Stochastic deformation of curves

Multi-type exclusion processes

Hydrodynamic limits

Functional integration

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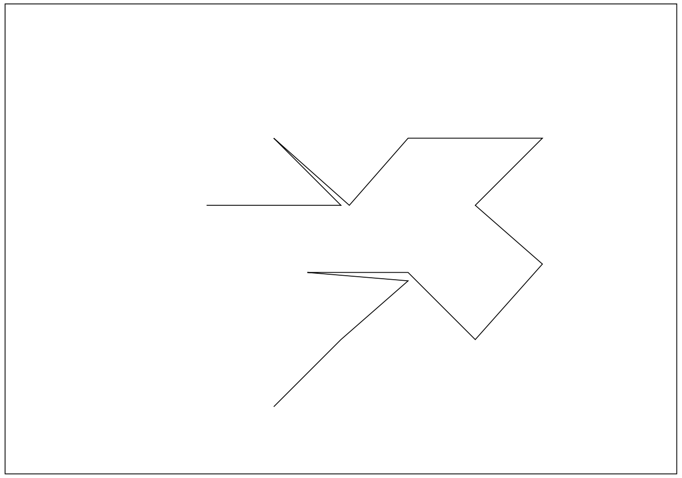
Philippe Flajolet's 60th birthday, ENS Paris, December 1-2, 2008.

Stochastic deformation of curves

Consider an oriented sample path of a planar random walk in $\mathbb{R}^2$, consisting of N steps (or links).

- Each step can have n discrete possible orientations $\theta_k = \frac{2k\pi}{n}, k = 0\ldots, n-1$.
- The stochastic dynamics consists in displacing one single point at a time, without breaking the path, so that 2 links are simultaneously displaced.



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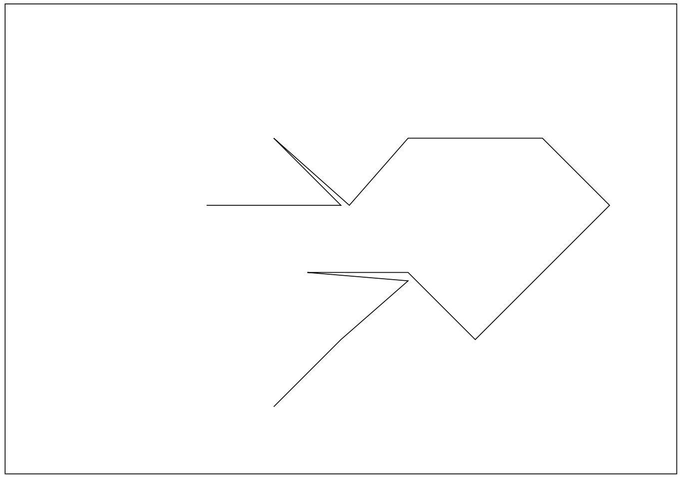

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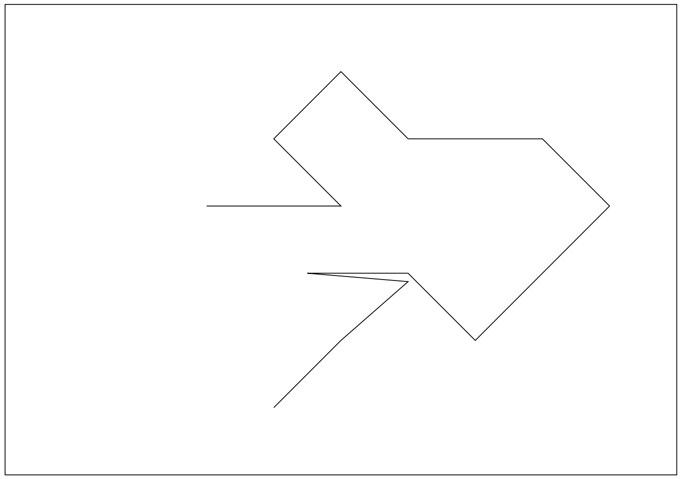

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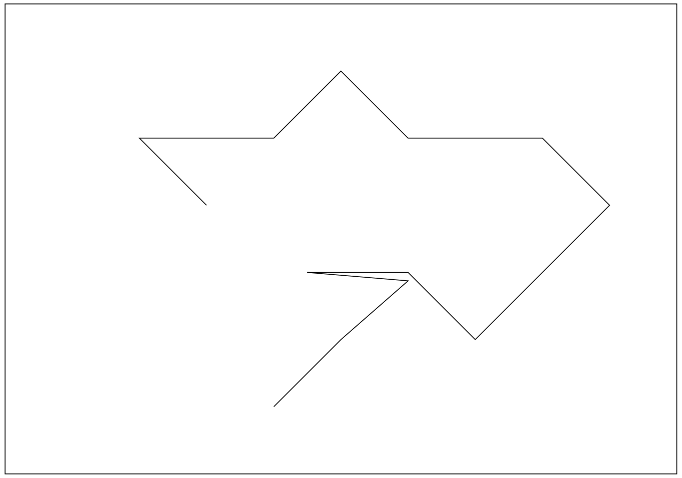

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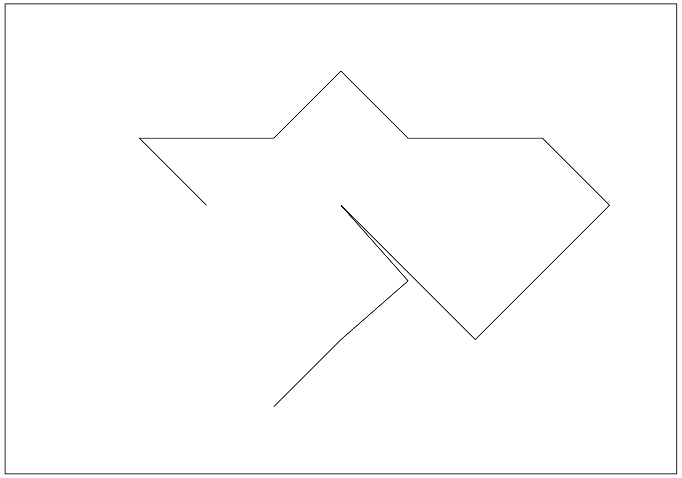

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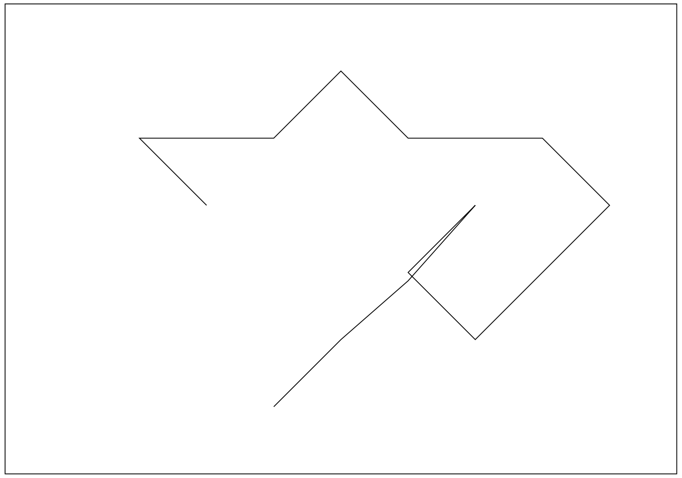

