Philippe Flajolet

and

Dynamical Combinatorics.

Brigitte Vallée
GREYC (CNRS and Université de Caen)

Conference in the memory of Philippe
Philippe Flajolet and Analytical Combinatorics

Paris, December 2011.

A long joint scientific story, which lasted twenty years ...

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... with four main steps

– Beginnings : analysis of the Gauss Algorithm (1990–1995)

– Developments (1995–1998)

– Foundation of dynamical combinatorics (1998–2006)

– Towards a realistic analysis of algorithms? (2008–2011)

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Analysis of the Gauss Algorithm

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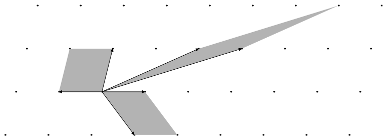
The Gauss algorithm

A lattice of $\mathbb{R}^2$

$=$ a discrete additive subgroup of $\mathbb{R}^2$

$$\mathcal{L} := \{w \in \mathbb{R}^2;\ w = xu + yv,\ x, y \in \mathbb{Z}\}$$



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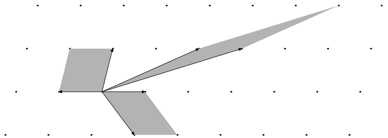

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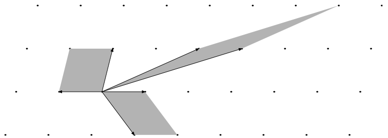

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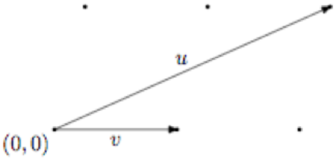

The Gauss alg. performs integer translations seen as "vectorial" divisions

$$u = mv + r \qquad \text{with} \qquad m = \left\lfloor \frac{u \cdot v}{|v|^2} \right\rfloor,$$

Here $m = 2$

Our main idea: A projective point of view.

$$(u, v) \in \mathbb{C}^2, \text{ with } u \neq 0 \longrightarrow z := \frac{v}{u} = \frac{(u \cdot v)}{|u|^2} + i\frac{\det(u, v)}{|u|^2}.$$

Up to similarity, the lattice $\mathcal{L}(u, v)$ becomes $\mathcal{L}(1, z) =: L(z)$.

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Euclid's algorithm	Gauss' algorithm
Division between real numbers $v = mu + r$ with $m = \left\lfloor \frac{u}{v} \right\rfloor$	Division between complex vectors $v = mu + r$ with $m = \left\lfloor \Re\left(\frac{u}{v}\right) \right\rfloor$

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Stopping condition: $x = 0$	Stopping condition: $z \in \mathcal{F}$ $\mathcal{F} := \{z; \ \ \|z\| \geq 1, \ \ \|\Re z\| \leq 1/2\}$

$\mathcal{D}$ is the disk of diameter $[0,1]$, and $$U(z) = \frac{1}{z} - \left\lfloor \Re\left(\frac{1}{z}\right) \right\rfloor$$

On an input $z \in \mathcal{D}$, the `CoreGauss` Alg performs $z := U(z)$, until $z \notin \mathcal{D}$.
It uses at each step the set of (inverse) branches of mapping U

$$\mathcal{H} := \{z \mapsto \frac{1}{m+z};\ m \geq 1\},$$ the same set as the Euclid Alg.

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An alternative expression involves ....
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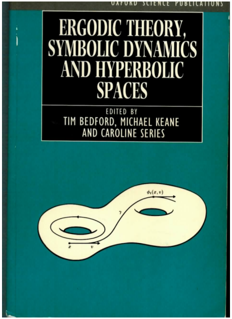

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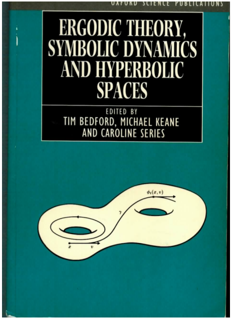

