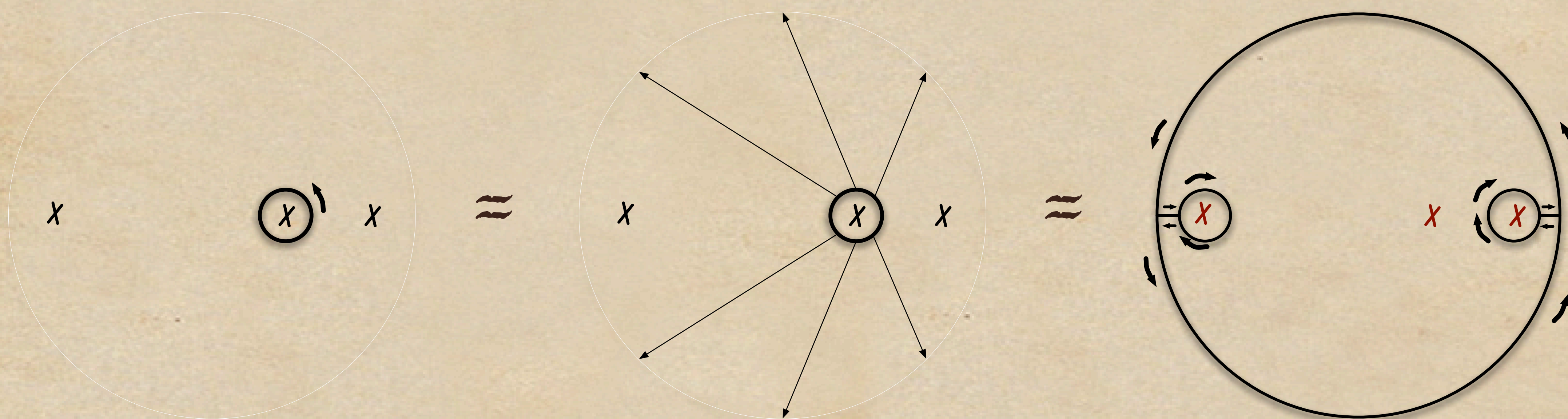
$$B(z) = \frac{1-\sqrt{1-4z}}{2z}$$

[➥AO tomorrow]

[86] PF & A. Odlyzko (1990).

Example 1: singularity analysis

$$[z^n]f(z) = \frac{1}{2\pi i}\oint f(z)\,\frac{dz}{z^{n+1}}$$



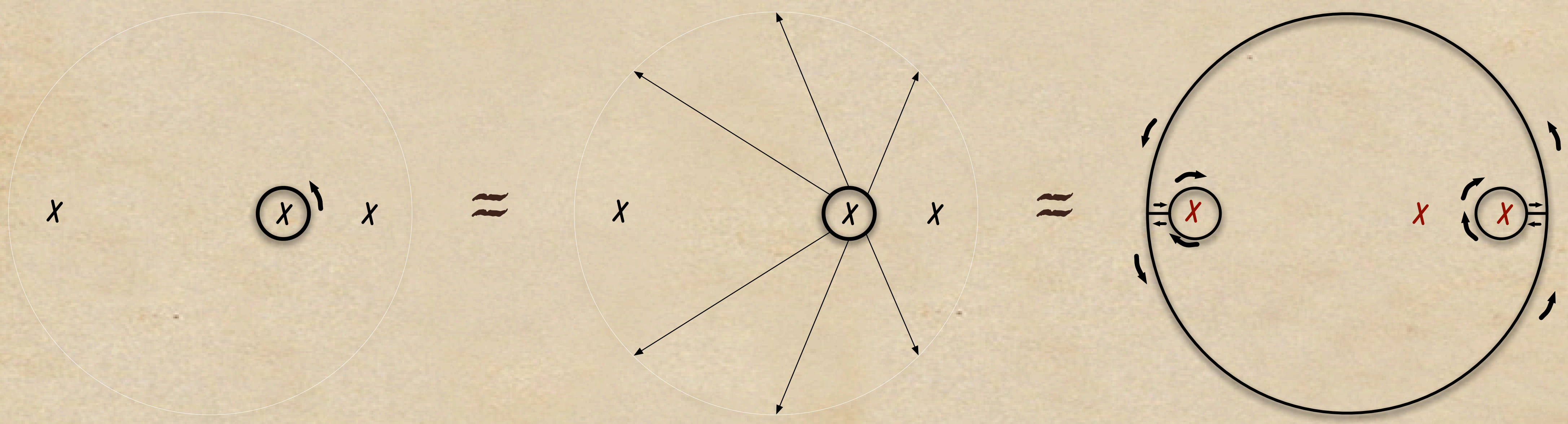
$$B(z) = \frac{1-\sqrt{1-4z}}{2z} = 2 - 2\sqrt{1-4z} + O(1-4z)$$

[➥AO tomorrow]

[86] PF & A. Odlyzko (1990).

Example 1: singularity analysis

$$[z^n]f(z) = \frac{1}{2\pi i}\oint f(z)\,\frac{dz}{z^{n+1}}$$



$$B(z) = \frac{1-\sqrt{1-4z}}{2z} = 2 - 2\sqrt{1-4z} + O(1-4z) \mapsto [z^n]B(z) = \frac{4^n}{\sqrt{\pi}n^{3/2}}(1 + O(\frac{1}{n}))$$

[➥AO tomorrow]

[86] PF & A. Odlyzko (1990).