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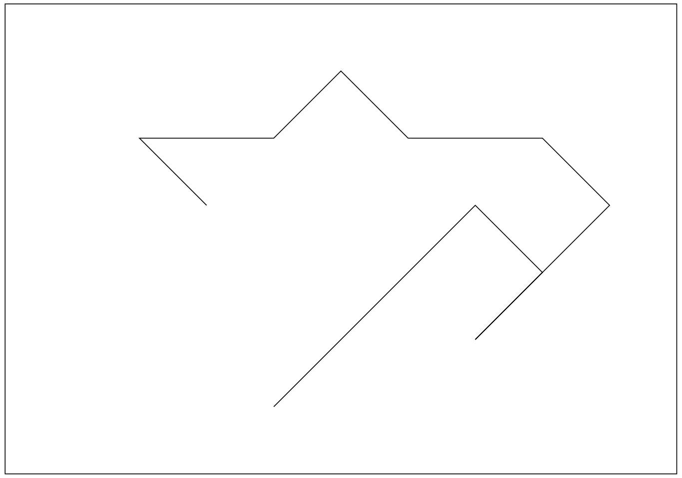

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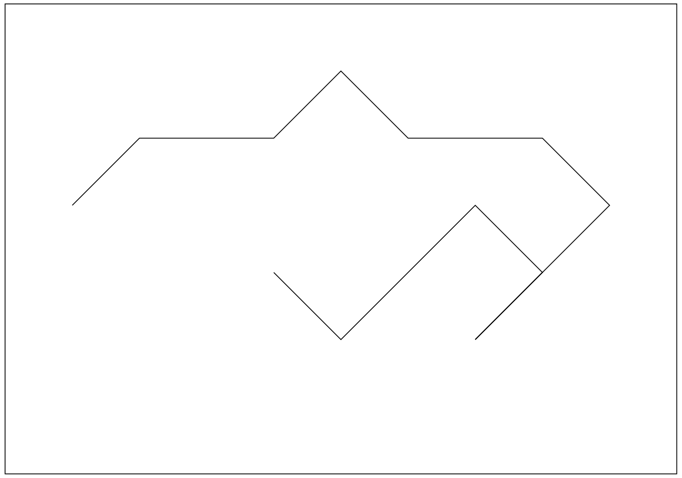

A general stochastic *clock model*

Construction of a continuous-time Markov chain

- Jumps are produced by independent exponential events.
- Periodic boundary conditions.
- Dynamical rules are given by a set of reactions between consecutive links [equivalent formulation in terms of *random grammar*].

With each link is associated a type, i.e. a letter of an alphabet.

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The set of reactions

For $i \in [1, N]$ and $k \in [1, n]$, let X_i^k represent a link of type k at site i.

Define the following set of reactions.

$$
\begin{cases}
X_i^k X_{i+1}^l \underset{\lambda^{lk}}{\overset{\lambda^{kl}}{\rightleftarrows}} X_i^l X_{i+1}^k, \qquad k = 1, \ldots, n, \quad l \neq k + \dfrac{n}{2}, \\
X_i^k X_{i+1}^{k+n/2} \underset{\delta^{k+1}}{\overset{\gamma^k}{\rightleftarrows}} X_i^{k+1} X_{i+1}^{k+n/2+1}, \qquad k = 1, \ldots, n.
\end{cases}
\tag{1}
$$

The red equations does exist only for even n, because of the existence of folds [two consecutive links with opposite directions], which yield a richer dynamics.

X_i^k can also be viewed as a *binary random variable* describing the occupation of site i by a letter of type k. Hence, the state space of the system is represented by the array $\quad \eta \stackrel{\text{def}}{=} \{X_i^k, i = 1, \ldots, N; k = 1, \ldots, n\}$.

Examples

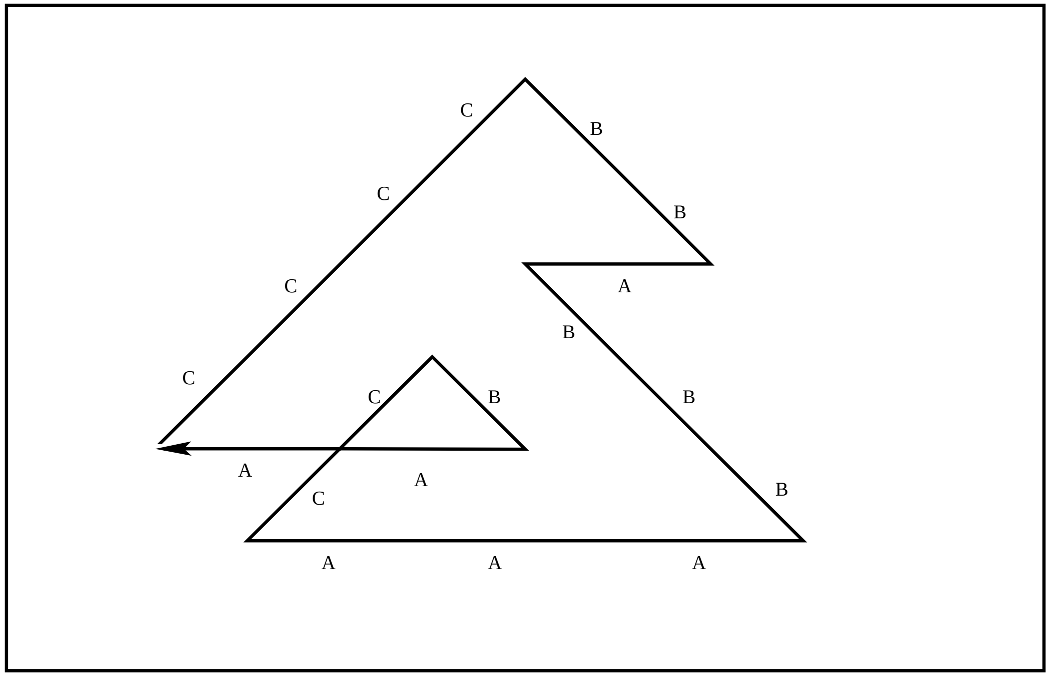
(0) ***ASEP***, **the basic** ***Asymmetric Simple Exclusion Process.***

(1) **The triangular lattice :** ***ABC model*** **[3 letter alphabet].**

(2) **The square lattice : a special** ***ABCD model*** **[4 letter alphabet] reducible to** ***2 coupled ASEP*****.**

Example I : the ABC model (3 letter alphabet)

$$AB \underset{\lambda_{ab}}{\overset{\lambda_{ba}}{\leftrightarrows}} BA, \qquad BC \underset{\lambda_{bc}}{\overset{\lambda_{cb}}{\leftrightarrows}} CB, \qquad CA \underset{\lambda_{ca}}{\overset{\lambda_{ac}}{\leftrightarrows}} AC,$$



