Random generation of combinatorial structures

Uniform random maps and graphs on surfaces using Boltzmann sampling

Gilles Schaeffer

CNRS / Ecole Polytechnique,
Palaiseau, France

Uniform random generation?

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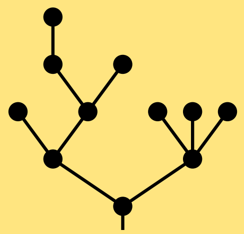
Ex: ordered trees (n edges)



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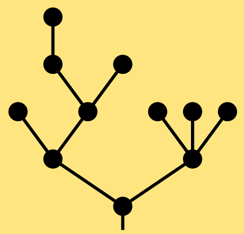

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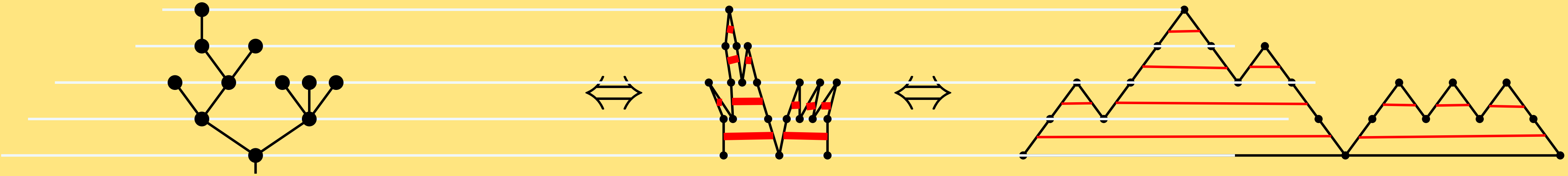

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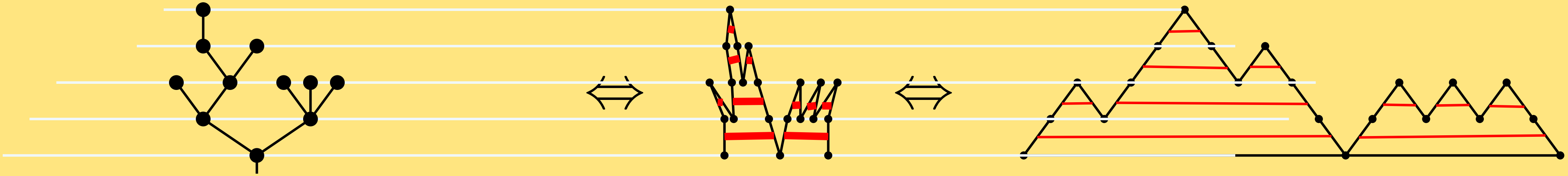

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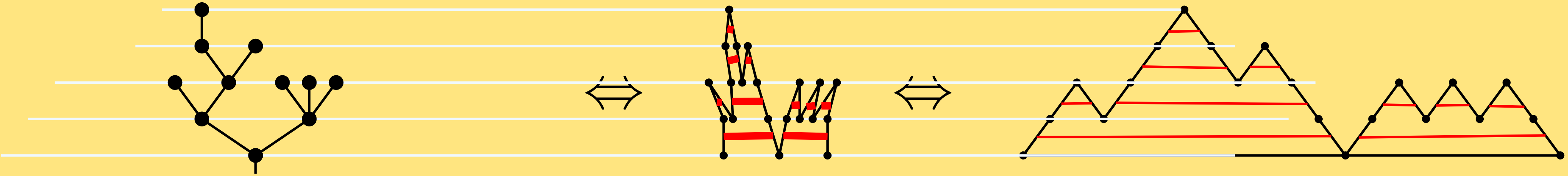

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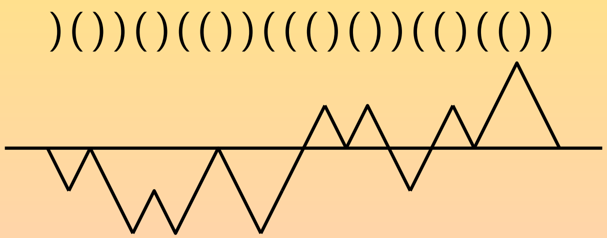

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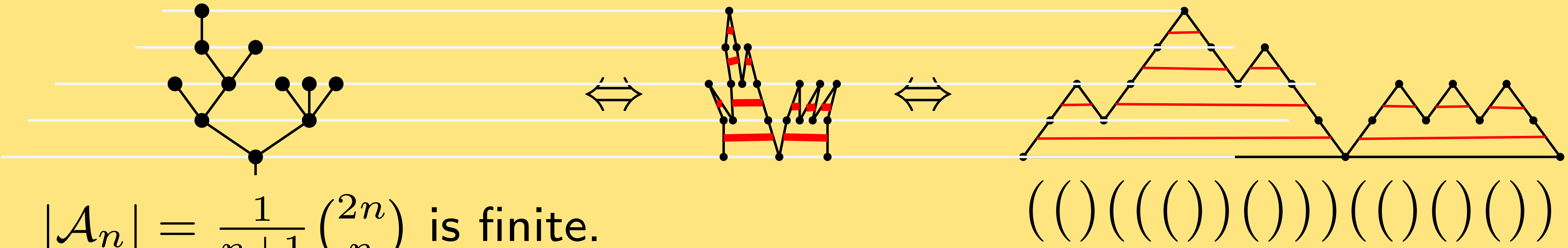

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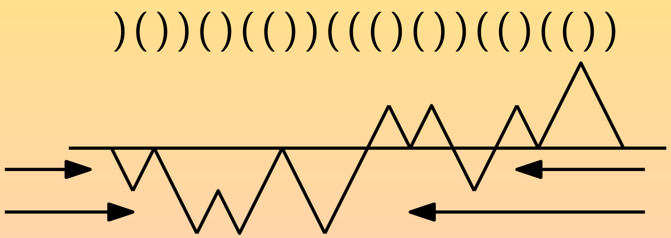

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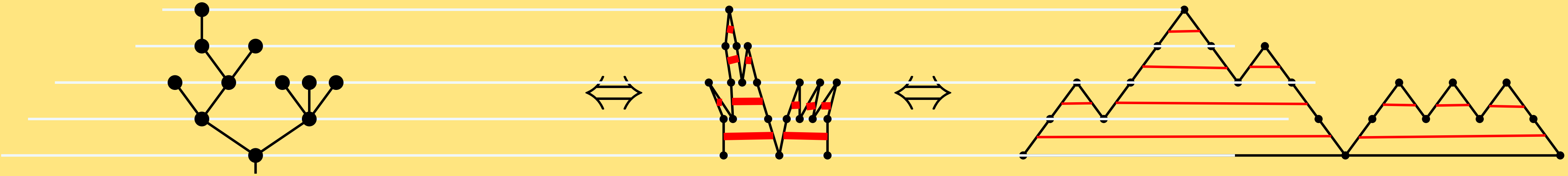