Example 2: Mellin transform

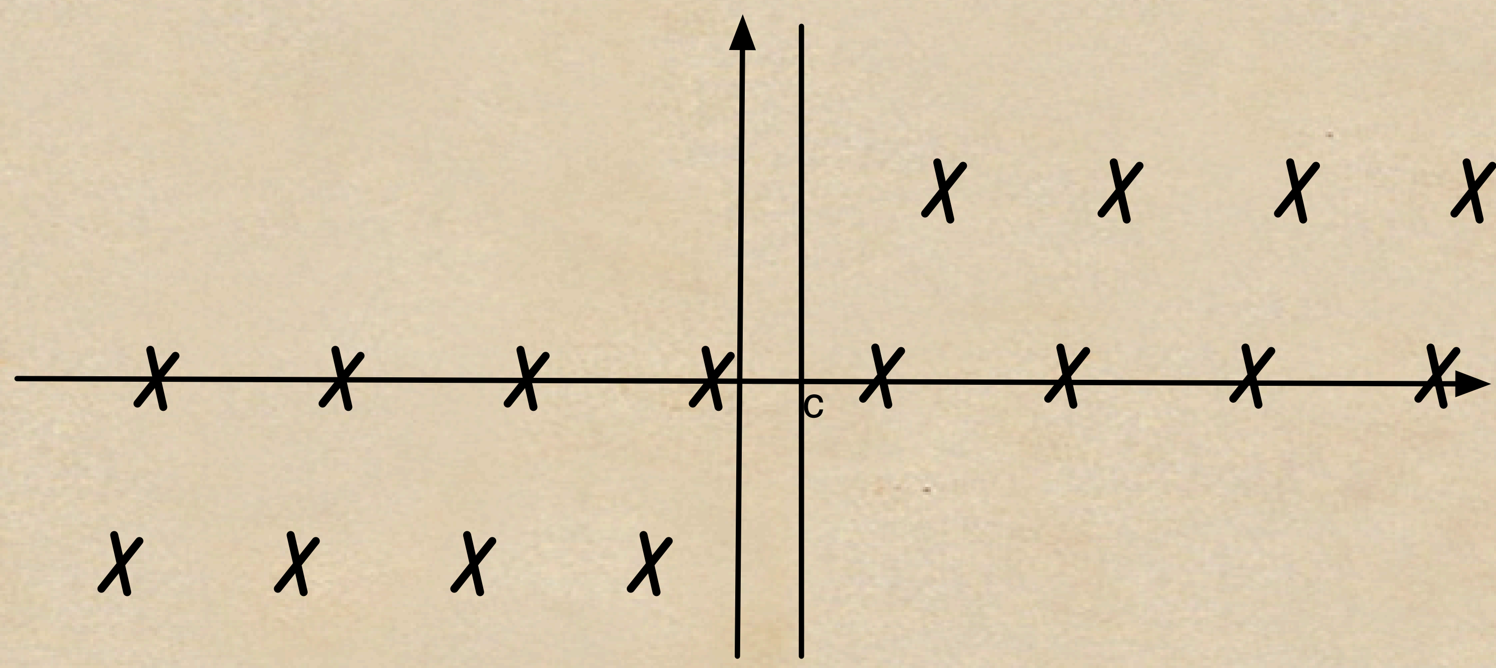
$$f(z) = \frac{1}{2\pi i} \int_{(c)} f^\star(s) z^{-s}\, ds$$ [➥PD tomorrow]

+ technical conditions

Symbolic computation again

Example 2: Mellin transform

$$f(z) = \frac{1}{2\pi i}\int_{(c)} f^{\star}(s) z^{-s}\, ds$$ [➥PD tomorrow]

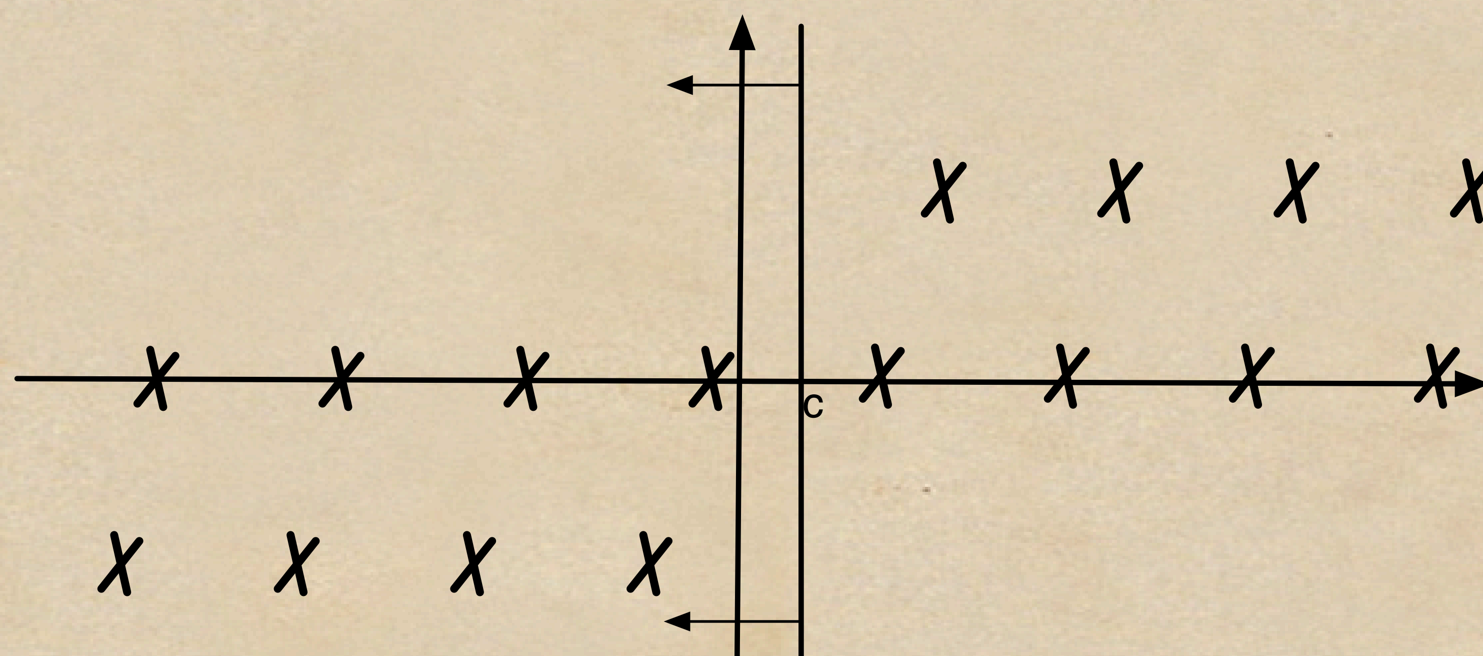


+ technical conditions

Symbolic computation again

Example 2: Mellin transform

$$f(z) = \frac{1}{2\pi i}\int_{(c)} f^\star(s) z^{-s}\, ds$$ [➥PD tomorrow]