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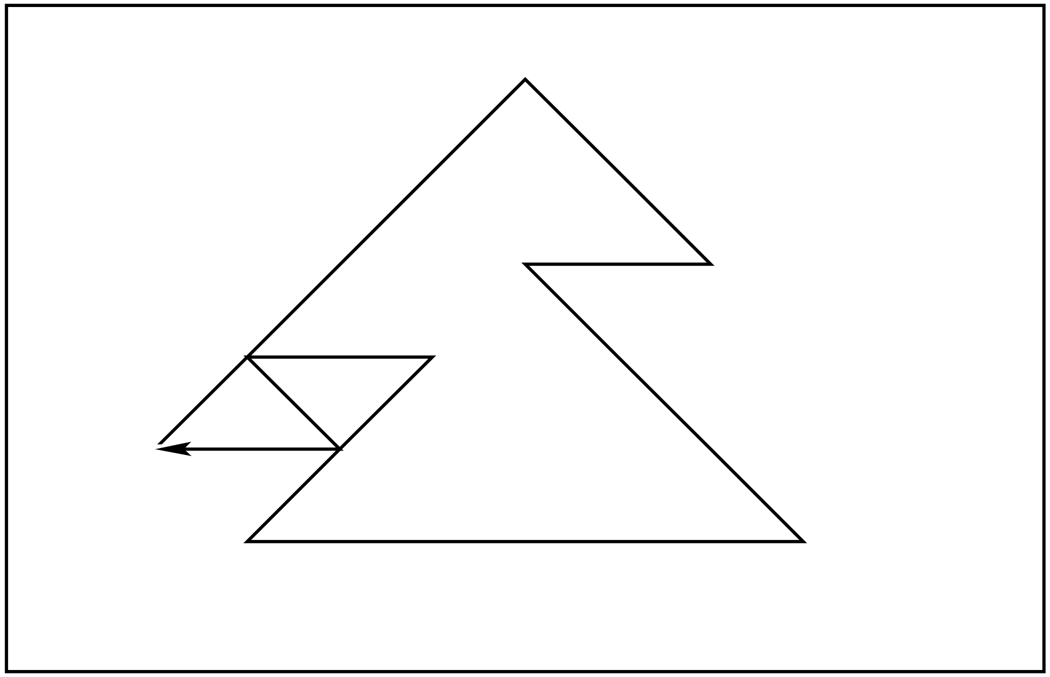

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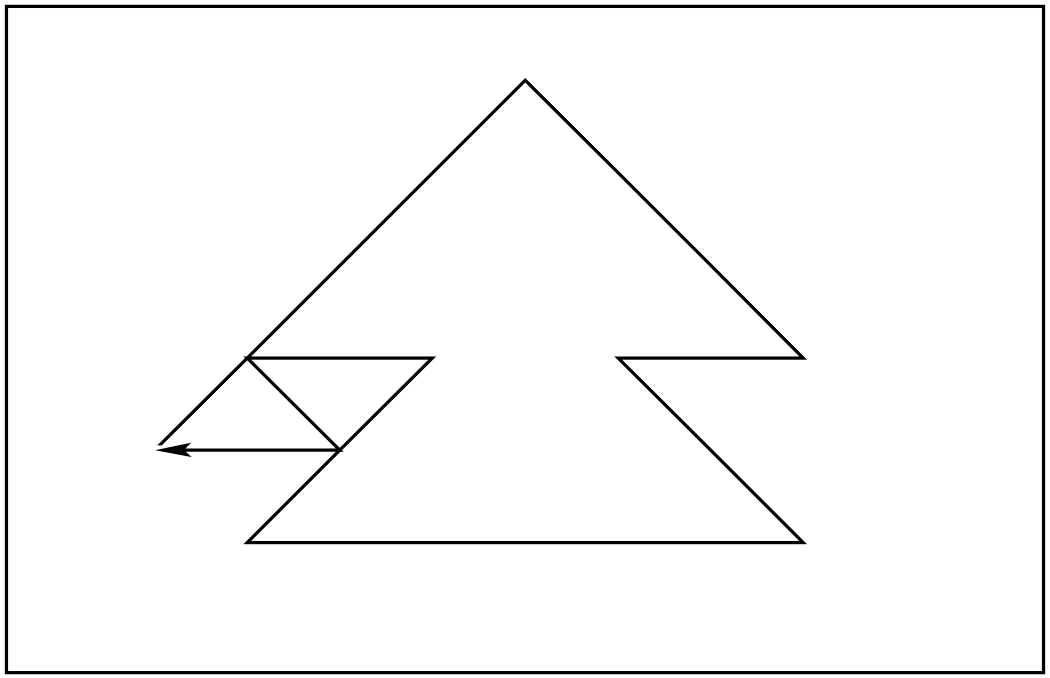

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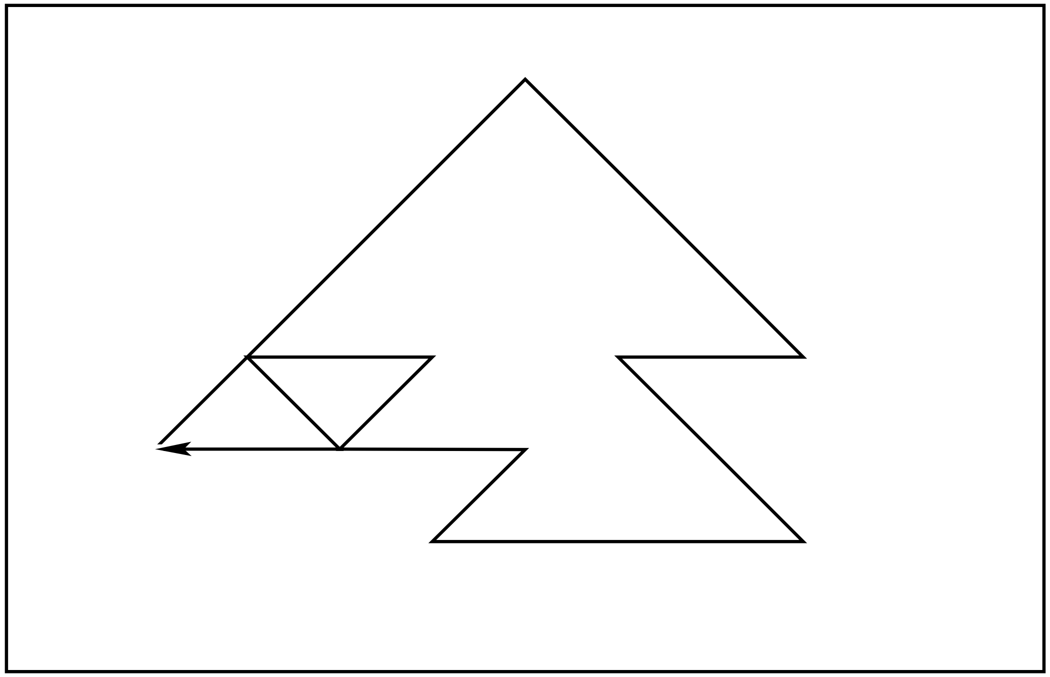

