Philippe Flajolet and Dynamical Combinatorics.

Brigitte Vallée
GREYC (CNRS and Université de Caen)

Conference in the memory of Philippe Flajolet and Analytical Combinatorics
A long joint scientific story, which lasted twenty years ...
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... with four main steps

I – Beginnings
Analysis of the Gauss Algorithm

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The Gauss algorithm

A lattice of $\mathbb{R}^2$

$= \text{a discrete additive subgroup of } \mathbb{R}^2$

$\mathcal{L} := \{ w \in \mathbb{R}^2; w = xu + yv, x, y \in \mathbb{Z} \}$

A lattice with three possible bases $(u, v)$
The Gauss algorithm

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The Gauss algorithm is a main tool for finding short bases in any dimensions.
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The Gauss alg. performs integer translations

seen as “vectorial” divisions

\[ u = mv + r \quad \text{with} \quad m = \left\lfloor \frac{u \cdot v}{|v|^2} \right\rfloor, \]

Here \( m = 2 \)
Our main idea: A projective point of view.

\[(u, v) \in \mathbb{C}^2, \text{ with } u \neq 0 \longrightarrow z := \frac{v}{u} = \frac{(u \cdot v)}{|u|^2} + i \frac{\text{det}(u, v)}{|u|^2}.\]

Up to similarity, the lattice \( \mathcal{L}(u, v) \) becomes \( \mathcal{L}(1, z) =: L(z) \).
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Stopping condition:

- Euclid: \( x = 0 \)
- Gauss: \( z \in \mathbb{F} := \{z; |z| \geq 1, |\Re(z)| \leq 1/2\} \)
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\(\mathcal{F} := \{z; |z| \geq 1, |\Re z| \leq 1/2\}\)
\( \mathcal{D} \) is the disk of diameter \([0, 1]\), and

\[
U(z) = \frac{1}{z} - \left\lfloor \Re\left(\frac{1}{z}\right) \right\rfloor
\]

On an input \( z \in \mathcal{D} \), the CoreGauss Alg performs \( z := U(z) \), until \( z \not\in \mathcal{D} \). It uses at each step the set of (inverse) branches of mapping \( U \)

\[
\mathcal{H} := \{ z \mapsto \frac{1}{m + z}; \ m \geq 1 \}, \quad \text{the same set as the Euclid Alg.}
\]

The domain \([R \geq k + 1]\) is a union of disks:

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Inside disk \( \mathcal{D} \),
the domains \([R = k]\)
alternatively
in black and white
\(D\) is the disk of diameter \([0, 1]\), and

\[U(z) = \frac{1}{z} - \left\lfloor \Re \left( \frac{1}{z} \right) \right\rfloor\]

On an input \(z \in D\), the CoreGauss Alg performs \(z := U(z)\), until \(z \notin D\). It uses at each step the set of (inverse) branches of mapping \(U\)

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The domain \([R \geq k + 1]\) is a union of disks : \([R \geq k + 1] = \bigcup_{h \in \mathcal{H}^k} h(D)\)

The disks \(h(D)\) are the “fundamental” disks

\[\mathbb{E}[R] = \sum_{h \in \mathcal{H}^*} [h(0) - h(1)]^2,\]

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A simple characterization of $\mathcal{H}^*$ leads to

$$\mathbb{E}[R] = \frac{3}{4} + \frac{2}{\zeta(4)} \sum_{d \geq 1} \frac{1}{d^2} \sum_{d < c \leq 2d} \frac{1}{d^2} = 1.351094...$$

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\Pr[R \geq k + 1] = \sum_{h \in \mathcal{H}^k} |h(0) - h(1)|^2
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But how to characterize \(\mathcal{H}^k\)? More difficult! ...
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(EDS)

The Ruelle-Mayer operator $G_s$, defined as

$$G_s[f](x) := \sum_{h \in H} |h'(x)|^s f \circ h(x)$$

is the transfer operator related to the EDS. It generates (with its $k$–th iterate) the subset $H_k$.

$$\Pr[R \geq k + 1] = G_k^2[1](1 + x)^2(0)$$

Spectral properties of $G_k^2$ on a convenient functional space prove:

$$\Pr[R \geq k + 1] \sim C \lambda^k(2)$$

with $\lambda(2)$ := the dominant eigenvalue of $G_k^2$

We "discovered" $\lambda(2)$ in 1994, its value is $\lambda(2) \sim 0.1994...$

Philippe wished to call it the Vallée constant... He described 1994 as "the year of the constant $\lambda(2)$..."
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II – Developments.