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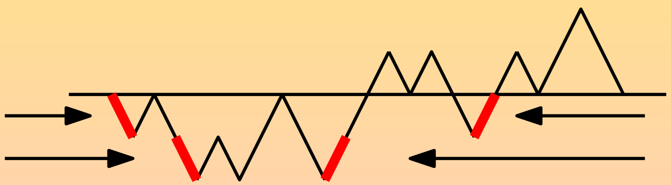

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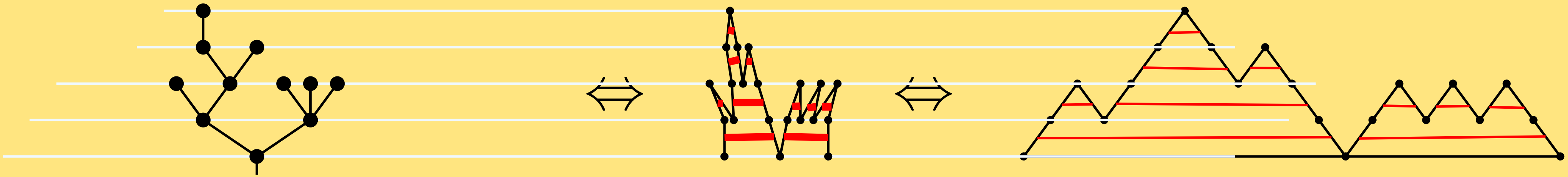

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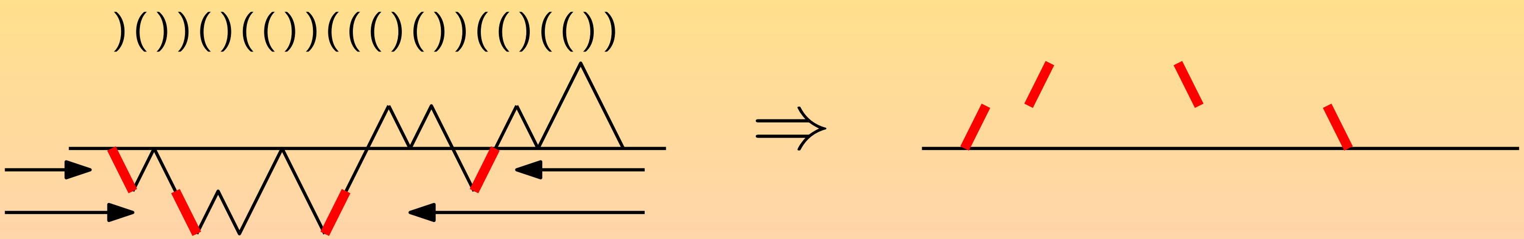

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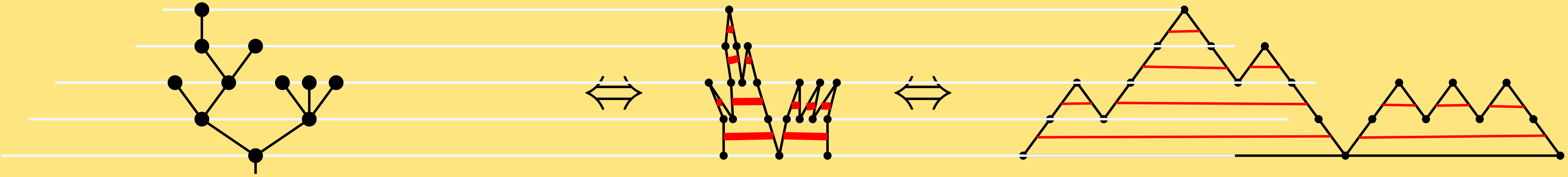

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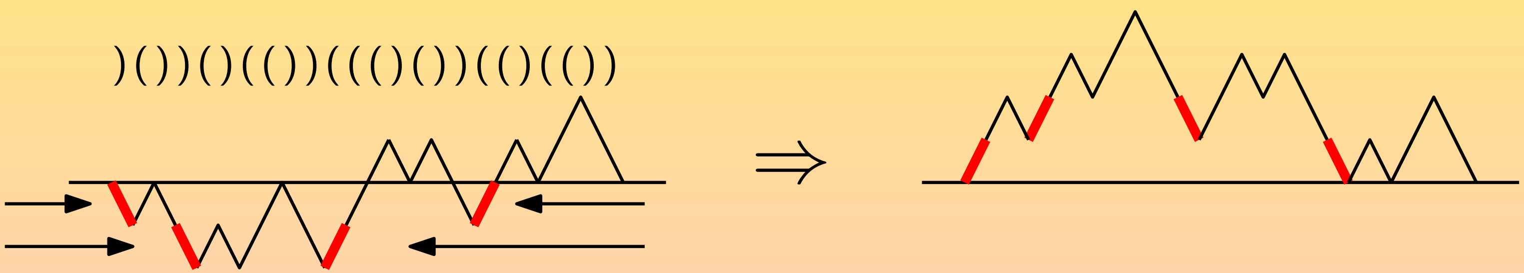

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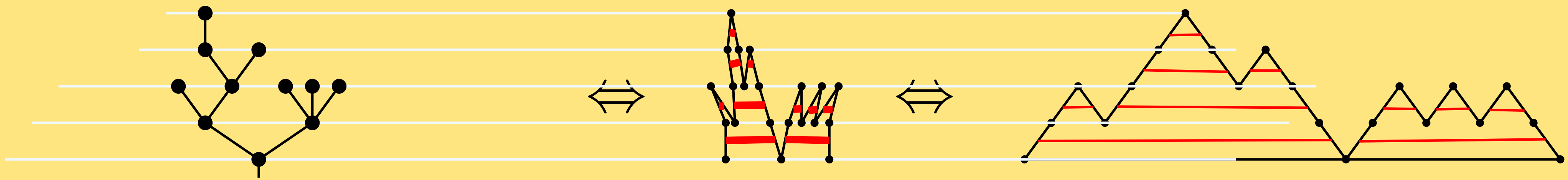

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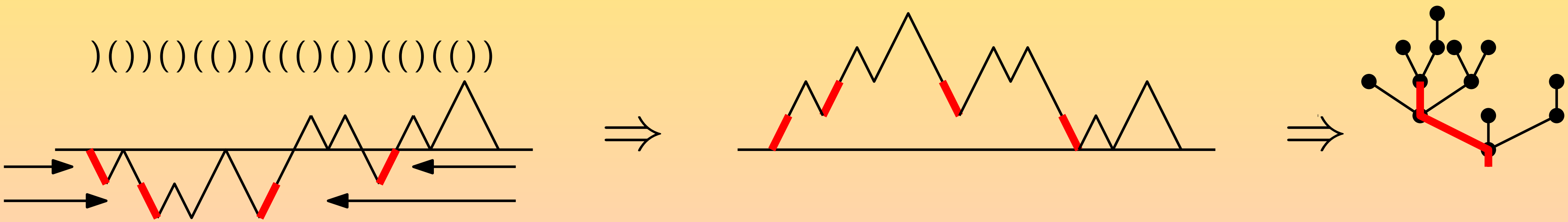