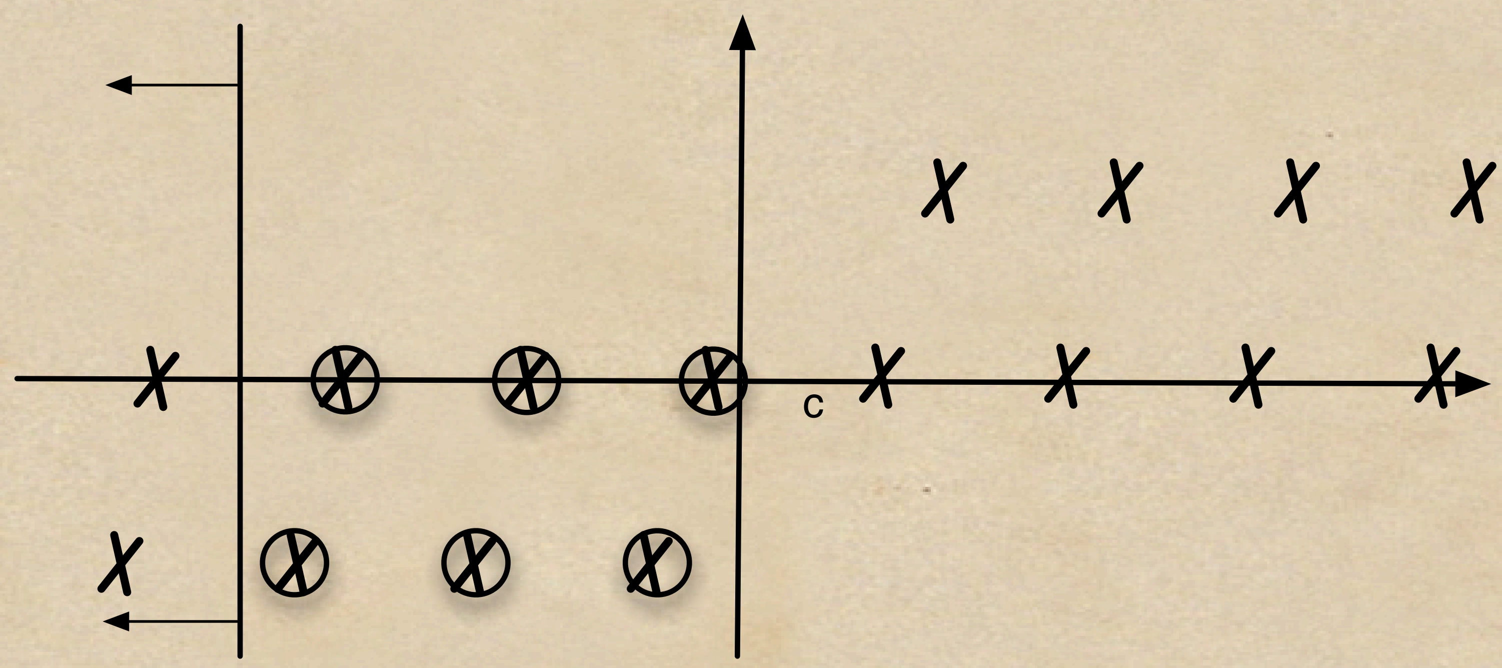


+ technical conditions

Symbolic computation again

Example 2: Mellin transform

$$f(z) = \frac{1}{2\pi i} \int_{(c)} f^\star(s) z^{-s}\, ds$$ [➥PD tomorrow]



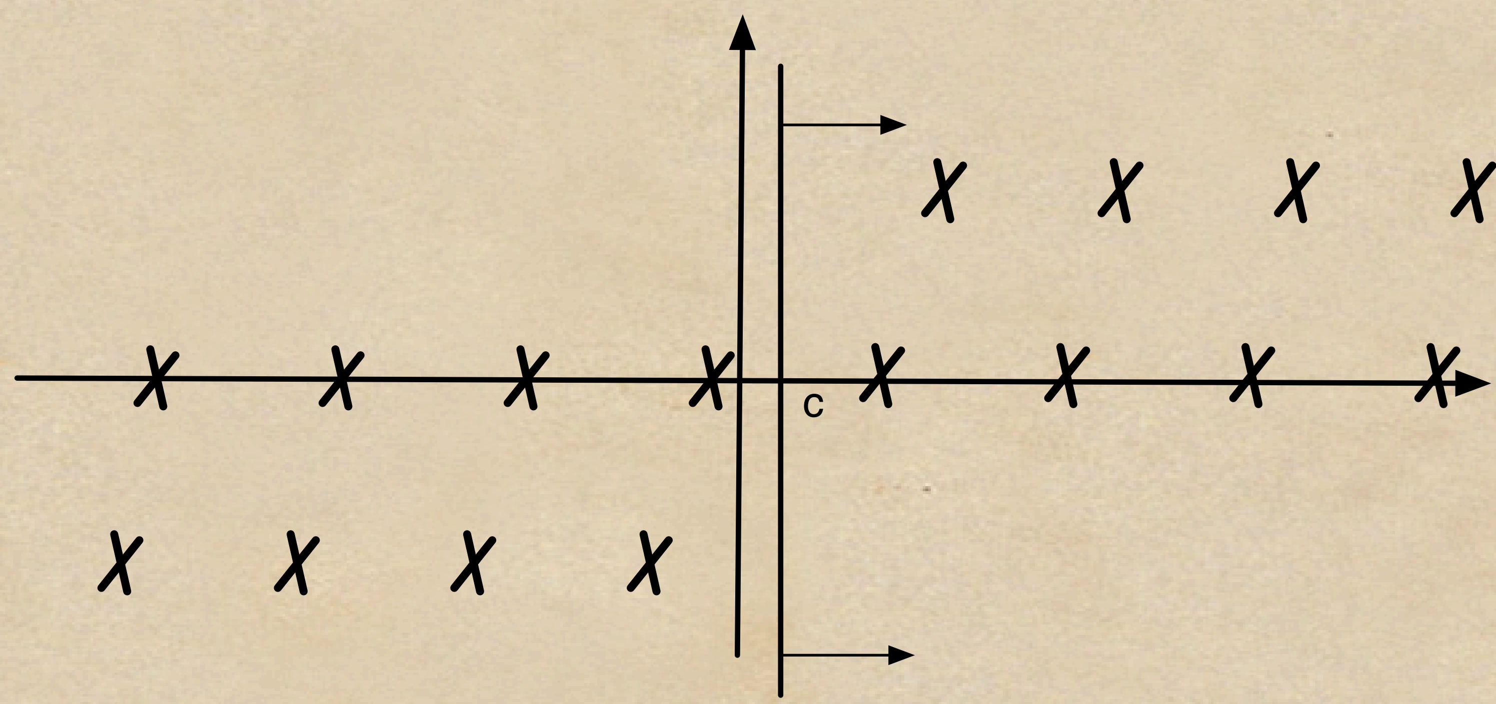
$$f(z) \underset{z \to 0}{\sim} \sum_{\substack{\alpha \text{ pole of } f^\star \\ \Re\alpha < c}} \operatorname{Res}_{s=\alpha}(f^\star(s)) z^{-\alpha}$$