How to use the Ruelle-Mayer operator in (almost) any Euclidean problem.

– Any problem related to continued fraction expansions
  (continuous world, real variable)

– Any problem related to the Euclid Algorithm
  (discrete world, rational variable)
Some instances of easy solutions for Euclidean problems – Real case (I)

Continued fraction expansion of $x$

$$x = \frac{1}{m_1 + \frac{1}{m_2 + \frac{1}{\ddots + \frac{1}{m_k + \frac{1}{\ddots}}}}}$$
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Iterates $G_s^k$ of the operator $G_s$ generate – the truncated $CFE's$ at depth $k$,
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Iterates $G_s^k$ of the operator $G_s$ generate
- the truncated $CFE'$s at depth $k$,
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  \[ h(\mathcal{I}) := [h(0), h(1)] \text{ with } h \in H^k. \]
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– the truncated $CFE'$s at depth $k$,
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  $$h(I) := [h(0), h(1)]$$ with $h \in \mathcal{H}^k$.
– the best rational approximation of a real $x$,
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Distance $\delta_k(x)$ between the real $x$ and its best approximation $p_k/q_k(x)$

\[ \mathbb{E}[\delta_k] \text{ expressed with } G_2^k \quad \implies \quad \mathbb{E}[\delta_k] \sim C \lambda(2)^k \]
Some instances of **easy** solutions for Euclidean problems – **Real** case (I)

Continued fraction expansion of \(x\)

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\frac{1}{m_1+ \frac{1}{m_2+ \frac{1}{\ddots + \frac{1}{m_k+ \frac{1}{\ddots }}}} }
\]

Iterates \(G_s^k\) of the operator \(G_s\) generate

- the truncated \(CFE's\) at depth \(k\),
- the fundamental intervals
  \[ h(I) := [h(0), h(1)] \text{ with } h \in \mathcal{H}_k. \]
- the best rational approximation of a real \(x\),
  namely the rational \(h(0) = \frac{p_k}{q_k}(x)\)

Distance \(\delta_k(x)\) between the real \(x\) and its best approximation \(p_k/q_k(x)\)

\[\mathbb{E}[\delta_k] \text{ expressed with } G_{k}^{2} \quad \implies \quad \mathbb{E}[\delta_k] \sim C \lambda(2)^k\]

Behaviour of \(\log q_k(x)\):
Some instances of easy solutions for Euclidean problems – Real case (I)

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Behaviour of $\log q_k(x)$ : $\mathbb{E}[q_k^{2s}]$ expressed with $G_{1-s}^k$

Then, [spectral dominant properties + quasi-power theorem]

$$\implies \text{an asymptotic normal law for } \log q_k$$
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$$\mathbb{E}[\log q_k] \sim k|\Lambda'(1)|, \quad \forall [\log q_k] \sim k\Lambda''(1) \quad \text{with} \quad \Lambda(s) := \log \lambda(s),$$

$\Lambda'(1)$ is the entropy, and $\Lambda''(1)$ is the Hensley constant.
Other Euclidean problems – Real case (II)
Comparing reals via their **continued fraction expansion**
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Comparing reals via their continued fraction expansion

Comparison of two real numbers:
– related to the sign of the determinant,
– and the Hakmem algorithm.
The Hakmem Alg. and the Gauss Alg: the same geometry.

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[J. Clément, Master Thesis, 98]
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Typical depth \(D_n\), Height \(H_n\)

\[
\mathbb{E}[D_n] \sim \frac{1}{|\lambda'(1)|} \log n
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\[
\mathbb{E}[H_n] \sim \frac{1}{|\log \lambda(2)|} \log n
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Euclidean problems – Rational case. \( CFE(u/v) \sim \text{Euclid}(u, v) \)

Study of the number of iterations \( P \) of the Euclid Algorithm.
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For any integer pair of \( \Omega := \{(u,v); \ u < v, \ u,v \text{ coprime}\} \)

\[ \exists k, \ \exists h \in \mathcal{H}^k, \ \frac{u}{v} = h(0), \ \frac{1}{v^2} = |h'(0)|, \ P(u,v) = k. \]
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**Main idea:** we introduce a generating function \( S(s) \) of Dirichlet type and we relate it to the Ruelle-Mayer operator \( G_s \):

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S(s) := \sum_{(u,v) \in \Omega} \frac{P(u,v)}{v^{2s}} = \sum_{k \geq 1} k \sum_{h \in \mathcal{H}^k} |h'(0)|^s = \sum_{k \geq 1} k G_s^k [1](0) = G_s \circ (I - G_s)^{-2}[1](0)
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Two properties of \( G_s \) (on a convenient functional space)

– The map \( s \mapsto (I - G_s)^{-1} \) is analytic on \( \Re s \geq 1; s \neq 1 \).