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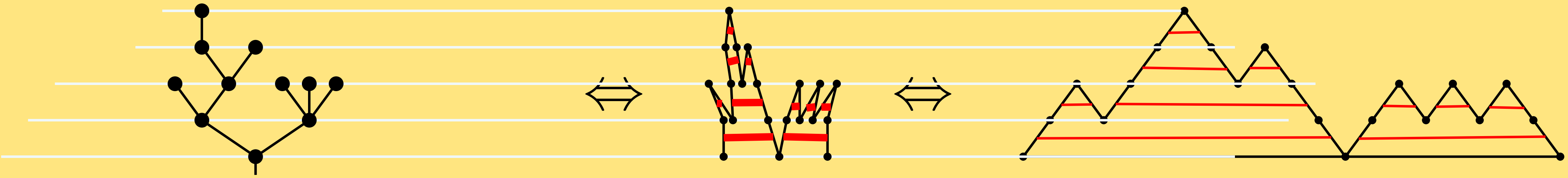

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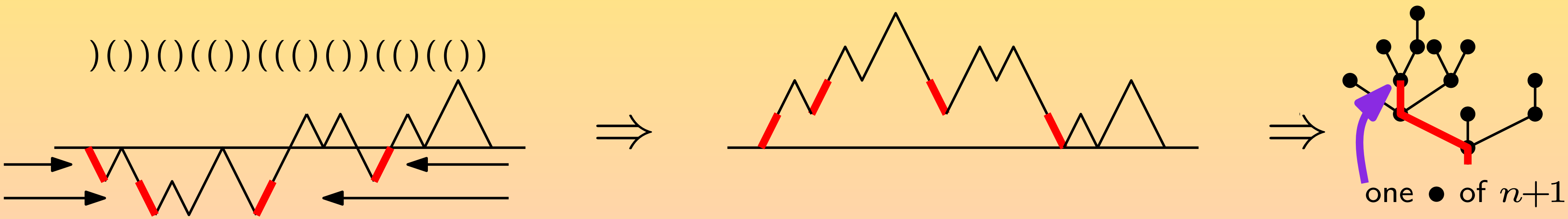

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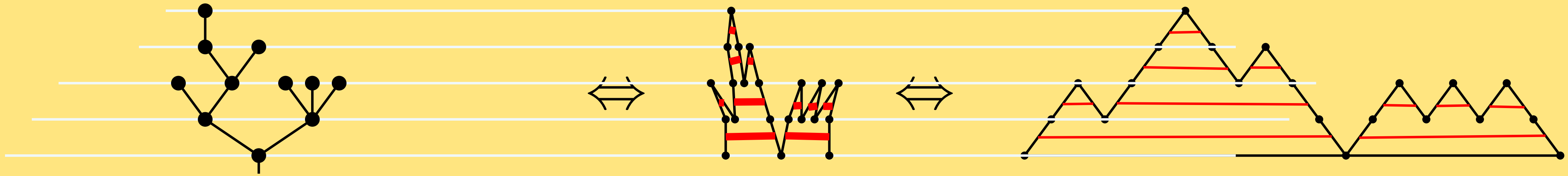

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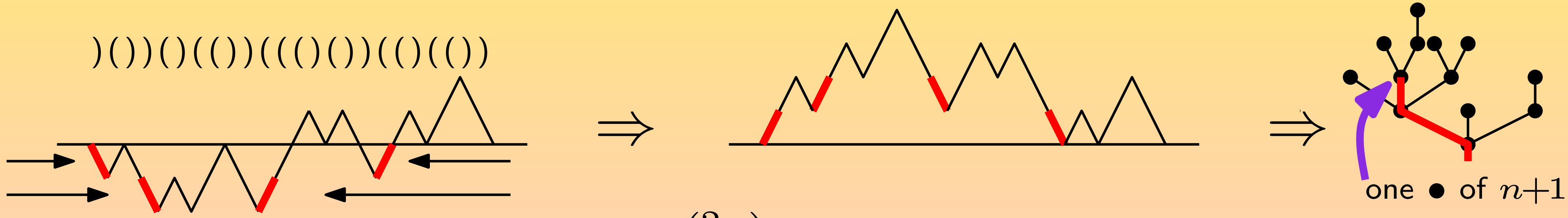

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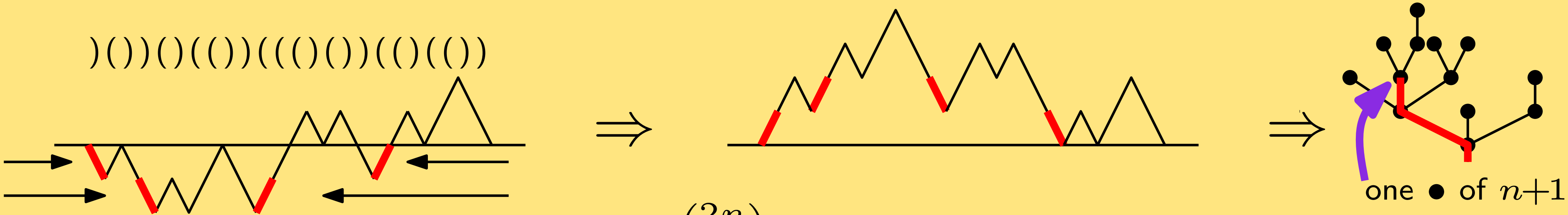

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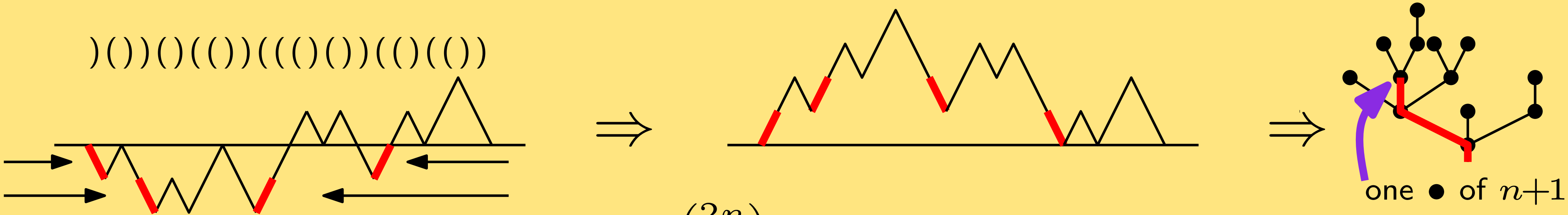

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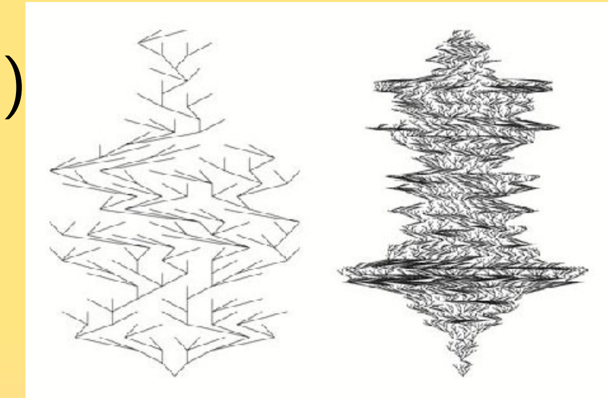