+ technical conditions

Symbolic computation again

Example 2: Mellin transform

$$f(z) = \frac{1}{2\pi i} \int_{(c)} f^\star(s) z^{-s}\, ds$$ [➥PD tomorrow]



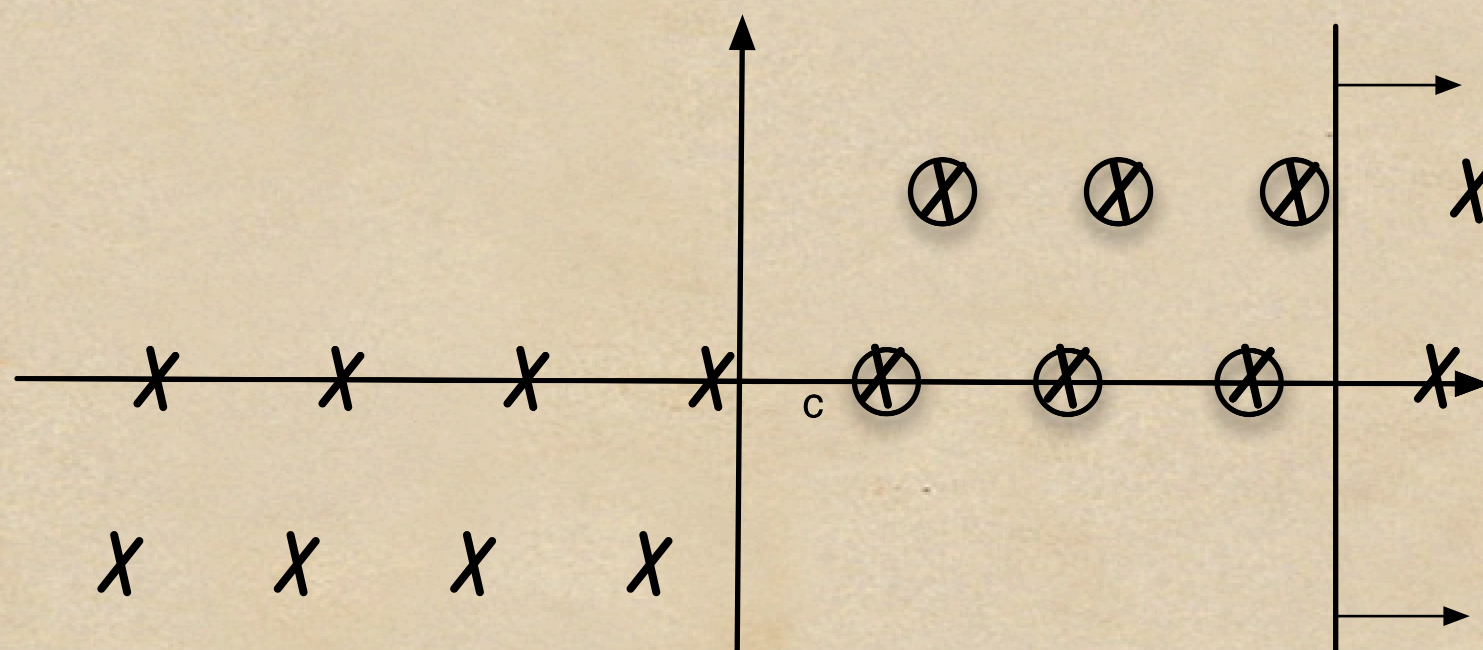
$$f(z) \underset{z\to 0}{\sim} \sum_{\substack{\alpha \text{ pole of } f^\star \\ \Re\alpha < c}} \mathrm{Res}_{s=\alpha}(f^\star(s)) z^{-\alpha}$$

+ technical conditions

Symbolic computation again

Example 2: Mellin transform

$$f(z) = \frac{1}{2\pi i}\int_{(c)} f^{\star}(s) z^{-s}\, ds$$ [➥PD tomorrow]