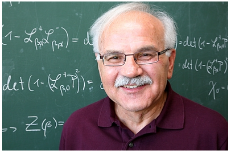

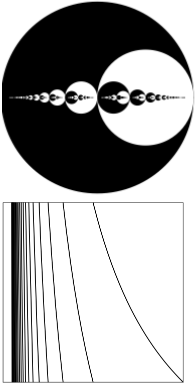

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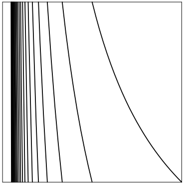

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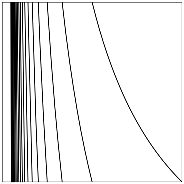

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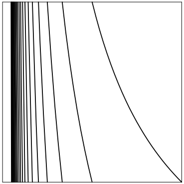

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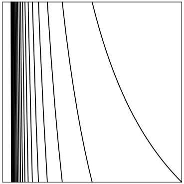

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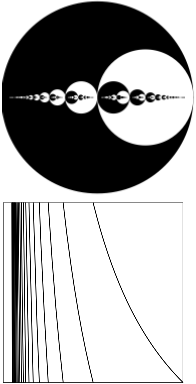

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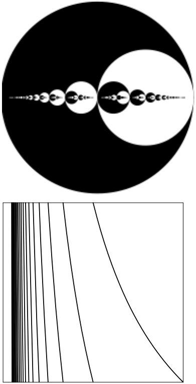

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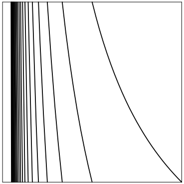

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Philippe wished to call it the Vallée constant...

He described 1994 as "the year of the constant $\lambda(2)$"...

II – Developments.

How to use the Ruelle-Mayer operator
in (almost) any Euclidean problem.

– Any problem related to continued fraction expansions
(continuous world, real variable)

– Any problem related to the Euclid Algorithm
(discrete world, rational variable)

Some instances of easy solutions for Euclidean problems – Real case (I)

Continued fraction expansion of x

$$x = \cfrac{1}{m_1 + \cfrac{1}{m_2 + \cfrac{1}{\ddots + \cfrac{1}{m_k + \cfrac{1}{\ddots}}}}}$$

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Iterates $\mathbf{G}_s^k$ of the operator $\mathbf{G}_s$ generate

– the truncated $CFE's$ at depth k,

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$$\mathbb{E}[\log q_k] \sim k|\Lambda'(1)|, \quad \mathbb{V}[\log q_k] \sim k\Lambda''(1) \qquad \text{with} \qquad \Lambda(s) := \log \lambda(s),$$

$\Lambda'(1)$ is the entropy, and $\Lambda''(1)$ is the Hensley constant.

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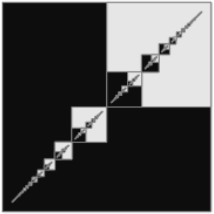

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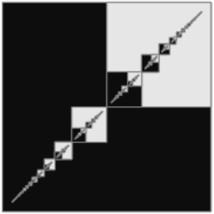

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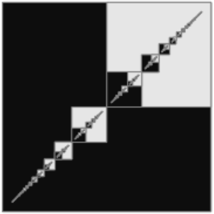

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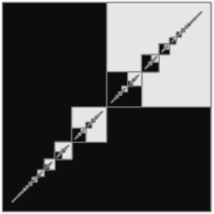

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Typical depth D_n, Height H_n

$$\mathbb{E}[D_n] \sim \frac{1}{|\lambda'(1)|} \log n$$

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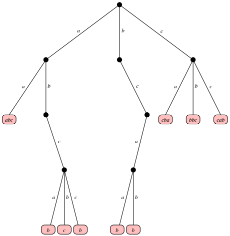

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$$S(s) := \sum_{(u,v)\in\Omega} \frac{P(u,v)}{v^{2s}} = \sum_{k\geq 1} k \sum_{h\in\mathcal{H}^k} |h'(0)|^s = \sum_{k\geq 1} k\mathbf{G}_s^k[1](0) = \mathbf{G}_s \circ (I - \mathbf{G}_s)^{-2}[1]($$

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Two properties of $\mathbf{G}_s$ (on a convenient functional space)

– The map $s \mapsto (I - \mathbf{G}_s)^{-1}$ is analytic on $\Re s \geq 1; s \neq 1$.