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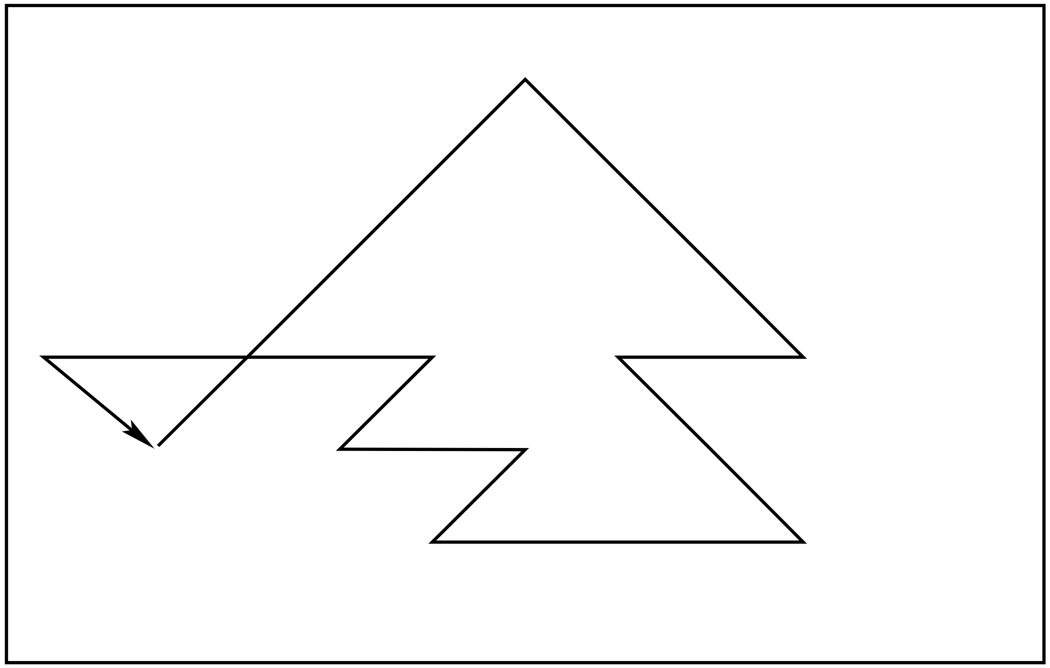


Thermodynamic limit and phase transition in the ABC model

[Evans, Kafri, Koduvely, Mukamel; *Phys. Rev. E, 58 (1998)*]

[Clincy, Derrida, Evans; *Phys. Rev. E, 67 (2003)*]

[Fayolle, Furtlehner; *Math. & Comp. Science III (2004)*]

[Fayolle, Furtlehner; *J. Stat. Phys., Vol. 127, No 5, (2007)*]

Let $\mathbb{X} = \{A_i, B_i, C_i\,; i = 1, \ldots, N\}$, where $A_i \in \{0,1\}$, $B_i \in \{0,1\}$, $C_i \in \{0,1\}$, denote binary random variables with the exclusion constraint $A_i + B_i + C_i = 1$.

$$\alpha^{ab} \stackrel{\text{def}}{=} \log \frac{\lambda_{ab}}{\lambda_{ba}}, \quad \alpha^{bc} \stackrel{\text{def}}{=} \log \frac{\lambda_{bc}}{\lambda_{cb}}, \quad \alpha^{ca} \stackrel{\text{def}}{=} \log \frac{\lambda_{ca}}{\lambda_{ac}}.$$

Under reversibility conditions

$$\frac{N_A}{N_B} = \frac{\alpha^{bc}}{\alpha^{ca}}, \quad \frac{N_B}{N_C} = \frac{\alpha^{ca}}{\alpha^{ab}}, \quad \frac{N_C}{N_A} = \frac{\alpha^{ab}}{\alpha^{bc}},$$

the invariant measure has the Gibbs form

$$\pi(\mathbb{X}) = \frac{1}{Z} \exp\Big[-\sum_{i<j} \alpha^{ab} A_i B_j + \alpha^{bc} B_i C_j + \alpha^{ca} C_i A_j\Big].$$

Fundamental scaling

$$\alpha^{bc} = \frac{\alpha}{N} + o\Big(\frac{1}{N}\Big), \quad \alpha^{ca} = \frac{\beta}{N} + o\Big(\frac{1}{N}\Big), \quad \alpha^{ab} = \frac{\gamma}{N} + o\Big(\frac{1}{N}\Big).$$

Then, **as $N \to \infty$** with natural periodic boundary conditions, densities of particles satisfy the following **Lotka-Volterra** system.

$$\begin{cases} \dfrac{\partial \rho_a}{\partial x} = \rho_a(\beta\rho_c - \gamma\rho_b), \\ \dfrac{\partial \rho_b}{\partial x} = \rho_b(\gamma\rho_a - \alpha\rho_c), \\ \dfrac{\partial \rho_c}{\partial x} = \rho_c(\alpha\rho_b - \beta\rho_a), \end{cases} \tag{2}$$

where $\rho_u(x+1) = \rho_u(x), \forall u \in \{a, b, c\}$.

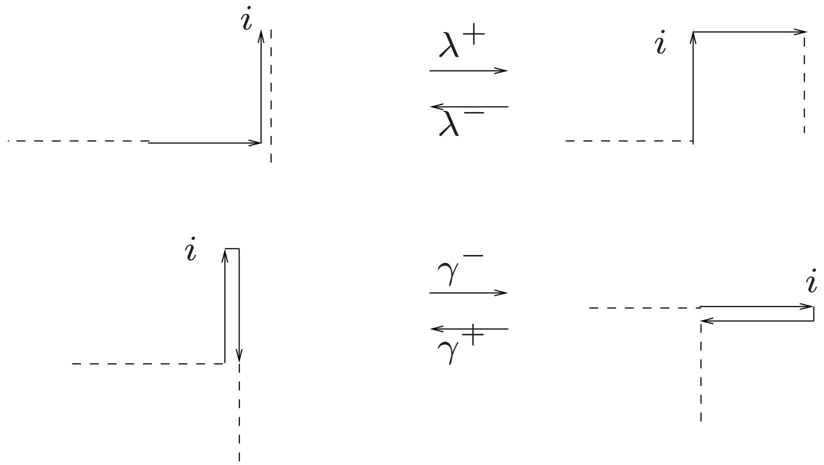
Theorem 1. *Let* $s \stackrel{\text{def}}{=} \alpha + \beta + \gamma, \quad \eta \stackrel{\text{def}}{=} \frac{s}{3}, \quad \widetilde{\rho}_a \stackrel{\text{def}}{=} \frac{\alpha}{s}, \quad \widetilde{\rho}_b \stackrel{\text{def}}{=} \frac{\beta}{s}, \quad \widetilde{\rho}_c \stackrel{\text{def}}{=} \frac{\gamma}{s}$. *Then there exists a critical value*

$$\eta_c \stackrel{\text{def}}{=} \frac{2\pi}{3\sqrt{\widetilde{\rho}_a\widetilde{\rho}_b\widetilde{\rho}_c}},$$

such that, for $\eta > \eta_c$, *there are non-degenerate trajectories of (2) with periods* $T_p = 1/p, p \in \{1, \ldots, [\frac{\eta}{\eta_c}]\}$. *The only admissible stable sytem corresponds*
– either to the trajectory associated with T_1 *if* $\eta > \eta_c$ *;*
– or to the degenerate one (a single point) if $\eta \leq \eta_c$. *Movie ABC1*

Example II : Coupled exclusion in $\mathbb{Z}^2$ (4 letter alphabet)

Consider a symmetric version of the $ABCD$ model, obtained by a rotation invariant version ($ABCD \longrightarrow BCDA$) with only 4 transition rates.



$$\text{The 4 main parameters} \quad \begin{cases} \lambda \stackrel{\text{def}}{=} \dfrac{\lambda^+ + \lambda^-}{2}, & \mu \stackrel{\text{def}}{=} \dfrac{\lambda^+ - \lambda^-}{2}, \\ \gamma \stackrel{\text{def}}{=} \dfrac{\gamma^+ + \gamma^-}{2}, & \delta \stackrel{\text{def}}{=} \dfrac{\gamma^+ - \gamma^-}{2}. \end{cases} \tag{3}$$

$\Leftrightarrow$ Reduction to 2 coupled simple exclusion processes.