– A simple pole at \( s = 1 \)

\[\implies \text{Possible extraction of coefficients with a Tauberian Theorem}\]

\[\implies \text{A short proof of the estimate} \quad \mathbb{E}[P_N] \sim \frac{2}{|\lambda'(1)|} \log N = \frac{12 \log 2}{\pi^2} \log N\]
In our previous studies, the Ruelle-Mayer operator $G_k^s$ generates the fundamental intervals $h(I)$ for $h \in H_k$ – the fundamental probabilities $p_w$, for $|w| = k$:

$$p_w = \text{the probability that } \text{CFE}(x) \text{ begins with the sequence } w$$

– the rationals $u/v$ of depth $k$ via their denominators $v$ as the branches are LFT’s:

$$u/v = h(0) \implies 1/v^2 = |h'(0)|$$

The role of $G_s^k$ is central because it is the transfer operator of the underlying dynamical system. We wished to extend this use of transfer operators to other contexts – Information theory: any dynamical system viewed as a process to produce symbols – Euclidean context: any dynamical system underlying a Euclid alg whose branches are LFT’s.
III– Foundations of Dynamical Combinatorics.

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Very often, these extensions began with Philippe, and
  – I mostly continued with other collaborators.
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P: The $x$, do you put it on the numerator or in the denominator?
B: [....]

P: I don’t know for this formula… But if you have another one …

[the formula is spelled], it would be easier...

B: The $i$, is it in the exponent?
Dynamical Combinatorics = **Extension** of Analytical Combinatorics

(Modified)transfer operators viewed as **Generating Operators**

In **Period II**, we studied the **Euclid dynamical system**, with its transfer operator

\[
G_s[f](x) = \sum_{h \in \mathcal{H}} |h'(x)|^s f \circ h(x)
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\[ G_s[f](x) = \sum_{h \in \mathcal{H}} |h'(x)|^s f \circ h(x) \]

In Period III, extension to any weighted dynamical system
with its set \( \mathcal{H} \) of branches weighted by some cost \( c \).
It is useful to consider two points \( x \) and \( y \),

\[ H_{s,w}[F](x, y) := \sum_{h \in \mathcal{H}} w^{c(h)} \left| \frac{h(x) - h(y)}{x - y} \right|^s F(h(x), h(y)) \]

\( s \) marks the input size, and \( w \) marks the cost \( c \)

The secant \[ \left| \frac{h(x) - h(y)}{x - y} \right| \] replaces the tangent \[ |h'(x)| \]
Dynamical Combinatorics = extension of Analytical Combinatorics. (Modified) transfer operator $H_{s,w}$ viewed as a generating operator.

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**Three main steps**

- Determine
  - the good *weighted dynamical system*
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- Translate the *geometric* properties of the weighted dynamical system into *spectral* properties of $H_{s,w}$, then into *analytical* properties of $(I - H_{s,w})^{-1}$
  (on a convenient functional space)
  [a functional analysis step]
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  [a functional analysis step]

– Translate these analytical properties into asymptotics of coefficients
  [a “classical” step in Analytical Combinatorics]
Euclidean dynamics: study of (almost) any algorithm of Euclid type

The branches are always LFT’s, but the dynamics may differ a lot.

Extensions of the methods of Period II in four directions

– to other types of Euclid Alg.
  the Binary Alg. – the Lyapounov tortoise and the dyadic hare.

– to other costs (for instance: bit complexity of Euclid Alg).

– to distributional analyses
  with bivariate transfer operators

– to a precise comparison between Euclid and Gauss Alg.
In Information Theory, the main object is:

[V. 2001]
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\[ \Lambda(s) := \sum_{w \in \Sigma^*} p_w^s \]

where \( p_w \) is the probability that a word begins with prefix \( w \).