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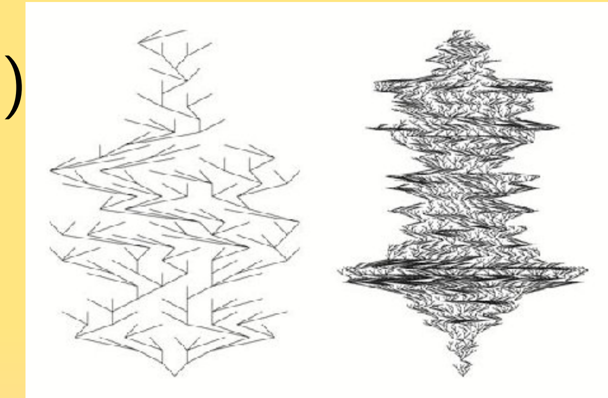

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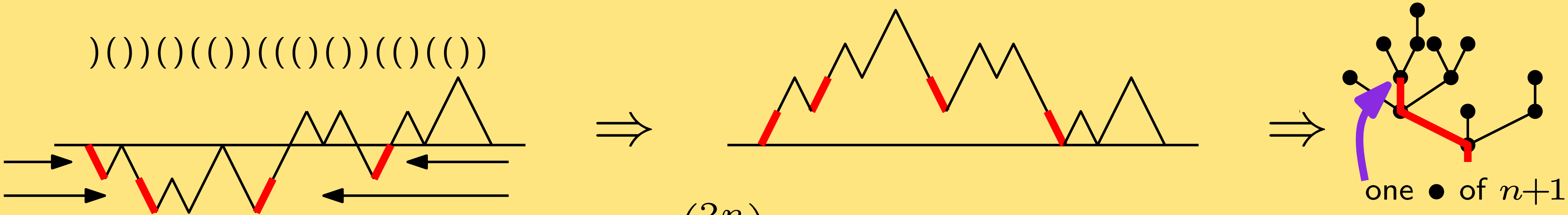

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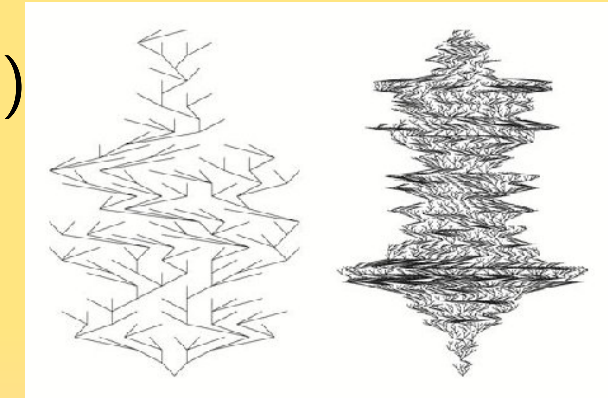

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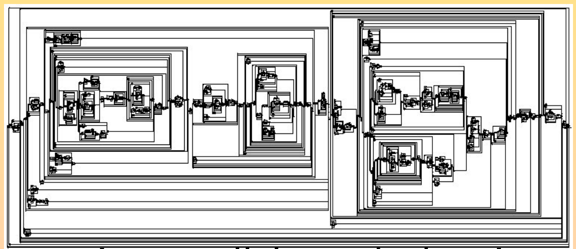

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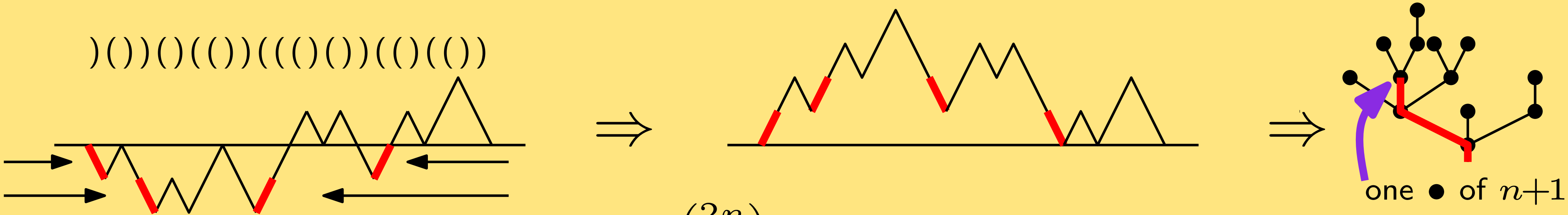

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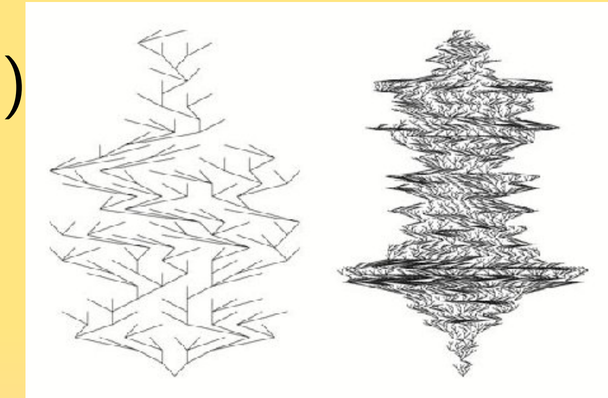

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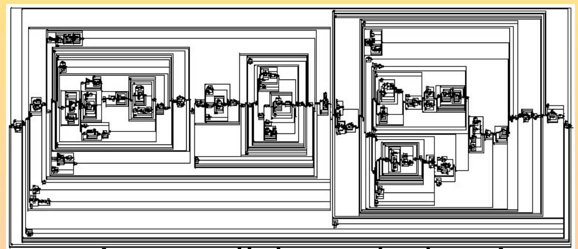

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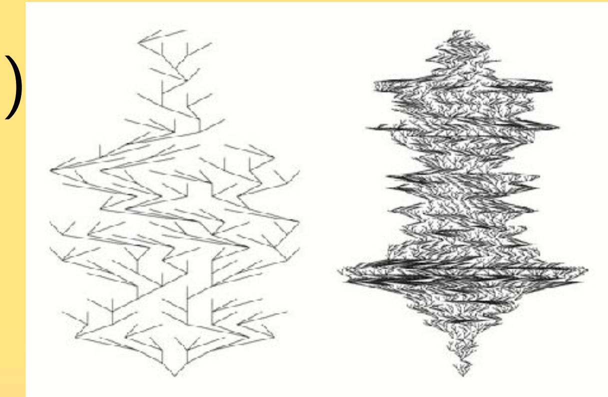