Define the mapping $(A, B, C, D) \rightarrow (\tau_i^a, \tau_i^b) \in \{0, 1\}^2$, such that

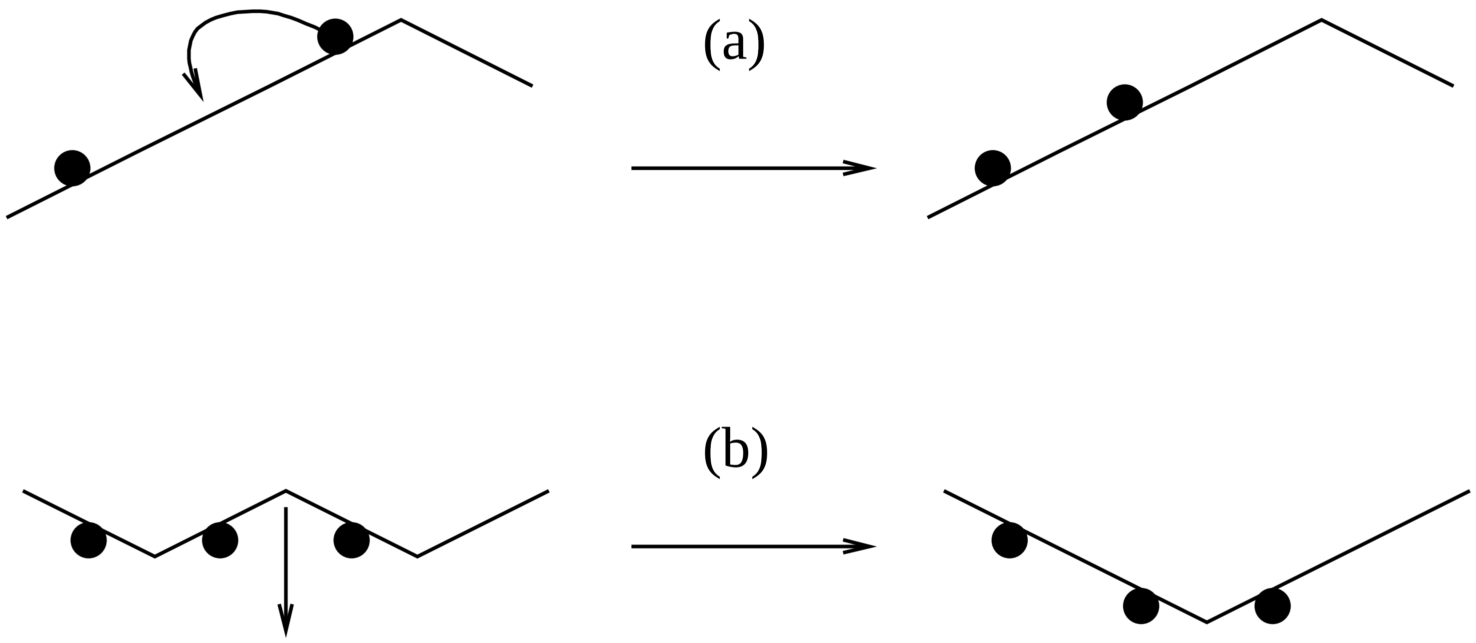
$$\begin{cases} A \rightarrow (0,0), \\ B \rightarrow (1,0), \\ C \rightarrow (1,1), \\ D \rightarrow (0,1). \end{cases}$$

Then $X_i \in \{A, B, C, D\} \rightarrow (\tau_i^a, \tau_i^b)$, a couple of binary random variables.

Each elementary transition corresponds to a jump of a particule (a) or (b).
In the case $\gamma^{\pm} = \lambda^{\pm}$, the conditional rates are given by

$$\begin{cases} \lambda_a^{\pm}(i) = \frac{\lambda^+ + \lambda^-}{2} + \frac{\lambda^+ - \lambda^-}{2}(2\bar{\tau}_i^b - 1), \\ \lambda_b^{\pm}(i) = \frac{\lambda^+ + \lambda^-}{2} - \frac{\lambda^+ - \lambda^-}{2}(2\bar{\tau}_i^a - 1). \end{cases}$$

⇔ Exclusion and fluctuating interface.



In figure (a), type (a) particles evolve on a profile defined by type (b) particles.

KPZ analogous

⇔ Fundamental scaling and phase transition phenomena.

Theorem 2. *(i) Under the scaling (see notation 3)*

$\frac{\mu(N)}{\lambda(N)} = \frac{\eta}{N} + o(\frac{1}{N}), \quad \frac{\delta(N)}{\gamma(N)} = \mathcal{O}(\frac{1}{N})$, *the following thermodynamic limit holds:*

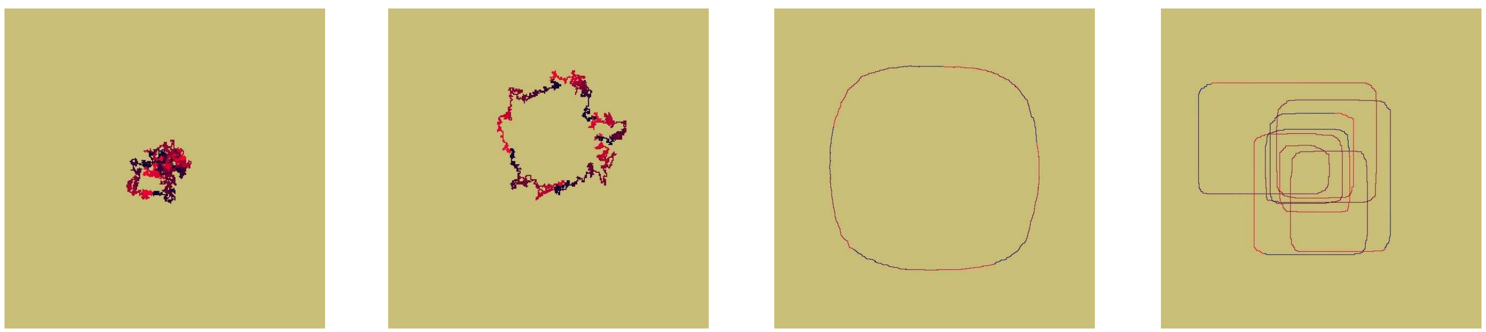
$$\begin{cases} \dfrac{\partial \rho_a(x)}{\partial x} = 2\eta\rho_a(x)(1-\rho_a(x))(2\rho_b(x)-1), \\ \dfrac{\partial \rho_b(x)}{\partial x} = -2\eta\rho_a(x)(1-\rho_b(x))(2\rho_a(x)-1). \end{cases}$$

(ii) Moreover, when $\lim_{N\to\infty} \frac{\lambda(N)}{N^2} = \frac{\gamma(N)}{N^2} \stackrel{def}{=} D$ *(diffusion constant), the two following coupled Burgers equations hold (under ad hoc initial conditions):*

$$\begin{aligned} \frac{\partial \rho_a}{\partial t} &= D\frac{\partial^2 \rho_a}{\partial x^2} - 2D\eta\frac{\partial}{\partial x}\big[\rho_a(1-\rho_a)(2\rho_b-1)\big], \\ \frac{\partial \rho_b}{\partial t} &= D\frac{\partial^2 \rho_b}{\partial x^2} + 2D\eta\frac{\partial}{\partial x}\big[\rho_b(1-\rho_b)(2\rho_a-1)\big]. \end{aligned}$$

Two last silent short films (not thrillers !)

- Phases in ABCD. η is the essential parameter.



$\eta = 0$ | $\eta \lesssim 2\pi$ | $2\pi \lesssim \eta \lesssim 17\pi$ | $\eta \gtrsim 17\pi$

- The non reversible ABC. Movie ABC2

Hydrodynamic limits for exclusion processes of type (1)

Consider again an oriented path consisting of N links of equal size, with periodic boundary conditions (e.g. closed).