A general dynamical source \( \Lambda(s) \) closely related to \( I - H_s \).

\[ A \text{ Markov chain, defined by} \]

- the vector \( R \) of initial probabilities \( (r_i) \)
- and the transition matrix \( P := (p_{i,j}) \)

\[ \Lambda(s) = 1 + t \left( I - P(s) \right)^{-1} R(s) \]
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where $H_s$ is the (secant) transfer operator of the dynamical system.

A memoryless source, with probabilities $(p_i)$

$$\Lambda(s) = \frac{1}{1 - \lambda(s)} \quad \text{with} \quad \lambda(s) = \sum_{i=1}^{r} p_i^s$$

$$M(x) = (c, b, a, c \ldots)$$
In Information Theory, the main object is:

[V. 2001]

the Dirichlet series of the source

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with \( P(s) = (p_{i,j}^s), \quad R(s) = (r_i^s) \).
Analysis of general tries

- All the possible node structures: array – list – or BST
- Words produced by a general (good) dynamical source.
- Possible infinite alphabet

Here, focus on the height and typical depth [2001]:
Both means are of order $\log n$ [FV with J. Clément]

$$
\mathbb{E}[D_n] \sim \frac{1}{|\lambda'(1)|} \log n \quad \mathbb{E}[H_n] \sim \frac{1}{|\log \lambda(2)|} \log n
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What about the remainder term for the typical depth?
It depends on the shape of a “tameness” region $R \subset \{Re s \leq 1, s \neq 1\}$
where $\Lambda(s)$ is analytic and of polynomial growth.
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where $\Lambda(s)$ is analytic and of polynomial growth.

Later, we study the tameness of a memoryless source,
We relate the shape of an optimal tameness region
with diophantine properties of ratios $\log p_i / \log p_j$.  

[FV with M. Roux, 2010]
IV– Towards a “realistic” analysis of algorithms?

– Concept of a general source, completely defined by the set $p_w :=$ the probability that the word begins with $w$.

$\implies$ Parametrization by the unit interval $I$

There exists a map $M : I \to \Sigma^N$, continuous, increasing, s.t each emitted word is (a.e uniquely) written as $M(u)$ with $u \in I$.

– Analysis of the basic algorithms (sorting and searching alg.) when

– the inputs are random words of a general source

– the cost of a comparison between words

$=$ the number of symbols needed in a lexicographic comparison.

Distribution of the coincidence $C$

$=$ the length of the longest common prefix

\[ \Pr[C \geq k + 1] = \sum_{w \in \Sigma^k} p_w^2 \]

coincidence=3; #comparisons=4.
Realistic analysis of an algorithm using comparisons on keys.

**Coincidence** $\gamma$ of the source $S$

$\gamma(u, t)$ is the coincidence between $M(u)$ and $M(t)$

**Density** $\phi$ of the algorithm $A$ (using comparisons):

$\phi(u, t) :=$ the “mean” number of key comparisons between $u$ and $t$ when they are given to Alg. $A$ after being inserted in a random sequence of $I^*$.

We wished to build a general dictionary (and began to do it)

<table>
<thead>
<tr>
<th>Source $S$, its coincidence $\gamma$ and Alg $A$, its density $\phi$</th>
<th>Mixed Dirichet series $\varpi(s)$ depends both on the source and the alg.</th>
<th>The number $S_n$ of symbol comp. performed by Alg $A$ on $n$ words</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\implies$</td>
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</table>

$$\implies: \quad \mathbb{E}[S_n] = \sum_{k=2}^{n} (-1)^k \binom{n}{k} \varpi(k)$$
A general dictionary

<table>
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<td>$\implies$</td>
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$$
\mathbb{E}[S_n] = \sum_{k=2}^{n} (-1)^k \left( \begin{array}{c} n \\ k \end{array} \right) \varpi(k)
$$

We [ = FV with Julien] have already determined the mixed series $\varpi(s)$
– for Tries [2001],
– for Binary Search Trees (or QuickSort) [with J.Fill, 2009]

$$
\varpi_T(s) = s\Lambda(s), \quad \varpi_B(s) = \frac{2\Lambda(s)}{s(s-1)}, \quad \text{with} \quad \Lambda(s) = \sum_{w \in \Sigma^*} p_w^s
$$
### A general dictionary

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And now? Determine series $\varpi(s)$ in other contexts!
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Philippe, we will continue this work.
This would have been your strong wish.