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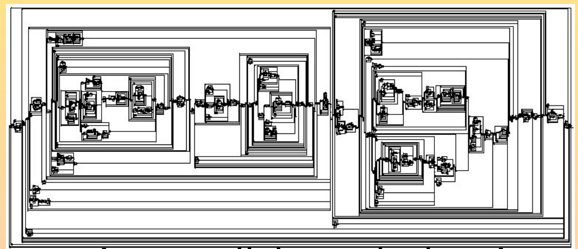

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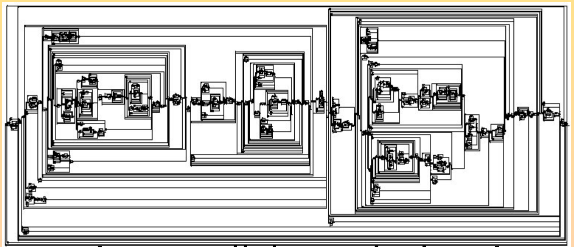

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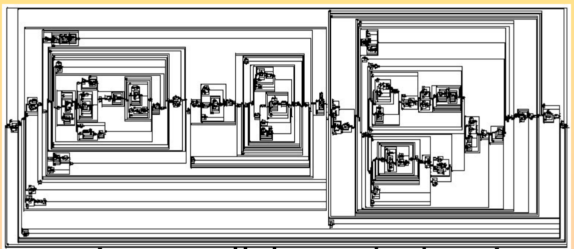

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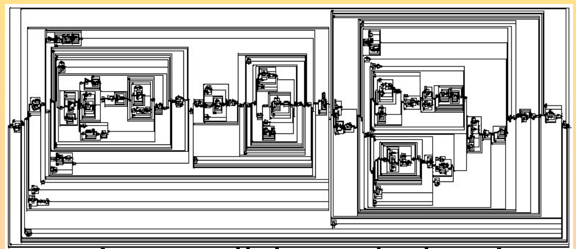

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Some possible tradeoffs: Imperfect sampling (distribution $\rightarrow$ uniform)

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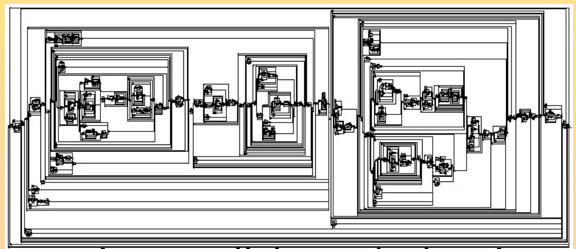

series parallel graph drawing

Pictures are courtesy of Carine Pivoteau and Philippe Flajolet

Uniform random generation?

Issues: complexity, genericity

– Bijective sampling requires efforts and luck... such results are rare

– but almost anything you can count by recurrence you can generate by recursive sampling: automatic for decomposable/constructive structures

systematized in *Flajolet, Zimmermann, Van Cutsem, (1994)*; limited to sizes of order 10^4

Some possible tradeoffs: Imperfect sampling (distribution $\rightarrow$ uniform)

– floating points in recursive sampling analyzed by *Denise, Zimmermann (1997)*

In general sampling aims at "in silico" experiments:

- – Average case complexity of algorithms
- " " quality " "
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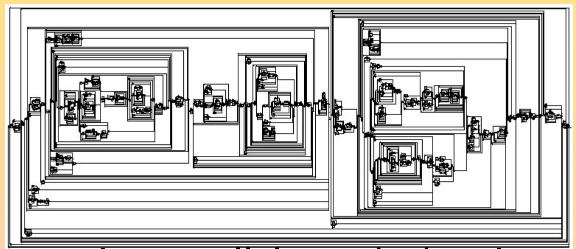

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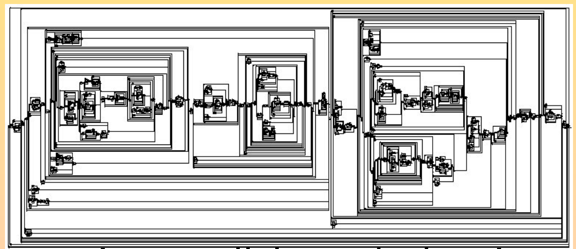

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Relax the exact size requirement: Boltzmann sampling (see later)

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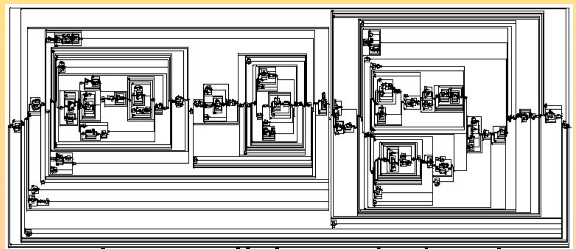

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My favorite random guys: maps and graphs

A planar graph G: there exists an embedding of G in the plane

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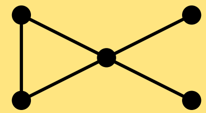

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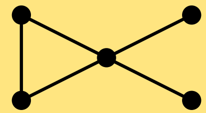

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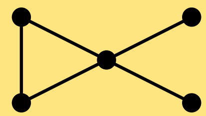

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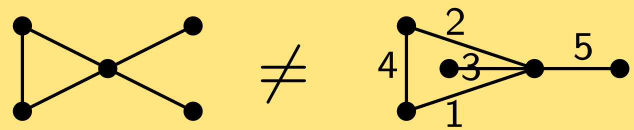

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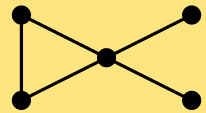

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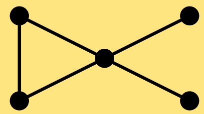

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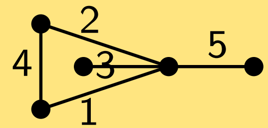

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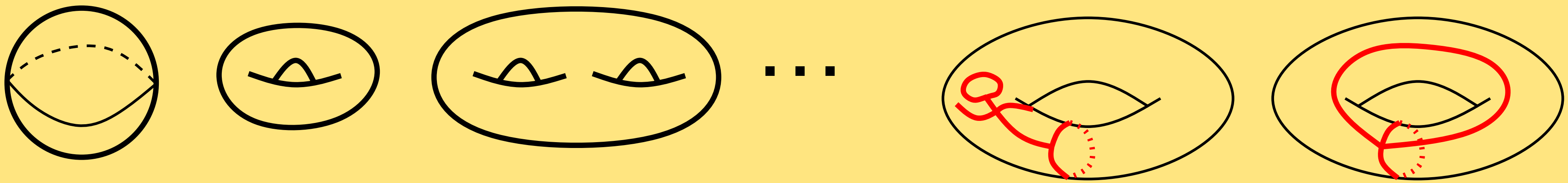
Surfaces: let $\mathcal{S}_g$ be the compact orientable surface of genus g.
$\mathcal{S}_0$ is the sphere, $\mathcal{S}_1$ the torus; in general $\mathcal{S}_g$ is a "sphere" with g handles.



A graph G of genus $\leq g$: there exists a proper embedding of G in $\mathcal{S}_g$.

A map of genus g: combinatorial description of a proper embedding in $\mathcal{S}_g$.

Proper = Faces must be topological disks: no handle inside a face.