- Dynamics. As in reactions (1). For an alphabet of size $n = 2p + 1$, we have

$$AB \underset{\lambda_{ba}}{\overset{\lambda_{ab}}{\rightleftarrows}} BA, \qquad a, b = 1, \ldots, n, \quad b \neq a. \tag{4}$$

- Fundamental scaling. For any pair (a,b),

$$\begin{cases} \lambda_{ab}(N) + \lambda_{ba}(N) = \lambda N^2 + o(N^2), \\ \lambda_{ab}(N) - \lambda_{ba}(N = \mu_{ab} N + o(N). \end{cases} \tag{5}$$

A (possibly new) method based on *functional integration*

Key example: the ASEP process.

Only two types of particles and the presence of a particle of type a [resp. b] at site i is equivalent to $A_i^{(N)}(t) = 1$ [resp. $B_i^{(N)}(t) = 1$], with the exclusion constraint $A_i^{(N)}(t) + B_i^{(N)}(t) = 1$.

The numbering of sites is implicitly taken modulo N, i.e. on the discrete torus $\mathbf{G}^{(N)} \stackrel{\text{def}}{=} \mathbb{Z}/N\mathbb{Z}$.

Problem: analyze, for $N \to \infty$, the sequence of random empirical measures

$$\mu_t^{(N)} = \frac{1}{N} \sum_{i \in \mathbf{G}^{(N)}} A_i^{(N)}(t) \delta_{\frac{i}{N}}.$$

Here $\left\{ \mathbf{A}^{(N)}(t) \stackrel{\text{def}}{=} \left(A_i^{(N)}(t), \ldots, A_N^{(N)}(t) \right), t \geq 0 \right\}$ is a Markov process.

An exponential transform

Let $\widetilde{\mathcal{C}[T]}$ denode the subset of functions $\in \mathcal{C}_0^\infty([0,1] \times [0, T_-])$ vanishing at $t = T$. Then, choose two arbitrary functions $\phi_a, \phi_b \in \widetilde{\mathcal{C}[T]}$ and define the following real-valued positive measure

$$Z_t^{(N)}[\phi_a, \phi_b] \stackrel{\text{def}}{=} \exp\left[\frac{1}{N} \sum_{i \in \mathbf{G}^{(N)}} \phi_a\Big(\frac{i}{N}, t\Big) A_i^{(N)}(t) + \phi_b\Big(\frac{i}{N}, t\Big) B_i^{(N)}(t)\right], \qquad (6)$$

The scaling being as in (5, and we assume the sequence of initial empirical measures $\log Z_0^{(N)}$, taken at time $t = 0$, converges in probability to some deterministic measure with a given density $\rho(x, 0)$, so that

$$\lim_{N \to \infty} \log Z_0^{(N)} = \int_0^1 [\rho(x,0)\phi_a(x,0) + (1 - \rho(x,0))\phi_b(x,0)]dx, \quad \text{in probability.}$$

Then, the following theorem holds.

Theorem 3. *For every $t > 0$, the sequence of random measures $\mu_t^{(N)}$ converges in probability, as $N \to \infty$, to a deterministic measure having a density $\rho(x,t)$ with respect to the Lebesgue measure, which is the unique* weak solution of the Cauchy problem

$$\int_0^T \int_0^1 \left[\rho(x,t)\Big(\frac{\partial \phi(x,t)}{\partial t} + \lambda \frac{\partial^2 \phi(x,t)}{\partial x^2}\Big) - \mu\rho(x,t)\big(1-\rho(x,t)\big)\frac{\partial \phi(x,t)}{\partial x}\right] dxdt + \int_0^1 \rho(x,0)\phi(x,0)dx = 0,$$

for any function $\phi \in \widetilde{\mathcal{C}[T]}$.

Proof. 3 Main steps.

(1) Sequential compactness.

(2) Characterization of limit points by a functional integral operator.

(3) Uniqueness. $\square$

Sequential compactness.

Obtained by some *classical* probabilistic arguments. [See e.g. H. Spohn (LSDIP), C. Kipnis & C. Landim (SLIPS), although for slightly different models].

Let $\Omega^{(N)}$ denote the generator of the underlying Markov process. Using the exponential form of $Z_t^{(N)}$ and a useful lemma in (SLIPS), one can easily check that the two following random processes

$$U_t^{(N)} \stackrel{\text{def}}{=} Z_t^{(N)} - Z_0^{(N)} - \int_0^t \left(\Omega^{(N)}[Z_s^{(N)}] + \theta_s^{(N)} Z_s^{(N)}\right) ds, \tag{7}$$

$$V_t^{(N)} \stackrel{\text{def}}{=} (U_t^{(N)})^2 - \int_0^t \left(\Omega^{(N)}[(Z_s^{(N)})^2] - 2Z_s^{(N)}\Omega^{(N)}[Z_s^{(N)}]\right) ds, \tag{8}$$

are bounded $\{\mathcal{F}_t^{(N)}\}$-martingales, with

$$\theta_t^{(N)} \stackrel{\text{def}}{=} \frac{1}{N} \sum_{i \in \mathbf{G}^{(N)}} \left[\frac{\partial \phi_a}{\partial t}\left(\frac{i}{N}, t\right) A_i^{(N)}(t) + \frac{\partial \phi_b}{\partial t}\left(\frac{i}{N}, t\right) B_i^{(N)}(t)\right]. \tag{9}$$

Sequential compactness (continuation)

Define the following quantities.

$$
\begin{aligned}
\psi_{xy} &\stackrel{\text{def}}{=} \phi_x - \phi_y = -\psi_{yx}, \\
\Delta\psi_{xy}\Big(\frac{i}{N}, t\Big) &\stackrel{\text{def}}{=} \psi_{xy}\Big(\frac{i+1}{N}, t\Big) - \psi_{xy}\Big(\frac{i}{N}, t\Big), \\
\widetilde{\lambda}^{(N)}_{xy}(i,t) &\stackrel{\text{def}}{=} \lambda_{xy}(N)\left[\exp\left(\frac{1}{N}\Delta\psi_{xy}\Big(\frac{i}{N}, t\Big)\right) - 1\right], \qquad xy = ab \text{ or } ba.
\end{aligned}
$$

Then

$$\Omega^{(N)}[Z^{(N)}_t] = L^{(N)}_t Z^{(N)}_t, \tag{10}$$

where

$$L^{(N)}_t = \sum_{i\in\mathbf{G}^{(N)}} \widetilde{\lambda}^{(N)}_{ab}(i,t) A_i B_{i+1} + \widetilde{\lambda}^{(N)}_{ba}(i,t) B_i A_{i+1}. \tag{11}$$

By using (11) and (10), it is straightforward to rewrite (8) in the form

$$V_t^{(N)} = (U_t^{(N)})^2 - \int_0^t (Z_s^{(N)})^2 R_s^{(N)} ds, \tag{12}$$

where the process $R_t^{(N)}$ is stricly positive and given by

$$R_t^{(N)} = \sum_{i \in \mathbf{G}^{(N)}} \frac{[\widetilde{\lambda}_{ab}^{(N)}(i,t)]^2}{\lambda_{ab}(N)} A_i B_{i+1} + \frac{[\widetilde{\lambda}_{ba}^{(N)}(i,t)]^2}{\lambda_{ba}(N)} B_i A_{i+1}.$$