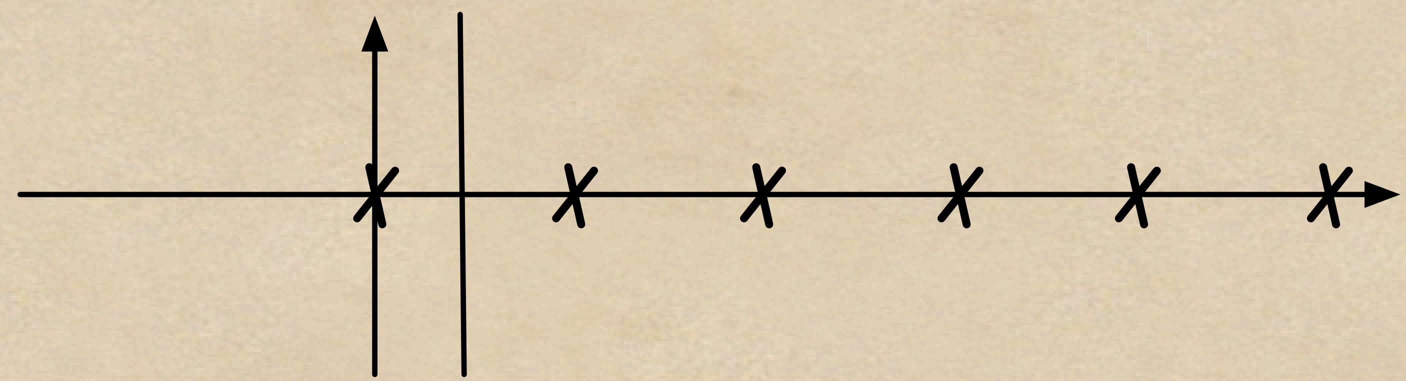


$$f(z) \underset{z\to 0}{\sim} \sum_{\substack{\alpha \text{ pole of } f^{\star} \\ \Re\alpha < c}} \mathrm{Res}_{s=\alpha}(f^{\star}(s))z^{-\alpha} \qquad f(z) \underset{z\to \infty}{\sim} \sum_{\substack{\alpha \text{ pole of } f^{\star} \\ \Re\alpha > c}} \mathrm{Res}_{s=\alpha}(f^{\star}(s))z^{-\alpha}$$

+ technical conditions

Symbolic computation again

Variant: summatory formulæ



$$-\operatorname{Res}_{s=0}(r(s)\phi(s)) = \sum_{n\in\mathbb{N}^\star} \operatorname{Res}_{s=n}(r(s)\phi(s))$$

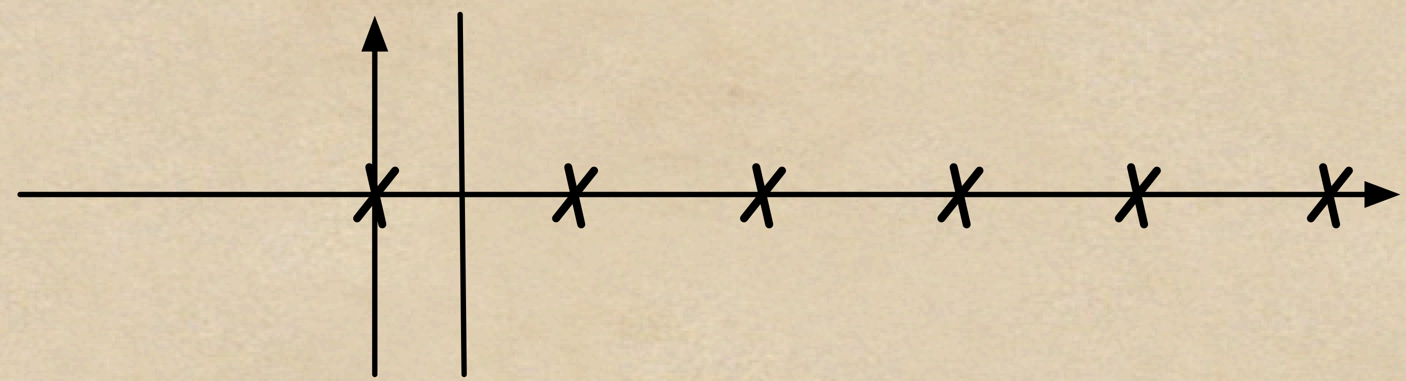
+ technical conditions

Examples:

- $r(s) = 1/s^2, \phi(s) = (\psi(-s)+\gamma)^2$ ➞ $\displaystyle\sum_{n\geq 1} \frac{H_n}{n^2} = 2\zeta(3)$ [Euler1742]
- $r(s) = 1/s^2, \phi(s) = (\psi(-s)+\gamma)^3$ ➞ $\displaystyle\sum_{n\geq 1} \frac{(H_n)^2}{n^2} = \frac{17}{4}\zeta(4)$ [Borwein et al.1995]

[143] PF & BS. Euler Sums (1998).

Variant: summatory formulæ



$$-\operatorname{Res}_{s=0}(r(s)\phi(s)) = \sum_{n\in\mathbb{N}^\star} \operatorname{Res}_{s=n}(r(s)\phi(s))$$

+ technical conditions

Examples:

- $r(s) = 1/s^2, \phi(s) = (\psi(-s)+\gamma)^2 \rightarrow \displaystyle\sum_{n\geq 1} \frac{H_n}{n^2} = 2\zeta(3)$ [Euler1742]
- $r(s) = 1/s^2, \phi(s) = (\psi(-s)+\gamma)^3 \rightarrow \displaystyle\sum_{n\geq 1} \frac{(H_n)^2}{n^2} = \frac{17}{4}\zeta(4)$ [Borwein et al.1995]

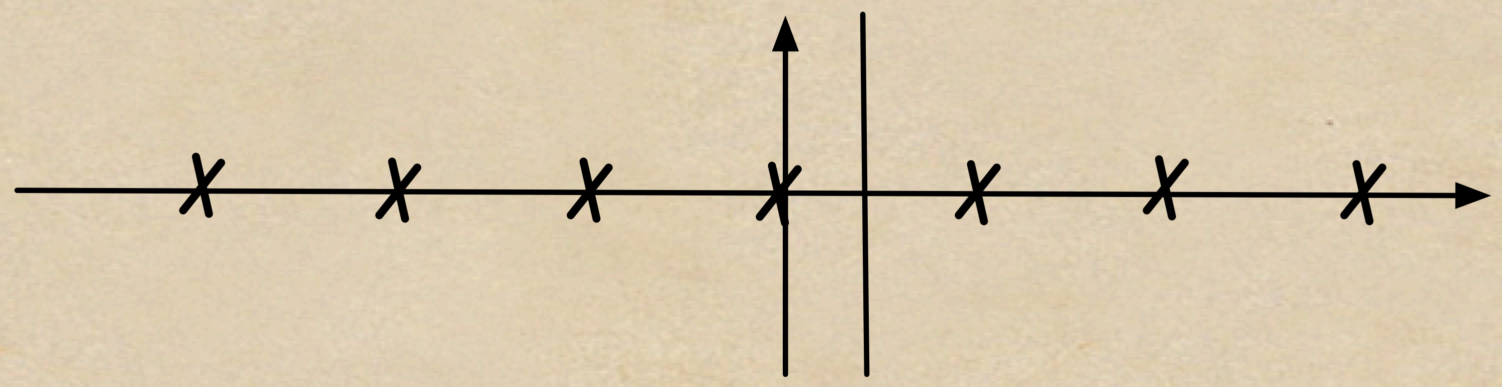
Where

$$\psi(-s) = -\frac{d}{ds}\log\Gamma(-s) \underset{s\to n}{=} \frac{1}{s-n} + H_n - \gamma + \sum_{k=1}^{\infty}((-1)^k H_n^{(k+1)} - \zeta(k+1))(s-n)^k$$

[143] PF & BS. Euler Sums (1998).