Euler's formula reads $v + f = e + 2 - 2g$.

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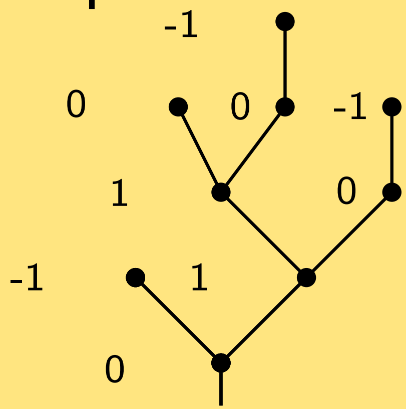

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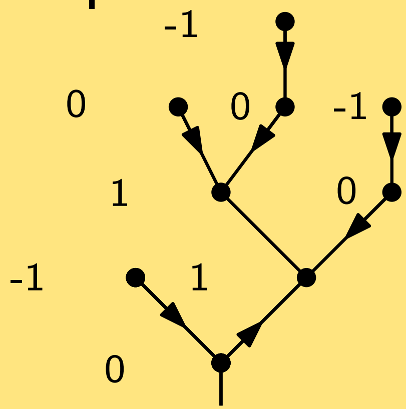

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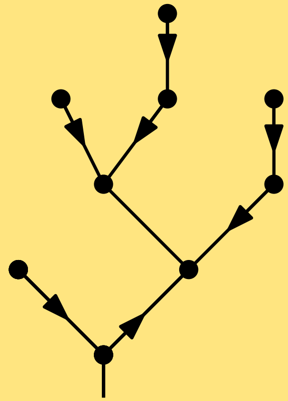

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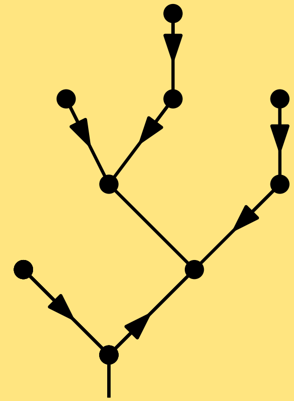

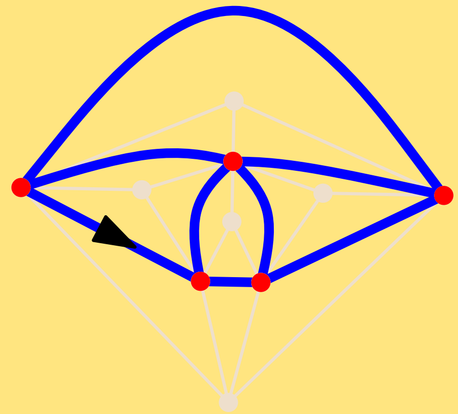

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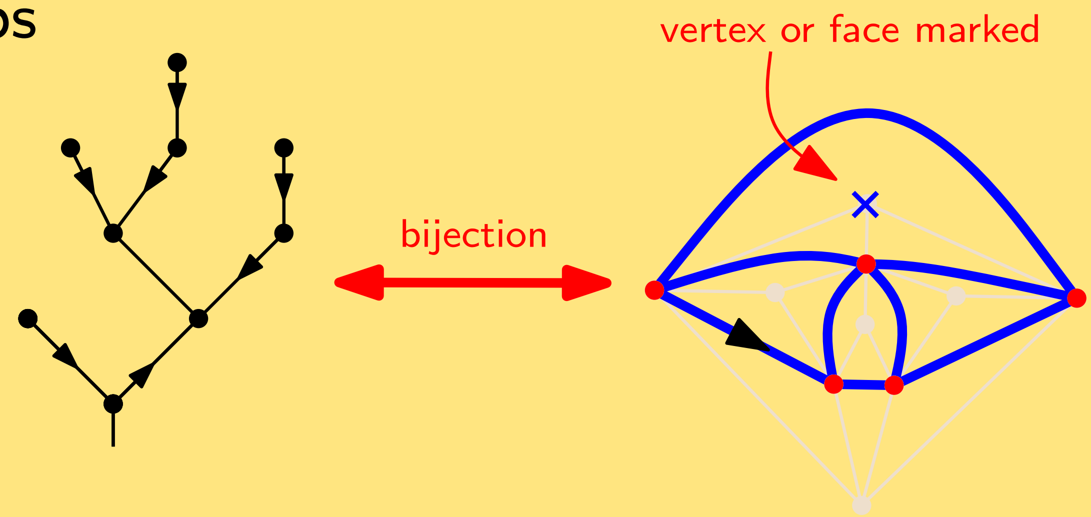

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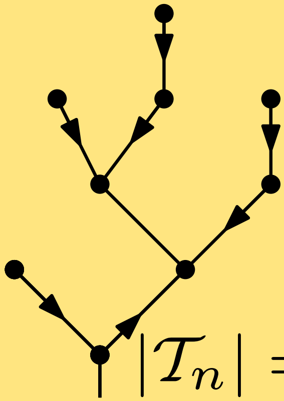

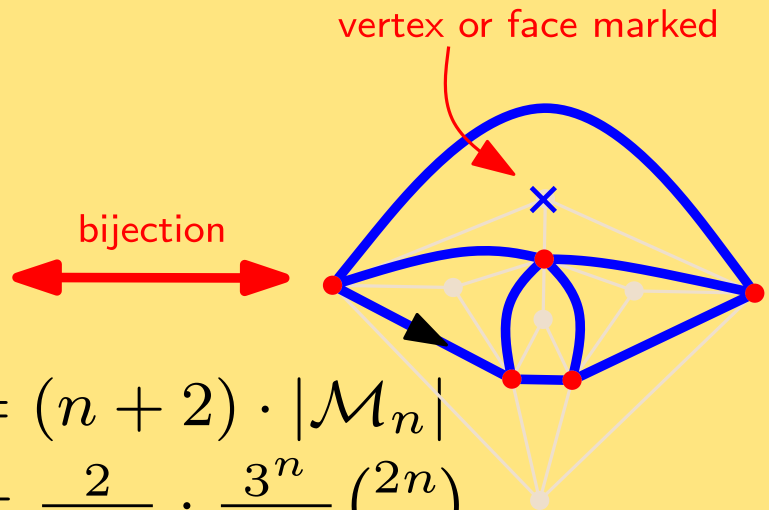

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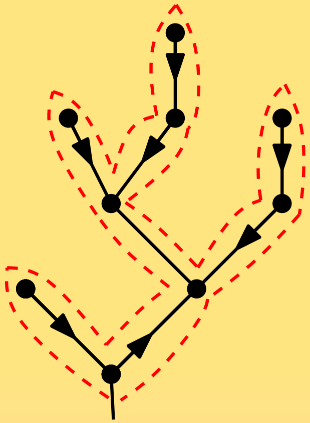