The integral term in (12) is nothing else but the increasing process associated with Doob's decomposition of the submartingale $(U_t^{(N)})^2$.
The following (crucial) estimates hold.

$$\begin{cases} L_t^{(N)} = \mathcal{O}(1), \\ R_t^{(N)} = \mathcal{O}\left(\frac{1}{N}\right). \end{cases} \tag{13}$$

Sequential compactness (end)

Doob's inequality for sub-martingales yields

$$\mathsf{E}\left[(U_{t+\delta}^{(N)} - U_t^{(N)})^2\right] = \mathsf{E}\left[\int_t^{t+\delta} (Z_s^{(N)})^2 R_s^{(N)} ds\right] \le \frac{C\delta}{N},$$

$$P\left[\sup_{t\le T} |U_t^{(N)}| \ge \epsilon\right] \le \frac{4}{\epsilon^2}\mathsf{E}\left[\int_0^T (Z_s^{(N)})^2 R_s^{(N)} ds\right] \le \frac{4CT}{N\epsilon^2},$$

where C is a positive constant depending only on ϕ. Hence $U_t^{(N)} \to 0$ in probability as $N \to \infty$. Then, writing

$$Z_{t+\delta}^{(N)} - Z_t^{(N)} = U_{t+\delta}^{(N)} - U_t^{(N)} + \int_t^{t+\delta} (L_s^{(N)} + \theta_s^{(N)}) Z_s^{(N)} ds,$$

we can apply Aldous's criterion, which gives a sufficient condition for the tightness of the sequence $Z_t^{(N)} \in \mathcal{D}_{\mathcal{R}}[0, T] \ldots$

Form of the limit points

The sequence of probability measures $Q^{(N)}$, defined on $\mathcal{D}_{\mathcal{M}}[0,T]$ and corresponding to the process $\mu_t^{(N)}$, is also relatively compact [classical projection theorems for measure-valued processes (see e.g. Kallenberg)]. Let Q [resp. Z_t] the limit point of some arbitrary subsequence $Q^{(N_k)}$ [resp. $Z_t^{(N_k)}$] , as $N_k \to \infty$. The mapping $\mu_t \to \sup_{t \le T} \log Z_t$ is continuous and the support of Q is a set of sample paths absolutely continuous with respect to the Lebesgue measure. Indeed

$$\sup_{t \le T} \log Z_t \le \int_0^1 [|\phi_a(x,t)| + |\phi_b(x,t)|]dx,$$

for all $\psi_a, \psi_b \in \mathbf{C}^2[0,1]$. Hence, by weak convergence, any limit point Z_t has the form

$$Z_t[\phi_a, \phi_b] = \exp\Big[\int_0^1 [\rho(x,t)\phi_a(x,t) + (1 - \rho(x,t)\phi_b(x,t)]dx\Big]. \tag{14}$$

Toward a functional integral operator

Consider for a while the $2N$ quantities $\phi_a\left(\frac{i}{N}, t\right), \phi_b\left(\frac{i}{N}, t\right)$, $1 \leq i \leq N$, as *ordinary free variables*, denoted by $x_i^{(N)}$ and $y_i^{(N)}$ respectively. Set

$$\begin{aligned}
\alpha_{xy}^{(N)}(i,t) &\stackrel{\text{def}}{=} \lambda_{ab}(N)\left[\exp\left(\frac{x_{i+1}^{(N)} - x_i^{(N)} + y_i^{(N)} - y_{i+1}^{(N)}}{N}\right) - 1\right], \\
\alpha_{yx}^{(N)}(i,t) &\stackrel{\text{def}}{=} \lambda_{ba}(N)\left[\exp\left(\frac{y_{i+1}^{(N)} - y_i^{(N)} + x_i^{(N)} - x_{i+1}^{(N)}}{N}\right) - 1\right],
\end{aligned}$$

and let $\mathcal{L}_t^{(N)}$ be the operator

$$\mathcal{L}_t^{(N)}[h] \stackrel{\text{def}}{=} N^2 \sum_{i \in \mathbf{G}^{(N)}} \alpha_{xy}^{(N)}(i,t) \frac{\partial^2 h}{\partial x_i^{(N)} \partial y_{i+1}^{(N)}} + \alpha_{yx}^{(N)}(i,t) \frac{\partial^2 h}{\partial y_i^{(N)} \partial x_{i+1}^{(N)}}.$$

$\Leftrightarrow$ Then $Z_t^{(N)}$ (see 6) satisfies the functional partial derivative equation (FPDE)

$$\frac{d(Z_t^{(N)} - U_t^{(N)})}{dt} = \mathcal{L}_t^{(N)}[Z_t^{(N)}] + \theta_t^{(N)} Z_t^{(N)}, \tag{15}$$

where θ_t was defined in (9). In fact, a brute force analysis of (15) would lead to a dead end, and one must use the estimates (13).

Lemma. *The following FPDE holds.*

$$\frac{d(Z_t^{(N)} - U_t^{(N)})}{dt} \stackrel{\mathit{def}}{=} \mathcal{A}_t^{(N)}[Z_t^{(N)}] + \theta_t^{(N)} Z_t^{(N)} + \mathcal{O}\Big(\frac{1}{N}\Big),$$

where $\mathcal{A}_t^{(N)}$ is viewed as an operator with domain $\in \mathcal{C}_0^\infty(\mathcal{V}^{(N)})$ such that

$$\begin{aligned}\mathcal{A}_t^{(N)}[g] &\stackrel{\mathit{def}}{=} \sum_{i\in\mathbf{G}^{(N)}} \mu\psi'_{ab}\Big(\frac{i}{N}, t\Big) \left[\frac{1}{2}\left(\frac{\partial g}{\partial x_i^{(N)}} + \frac{\partial g}{\partial x_{i+1}^{(N)}}\right) - N\frac{\partial^2 g}{\partial x_i^{(N)}\partial x_{i+1}^{(N)}}\right] \\ &+ \lambda \sum_{i\in\mathbf{G}^{(N)}} \psi''_{ab}\Big(\frac{i}{N}, t\Big)\frac{\partial g}{\partial x_{i+1}^{(N)}}.\end{aligned} \tag{16}$$

Analysis of $\mathcal{A}_t^{(N)}[g]$

- Skohorod's coupling. This allows an interim reduction to an almost sure convergence setting on a new probability space, where $Z_t^{(N)}$ is rewritten as $Y_t^{(N)}$, so that

$$\frac{d(Y_t^{(N)} - U_t^{(N)})}{dt} \stackrel{\text{def}}{=} \mathcal{A}_t^{(N)}[Y_t^{(N)}] + \theta_t^{(N)} Y_t^{(N)} + \mathcal{O}\Big(\frac{1}{N}\Big), \tag{17}$$

and

$$\lim_{k\to\infty} Y_t^{(N_k)}[\phi_a, \phi_b] \stackrel{a.s.}{\to} Y_t[\phi_a, \phi_b].$$

- Reduction, for each finite N, to a partial differential operator with constant coefficients. For each finite N, the quantities

$$\psi'_{ab}\Big(\frac{i}{N}, t\Big),\ \psi''_{ab}\Big(\frac{i}{N}, t\Big),\ i = 1, \ldots, N,$$

can be viewed as *constant parameters*, while the $x_i^{(N)}$'s are the free variables from a variational calculus point of view.