Variant: Lindelöf integrals

$$F(z) = \frac{1}{2\pi i}\int_{\mathcal{C}} \phi(s) z^s \frac{\pi}{\sin \pi s}\, ds$$



Philippe's 'Magic Duality':

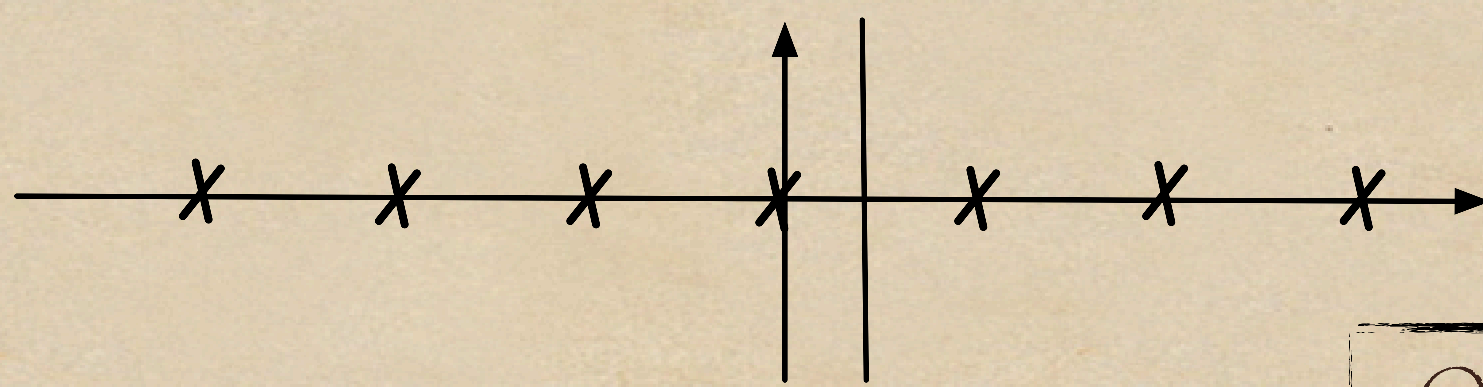
$$\sum_{n>0} \phi(n)(-z)^n \underset{z\to 0}{\equiv} F(z) \underset{z\to\infty}{\equiv} \sum_{n\geq 0} \phi(-n)(-z)^{-n} + \text{stuff}$$

+ technical conditions

[122] PF & G. Labelle & L. Laforest & B. Salvy (1995). [➡CM tomorrow]

Variant: Lindelöf integrals

$$F(z) = \frac{1}{2\pi i}\int_{\mathcal{C}} \phi(s) z^s \frac{\pi}{\sin \pi s}\, ds$$



Other variant: Rice's integrals.

Philippe's 'Magic Duality':

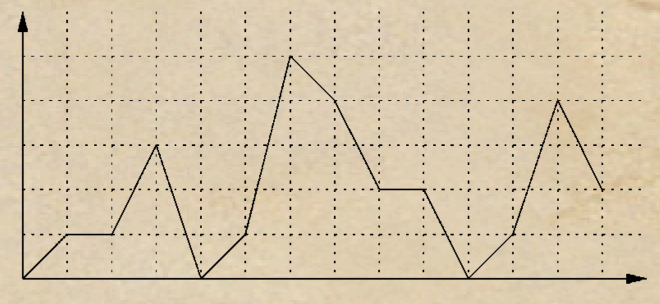
$$\sum_{n>0} \phi(n)(-z)^n \underset{z\to 0}{\equiv} F(z) \underset{z\to\infty}{\equiv} \sum_{n\geq 0} \phi(-n)(-z)^{-n} + \text{stuff}$$

+ technical conditions

[122] PF & G. Labelle & L. Laforest & B. Salvy (1995). [➥CM tomorrow]

III. Philippe's results in Symbolic Computation

Platypus algorithm



- Enumeration of lattice paths
  [➥MBM tomorrow]
- From $Q(u) = \prod_{\alpha} \left(1 - \frac{u}{\alpha}\right)$ (Q known, not its roots), compute

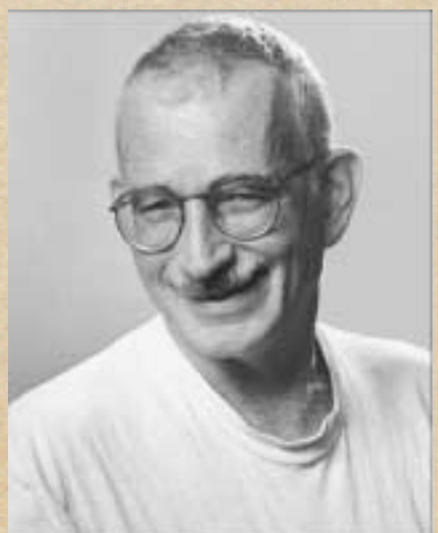
$$\prod_{\alpha,\alpha'} \left(1 - \frac{u}{\alpha\alpha'}\right)$$

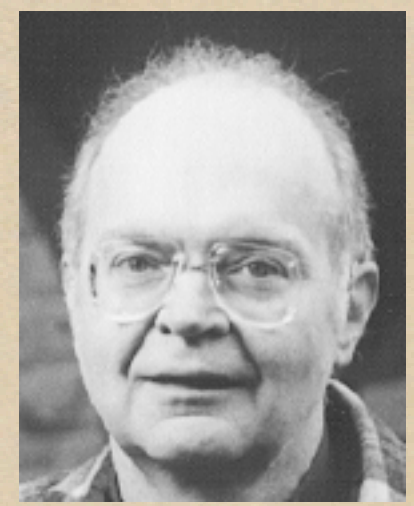
1. Compute Newton sums (Q'/Q is their gf)
2. Multiply coefficient by coefficient (and adjust).