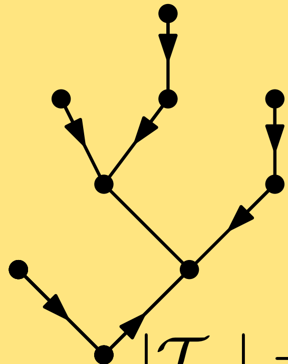

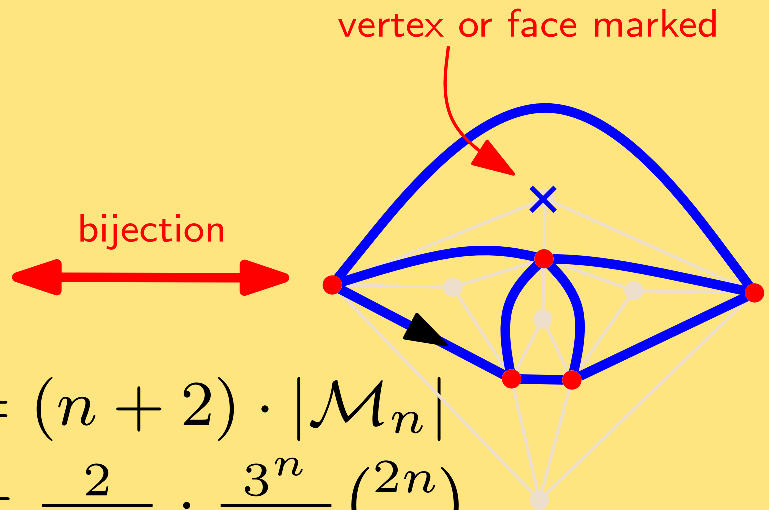

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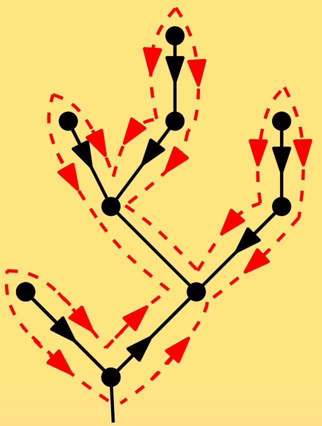

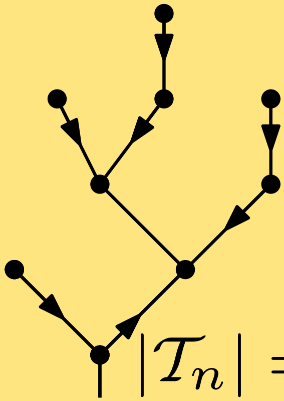

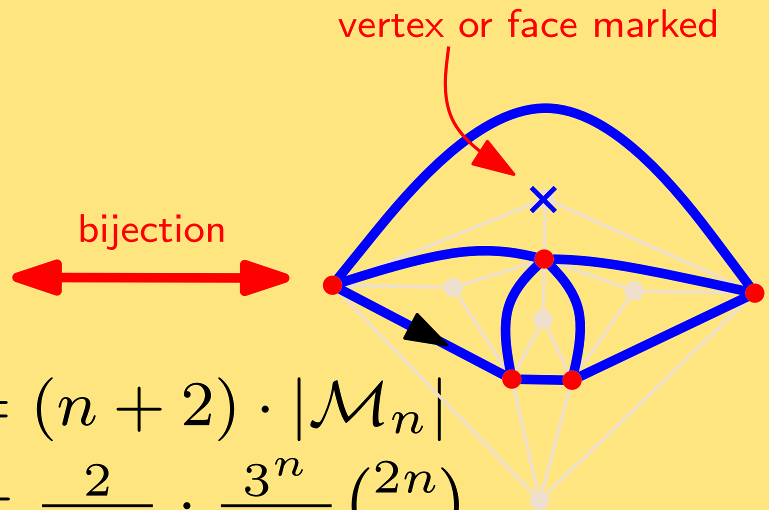

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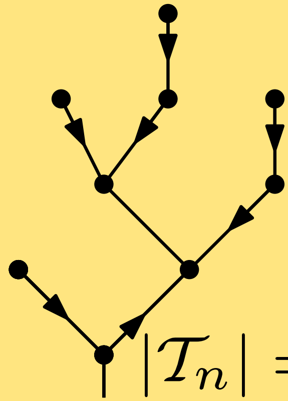

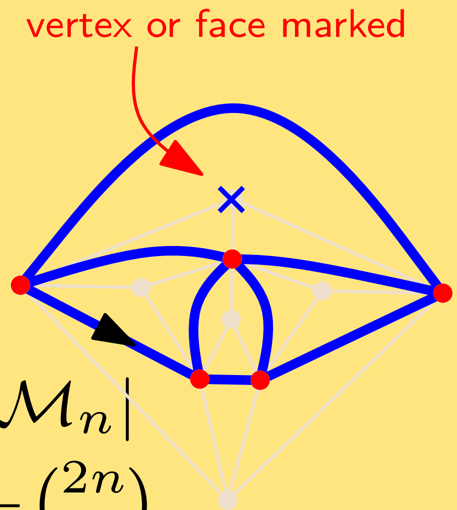

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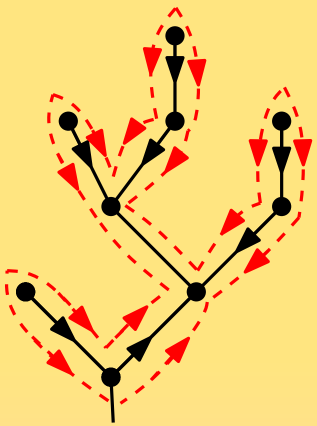

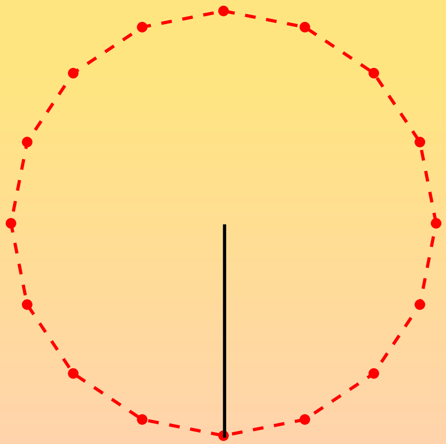

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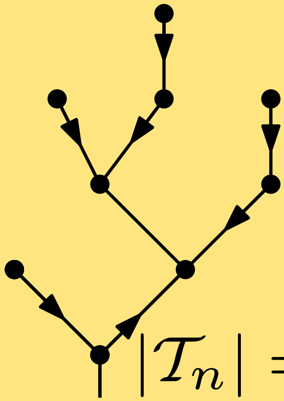