- Regularization and Functional Integration

Let $\vec{x}^{(N)} \stackrel{\text{def}}{=} (x_1^{(N)}, x_2^{(N)}, \ldots, x_N^{(N)})$, with $x_i^{(N)} = \phi_a\left(\frac{i}{N}, t\right), 1 \leq i \leq N$. Introduce the following family of positive test functions

$$\chi_\varepsilon^{(N)}(\vec{x}^{(N)}) = \omega\left(\frac{\sum_{i=1}^{N}(x_i^{(N)})^2}{N} - \varepsilon^2\right), \quad \varepsilon \geq 0, \tag{18}$$

where $\omega \in \mathcal{C}_0^\infty(\mathcal{R})$ is a function of the real variable z defined by

$$\omega(z) \stackrel{\text{def}}{=} \begin{cases} \exp\left(\frac{1}{z}\right) \text{ if } z < 0, \\ 0 \text{ if } z \geq 0. \end{cases}$$

For each $\phi \in \widetilde{\mathcal{C}[T]}$, we have

$$\chi_\varepsilon([\phi]) \stackrel{\text{def}}{=} \lim_{N\to\infty} \chi_\varepsilon^{(N)}(\vec{x}^{(N)}) = \omega\left(\int_0^1 \phi^2(x,t)dx - \varepsilon^2\right).$$

The next step is to proceed by regularization from the basic equation

$$\left(\frac{d(Y_t^{(N)} - U_t^{(N)})}{dt} \star \chi_\varepsilon^{(N)}\right)(\vec{x}^{(N)}) = \left(\widetilde{\mathcal{A}}_t^{(N)}[\chi_\varepsilon^{(N)}] \star Y_t^{(N)}\right)(\vec{x}^{(N)}) + \left((\theta_t^{(N)} Y_t^{(N)}) \star \chi_\varepsilon^{(N)}\right)(\vec{x}^{(N)}) + \mathcal{O}\left(\frac{1}{N}\right), \tag{19}$$

where $\widetilde{\mathcal{A}}_t^{(N)}$ is the *adjoint operator* of $\mathcal{A}^{(N)}$ defined, for any h of the form [see equation (6)]

$$h = \exp\left[\int_0^1 d\sigma_a^{(N)}(x) U[\phi_a(x,t)] + d\sigma_b^{(N)}(x) U[\phi_b(x,t)]\right],$$

by the formula

$$
\begin{aligned}
\widetilde{\mathcal{A}}_t^{(N)}[h] = &- \sum_{i \in \mathbf{G}^{(N)}} \mu \psi'_{ab}\Big(\frac{i}{N}, t\Big) \left[\frac{1}{2} \Bigg(\frac{\partial h}{\partial x_i^{(N)}} + \frac{\partial h}{\partial x_{i+1}^{(N)}} \Bigg) + N \frac{\partial^2 h}{\partial x_i^{(N)} \partial x_{i+1}^{(N)}} \right] \\
&- \lambda \sum_{i \in \mathbf{G}^{(N)}} \psi''_{ab}\Big(\frac{i}{N}, t\Big) \frac{\partial h}{\partial x_{i+1}^{(N)}},
\end{aligned}
$$

noting that

$$
\left(\mathcal{A}_t^{(N)}[Y_t^{(N)}] \star \chi_\varepsilon^{(N)} \right) (\vec{x}^{(N)}) = \left(\widetilde{\mathcal{A}}_t^{(N)}[\chi_\varepsilon^{(N)}] \star Y_t^{(N)} \right) (\vec{x}^{(N)}).
$$

Then, for each $\phi(x,t) \in \widetilde{\mathcal{C}[T]}$, the following limit holds uniformly.

$$\lim_{N\to\infty} \widetilde{\mathcal{A}}_t^{(N)}[\chi_\varepsilon^{(N)}](\vec{x}^{(N)}) = -\int_0^1 \left[\mu\psi'_{ab}(z,t)K([\phi],z) + \lambda\psi''_{ab}(z,t)H([\phi],z)\right]dz,$$

where

$$\begin{cases} H([\phi],z) = 2\phi(z,t)\omega'\left(\displaystyle\int_0^1 \phi^2(u,t)du - \varepsilon^2\right), \\ K([\phi],z) = H([\phi],z) + 4\phi^2(z,t)\omega''\left(\displaystyle\int_0^1 \phi^2(u,t)du - \varepsilon^2\right). \end{cases}$$

$\Leftrightarrow$ Variational derivatives appear, as expected. . .

The last *agendum* of the method is to give a rigourous meaning to integrals of the form

$$\lim_{N\to\infty} \int_{\mathcal{V}^{(N)}} f^{(N)}(\vec{x}^{(N)}) d\vec{x}^{(N)},$$

in order to to carry out *functional integration by parts* and *variational differentiation*. This can be done in several ways via *promeasures, prodistributions, integrators* *Non sumus in a Terra Incognita. . .*

- F. Riesz (1909) *Representation theorem*.
- P. J. Daniell (1919), A. N. Kolmogorov (1933).
- Y. V. Prokhorov (1956).
- N. Bourbaki ($\approx$ 1960-1969) Promeasures
- P. Cartier & C. De Witt-Morette (*Functional Integration*, 2006).

Hydrodynamic limit for multitype ASEP

Start again with the transform

$$Z_t^{(N)}[\phi] \stackrel{\text{def}}{=} \exp\left[\frac{1}{N}\sum_{a=1}^{n}\sum_{i\in\mathbf{G}^{(N)}}\phi_a\left(\frac{i}{N},t\right)A_i\right].$$

Then, under the scaling

$$\lambda_{ab}(N) = N^2\lambda + \frac{\mu_{ab}}{2}N, \quad \forall a,b \in \{1,\ldots,n\},$$

we have

$$\frac{d(Z_t^{(N)} - U_t^{(N)})}{dt} = \mathcal{A}_t^{(N)}[Z_t^{(N)}] + \theta_t^{(N)} Z_t^{(N)} + \mathcal{O}\left(\frac{1}{N}\right),$$

where $\mathcal{A}_t^{(N)}$ stands for the operator

$$\mathcal{A}_t^{(N)}[h] = \sum_{\substack{a,b=1 \\ a \neq b}}^{n} \sum_{i \in \mathbf{G}^{(N)}} \Big[\lambda \phi_a''(\frac{i}{N}, t) \frac{\partial h}{\partial a_i^{(N)}} + N \mu_{ab} \phi_a'(\frac{i}{N}, t) \Big(\frac{\partial^2 h}{\partial a_i^{(N)} \partial b_{i+1}^{(N)}} + \frac{\partial^2 h}{\partial a_{i+1}^{(N)} \partial b_i^{(N)}} \Big) \Big],$$

with the notation $a_i^{(N)} \overset{\text{def}}{=} \phi_a(\frac{i}{N}, t)$. The same tools can be applied. . .

Final remark: a functional of the form

$$Y_t^{(N)}[\phi] \overset{\text{def}}{=} \exp \left[\frac{1}{N} \sum_{a,b=1}^{n} \sum_{i=1}^{N} \phi_{ab}(\frac{i}{N}, t) A_i B_{i+1} \right]$$

can be used to analyze the *arrangements* of interfaces between particle domains (local correlations). However, the limit process of these interfaces evolves at a shorter time-scale [indeed by a factor N] than the one leading to particle density.

Apologies

The speaker would like to emphasize that this talk does not make use of expressions like *Generating Function* or *Functional Equation*, that Philippe Flajolet holds so dear ! For sure, Philippe's broad-mindedness will excuse these unfortunate oversights . . .