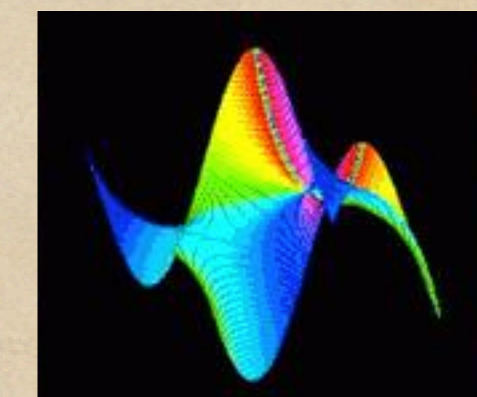
Extend to fast algorithms for various resultants.

[168] C. Banderier & PF. (2002)
[185] A. Bostan & PF & BS & É. Schost. (2006)

Holonomy







Zeilberger 1990

Algo Seminar 91-92

1st version of gfun in 92

(BS & P. Zimmermann)

Main properties

- Solutions of linear differential/recurrence equations closed under various operations + algorithms
- LDE ↔ LRE via generating functions.

Two exercises

```
> P:=(1+x)^(4*N)*(1+x+x^2)^(2*N)*(1+x+x^2+x^3+x^4)^N;
```

$$P := (1+x)^{4N}\left(1+x+x^2\right)^{2N}\left(1+x+x^2+x^3+x^4\right)^N$$

```
> deq:=gfun:-holexprtodiffeq(P,y(x)):
> rec:=gfun:-diffeqtorec(deq,y(x),u(n)):
> gfun:-rectoproc(eval(rec,N=10000),u(n))(60000):
> length(%),time();
```

28571, 36.093

Two exercises

$$\sum_{m=0}^{n} \binom{-1/4}{m}^2 \binom{-1/4}{n-m}^2$$

$$[x^{6N}](1+x)^{4N}(1+x+x^2)^{2N}(1+x+x^2+x^3+x^4)^N = ?, \qquad N = 500.$$

```
> P:=(1+x)^(4*N)*(1+x+x^2)^(2*N)*(1+x+x^2+x^3+x^4)^N;
```

$$P := (1+x)^{4N} (1+x+x^2)^{2N} (1+x+x^2+x^3+x^4)^N$$

```
> deq:=gfun:-holexprtodiffeq(P,y(x)):
> rec:=gfun:-diffeqtorec(deq,y(x),u(n)):
> gfun:-rectoproc(eval(rec,N=10000),u(n))(60000):
> length(%),time();
```

28571, 36.093

Two exercises

$$\sum_{m=0}^{n} \binom{-1/4}{m}^2 \binom{-1/4}{n-m}^2$$

```
> binomial(-1/4,n);
```

$$\mathrm{binomial}\left(-\frac{1}{4}, n\right)$$

```
> rec:={numer(u(n+1)/u(n)-expand(subs(n=n+1,%)/%)),u(0)=1};
```

$$rec := \{4\,u(n+1)\,n + 4\,u(n+1) + u(n) + 4\,u(n)\,n,\ u(0) = 1\}$$

```
> rec2:=gfun:-poltorec(u(n)^2,[rec],[u(n)],u(n));
```

$$rec2 := \{(-1 - 8\,n - 16\,n^2)\,u(n) + (16\,n^2 + 32\,n + 16)\,u(n+1),\ u(0) = 1\}$$

```
> collect(gfun:-cauchyproduct(rec2,rec2,u(n)),u,factor);
```

$$\{(2\,n+1)^3\,u(n) - 8\,(n+1)^3\,u(n+1),\ u(0) = 1\}$$

```
> convert(rsolve(%,u(n)),binomial);
```

$$\mathrm{binomial}\left(n - \frac{1}{2}, -\frac{1}{2}\right)^3$$

$$[x^{6N}](1+x)^{4N}(1+x+x^2)^{2N}(1+x+x^2+x^3+x^4)^N = ?, \qquad N = 500.$$

```
> P:=(1+x)^(4*N)*(1+x+x^2)^(2*N)*(1+x+x^2+x^3+x^4)^N;
```

$$P := (1+x)^{4N}\,(1+x+x^2)^{2N}\,(1+x+x^2+x^3+x^4)^N$$

```
> deq:=gfun:-holexprtodiffeq(P,y(x)):
> rec:=gfun:-diffeqtorec(deq,y(x),u(n)):
> gfun:-rectoproc(eval(rec,N=10000),u(n))(60000):
> length(%),time();
```

28571, 36.093

Two exercises

$$\sum_{m=0}^{n} \binom{-1/4}{m}^2 \binom{-1/4}{n-m}^2 = \binom{n-\frac{1}{2}}{-\frac{1}{2}}^3$$

```
> binomial(-1/4,n);
```

$$\operatorname{binomial}\left(-\frac{1}{4}, n\right)$$

```
> rec:={numer(u(n+1)/u(n)-expand(subs(n=n+1,%)/%)),u(0)=1};
```

$$rec := \{4\,u(n+1)\,n + 4\,u(n+1) + u(n) + 4\,u(n)\,n,\, u(0)=1\}$$

```
> rec2:=gfun:-poltorec(u(n)^2,[rec],[u(n)],u(n));
```

$$rec2 := \{(-1 - 8\,n - 16\,n^2)\,u(n) + (16\,n^2 + 32\,n + 16)\,u(n+1),\, u(0)=1\}$$

```
> collect(gfun:-cauchyproduct(rec2,rec2,u(n)),u,factor);
```

$$\{(2\,n+1)^3\,u(n) - 8\,(n+1)^3\,u(n+1),\, u(0)=1\}$$

```
> convert(rsolve(%,u(n)),binomial);
```

$$\operatorname{binomial}\left(n-\frac{1}{2}, -\frac{1}{2}\right)^3$$

$$[x^{6N}](1+x)^{4N}(1+x+x^2)^{2N}(1+x+x^2+x^3+x^4)^N = ?, \qquad N = 500.$$

```
> P:=(1+x)^(4*N)*(1+x+x^2)^(2*N)*(1+x+x^2+x^3+x^4)^N;
```

$$P := (1+x)^{4N}\,(1+x+x^2)^{2N}\,(1+x+x^2+x^3+x^4)^N$$

```
> deq:=gfun:-holexprtodiffeq(P,y(x)):
> rec:=gfun:-diffeqtorec(deq,y(x),u(n)):
> gfun:-rectoproc(eval(rec,N=10000),u(n))(60000):
> length(%),time();
```