– A simple pole at $s = 1$

$\Longrightarrow$ Possible extraction of coefficients with a Tauberian Theorem

$\Longrightarrow$ A short proof of the estimate $\mathbb{E}[P_N] \sim \dfrac{2}{|\lambda'(1)|} \log N = \dfrac{12 \log 2}{\pi^2} \log N$

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P: I don't know for this formula... But if you have another one ...
[*the formula is spelled*], it would be easier...

B: The i, is it in the exponent?

Dynamical Combinatorics = Extension of Analytical Combinatorics
(Modified)transfer operators viewed as Generating Operators

In Period II, we studied the Euclid dynamical system,
with its transfer operator

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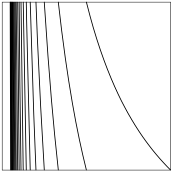

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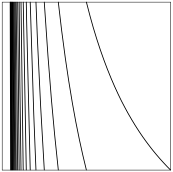

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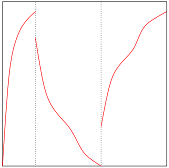
In Period III, extension to any weighted dynamical system with its set $\mathcal{H}$ of branches weighted by some cost c.

It is useful to consider two points x and y,



$$\mathbf{H}_{s,w}[F](x,y) := \sum_{h \in \mathcal{H}} w^{c(h)} \left| \frac{h(x) - h(y)}{x - y} \right|^s F(h(x), h(y))$$

s marks the input size, and w marks the cost c

The secant $\left| \frac{h(x) - h(y)}{x - y} \right|$ replaces the tangent $|h'(x)|$

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(on a convenient functional space)

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  [a functional analysis step]

– Translate these analytical properties into asymptotics of coefficients
  [a "classical" step in Analytical Combinatorics]

Euclidean dynamics: study of (almost) any algorithm of Euclid type

The branches are always LFT's, but the dynamics may differ a lot.

Extensions of the methods of Period II in four directions

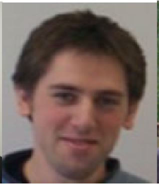
– to other types of Euclid Alg.
  the Binary Alg. – the Lyapounov tortoise and the dyadic hare.

– to other costs (for instance: bit complexity of Euclid Alg).

– to distributional analyses
  with bivariate transfer operators

– to a precise comparison between Euclid and Gauss Alg.



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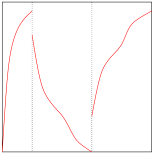

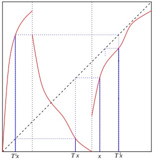

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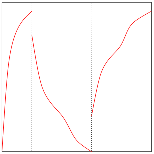
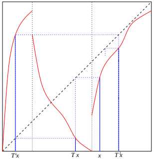
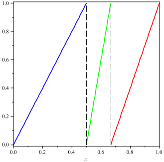




$M(x) = (c, b, a, c \ldots)$

A memoryless source, with probabilities (p_i)

$$\Lambda(s) = \frac{1}{1 - \lambda(s)} \qquad \text{with} \qquad \lambda(s) = \sum_{i=1}^{r} p_i^s$$



In Information Theory, the main object is : [V. 2001]

the Dirichlet series of the source $\Lambda(s) := \sum_{w \in \Sigma^\star} p_w^s$

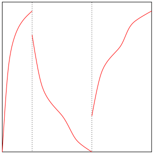
$p_w :=$ the probability that a word begins with prefix w

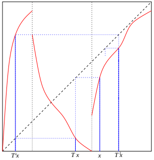
A general dynamical source

$\Lambda(s)$ closely related to $(I - \mathbf{H}_s)^{-1}$

where $\mathbf{H}_s$ is the (secant) transfer operator

of the dynamical system.