28571, 36.093

[109] PF & BS (1993)

Two exercises

$$\sum_{m=0}^{n} \binom{-1/4}{m}^2 \binom{-1/4}{n-m}^2 = \binom{n-\frac{1}{2}}{-\frac{1}{2}}^3$$

```
> binomial(-1/4,n);
```

$$\mathrm{binomial}\left(-\frac{1}{4}, n\right)$$

```
> rec:={numer(u(n+1)/u(n)-expand(subs(n=n+1,%)/%)),u(0)=1};
```

$$rec := \{4\,u(n+1)\,n + 4\,u(n+1) + u(n) + 4\,u(n)\,n,\, u(0) = 1\}$$

```
> rec2:=gfun:-poltorec(u(n)^2,[rec],[u(n)],u(n));
```

$$rec2 := \{(-1 - 8\,n - 16\,n^2)\,u(n) + (16\,n^2 + 32\,n + 16)\,u(n+1),\, u(0) = 1\}$$

```
> collect(gfun:-cauchyproduct(rec2,rec2,u(n)),u,factor);
```

$$\{(2\,n+1)^3\,u(n) - 8\,(n+1)^3\,u(n+1),\, u(0) = 1\}$$

```
> convert(rsolve(%,u(n)),binomial);
```

$$\mathrm{binomial}\left(n - \frac{1}{2}, -\frac{1}{2}\right)^3$$

$$[x^{6N}](1+x)^{4N}(1+x+x^2)^{2N}(1+x+x^2+x^3+x^4)^N = ?, \qquad N = 500.$$

```
> P:=(1+x)^(4*N)*(1+x+x^2)^(2*N)*(1+x+x^2+x^3+x^4)^N;
```

$$P := (1+x)^{4\,N}\,(1+x+x^2)^{2\,N}\,(1+x+x^2+x^3+x^4)^N$$

```
> deq:=gfun:-holexprtodiffeq(P,y(x)):
> rec:=gfun:-diffeqtorec(deq,y(x),u(n)):
> gfun:-rectoproc(eval(rec,N=10000),u(n))(60000):
> length(%),time();
```

28571, 36.093

For large N, expanding is not possible.

[109] PF & BS (1993) ; [136] PF & BS (1997).

Non-holonomy of simple sequences

$$\log n, \sqrt{n}, p_n, \frac{1}{H_n}, n^n, e^{\sqrt{n}}, e^{1/n}, \Gamma(n\sqrt{2}), \ldots$$

do not satisfy linear recurrences with polynomial coefficients.

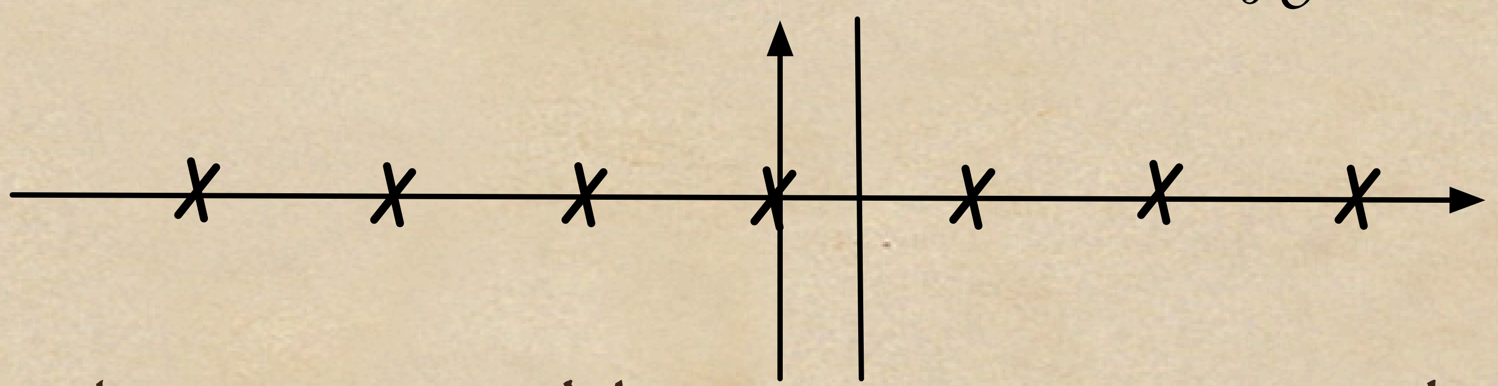
[184,207] PF & S. Gerhold & BS (2005 & 2010).

Non-holonomy of simple sequences

$$\log n, \sqrt{n}, p_n, \frac{1}{H_n}, n^n, e^{\sqrt{n}}, e^{1/n}, \Gamma(n\sqrt{2}), \ldots$$

do not satisfy linear recurrences with polynomial coefficients.

Proof by Lindelöf integrals. $F(z) = \frac{1}{2\pi i} \int_C \phi(s) z^s \frac{\pi}{\sin \pi s} \, ds$



- Use known possible asymptotic behaviours for solutions of LDEs.
- Compute the asymptotics of these integrals.

[184,207] PF & S. Gerhold & BS (2005 & 2010).

Conclusion

Was Philippe doing only symbolic computation?

- Not quite
  [➥all the other talks]
- But still, more than you'd have thought.

Conclusion

Was Philippe doing only symbolic computation?

- Not quite
  [➥all the other talks]
- But still, more than you'd have thought.

Thank you, Philippe.