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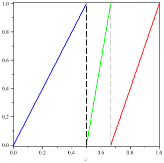


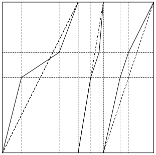
A Markov chain, defined by

– the vector R of initial probabilities (r_i)

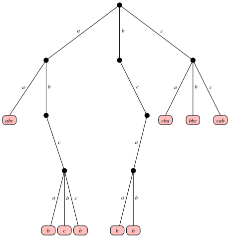
– and the transition matrix $P := (p_{i,j})$

$$\Lambda(s) = 1 + {}^t\mathbf{1}(I - P(s))^{-1} R(s)$$

with $\quad P(s) = (p_{i,j}^s), \quad R(s) = (r_i^s).$



Analysis of general tries



– All the possible node structures: array – list – or BST

– Words produced by a general (good) dynamical source.

– Possible infinite alphabet

Here, focus on the height and typical depth [2001]:
Both means are of order $\log n$ [FV with J. Clément]

$$\mathbb{E}[D_n] \sim \frac{1}{|\lambda'(1)|} \log n \qquad \mathbb{E}[H_n] \sim \frac{1}{|\log \lambda(2)|} \log n$$

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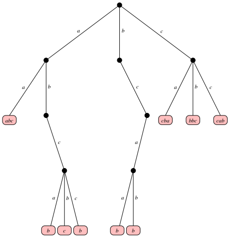

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It depends on the shape of a "tameness" region $\mathcal{R} \subset \{\Re s \leq 1, s \neq 1\}$
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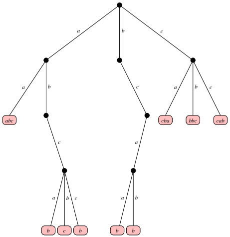

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Later, we study the tameness of a memoryless source,
We relate the shape of an optimal tameness region
with diophantine properties of ratios $\log p_i / \log p_j$.

[FV with M. Roux, 2010]

IV– Towards a "realistic" analysis of algorithms?

– Concept of a general source, completely defined by the set

$p_w :=$ the probability that the word begins with w.

$\Longrightarrow$ Parametrization by the unit interval $\mathcal{I}$

There exists a map $M : \mathcal{I} \to \Sigma^{\mathbb{N}}$, continuous, increasing, s.t each emitted word is (a.e uniquely) written as $M(u)$ with $u \in \mathcal{I}$.

– Analysis of the basic algorithms (sorting and searching alg.) when

- the inputs are random words of a general source
- the cost of a comparison between words
  $=$ the number of symbols needed in a lexicographic comparison.

a b a b b b...
a b a a b a...

coincidence=3; #comparisons=4.

Distribution of the coincidence C

$=$ the length of the longest common prefix

$$\Pr[C \geq k+1] = \sum_{w \in \Sigma^k} p_w^2$$

Realistic analysis of an algorithm using comparisons on keys.

Coincidence γ of the source $\mathcal{S}$

$\gamma(u,t)$ is the coincidence between $M(u)$ and $M(t)$

Density ϕ of the algorithm $\mathcal{A}$ (using comparisons):

$\phi(u,t)$:= the "mean" number of key comparisons between u and t when they are given to Alg. $\mathcal{A}$ after being inserted in a random sequence of $\mathcal{I}^\star$.

We wished to build a general dictionary (and began to do it)

Source $\mathcal{S}$, its coincidence γ and Alg $\mathcal{A}$, its density ϕ	$\Longrightarrow$	Mixed Dirichet series $\varpi(s)$ depends both on the source and the alg.	$\Longrightarrow$	The number S_n of symbol comp. performed by Alg $\mathcal{A}$ on n words

$\Longrightarrow$: $$\mathbb{E}[S_n] = \sum_{k=2}^{n} (-1)^k \binom{n}{k} \varpi(k)$$

A general dictionary

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$$\mathbb{E}[S_n] = \sum_{k=2}^{n} (-1)^k \binom{n}{k} \varpi(k)$$

We [$=$ FV with Julien] have already determined the mixed series $\varpi(s)$

– for Tries [2001],

– for Binary Search Trees (or QuickSort) [with J.Fill, 2009]

$$\varpi_T(s) = s\Lambda(s), \qquad \varpi_B(s) = \frac{2\Lambda(s)}{s(s-1)}, \qquad \text{with} \quad \Lambda(s) = \sum_{w \in \Sigma^\star} p_w^s$$

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And now? Determine series $\varpi(s)$ in other contexts!

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Philippe, we will continue this work.
This would have been your strong wish.