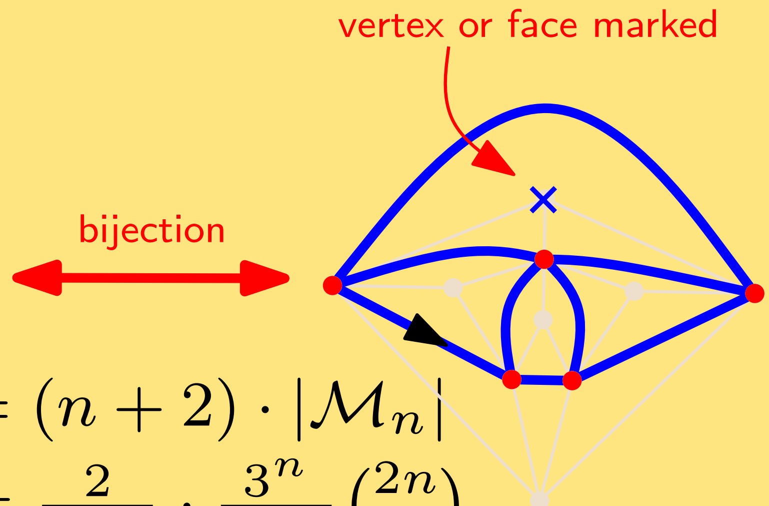

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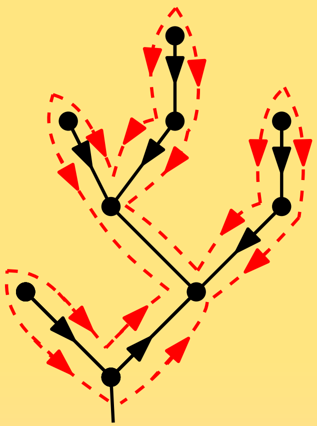

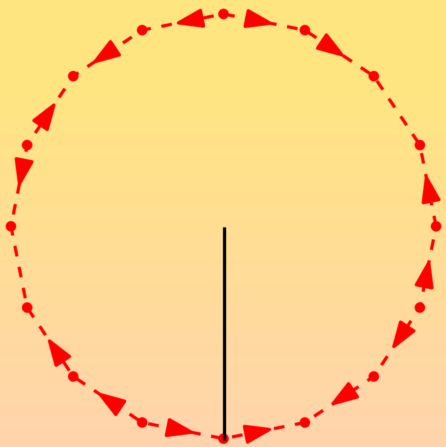

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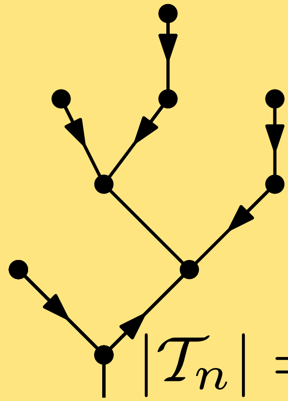

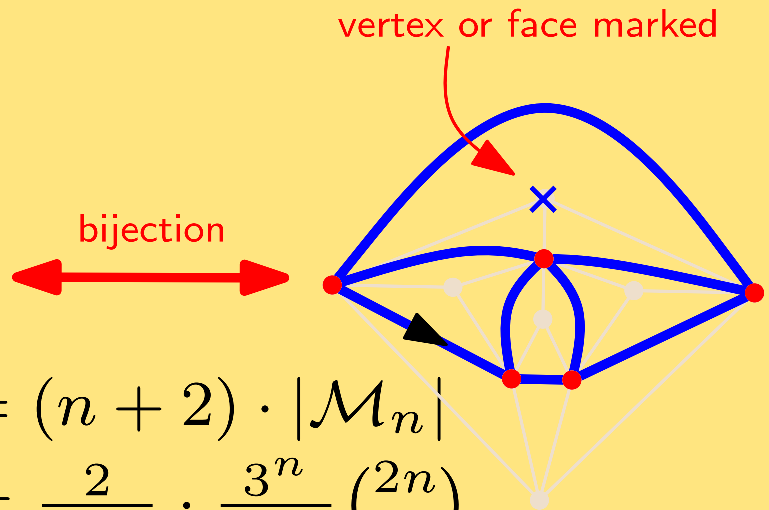

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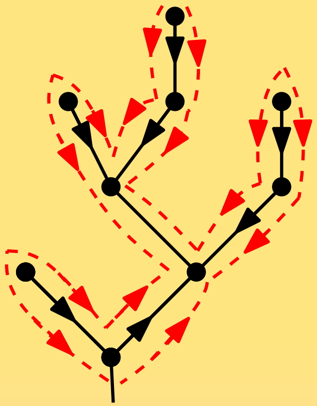

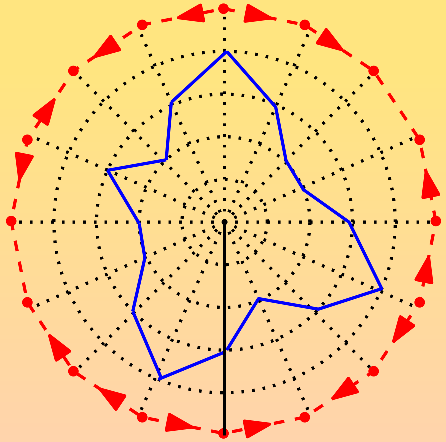

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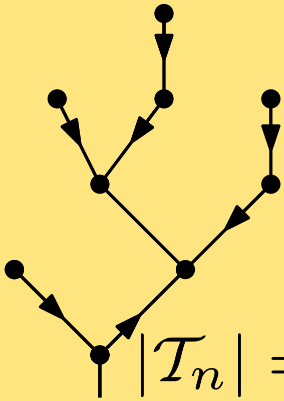

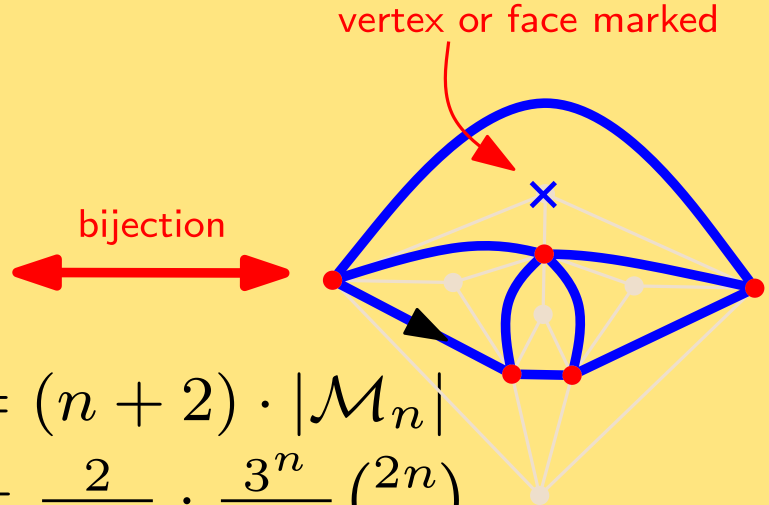

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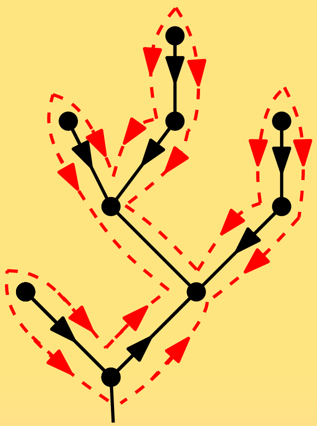

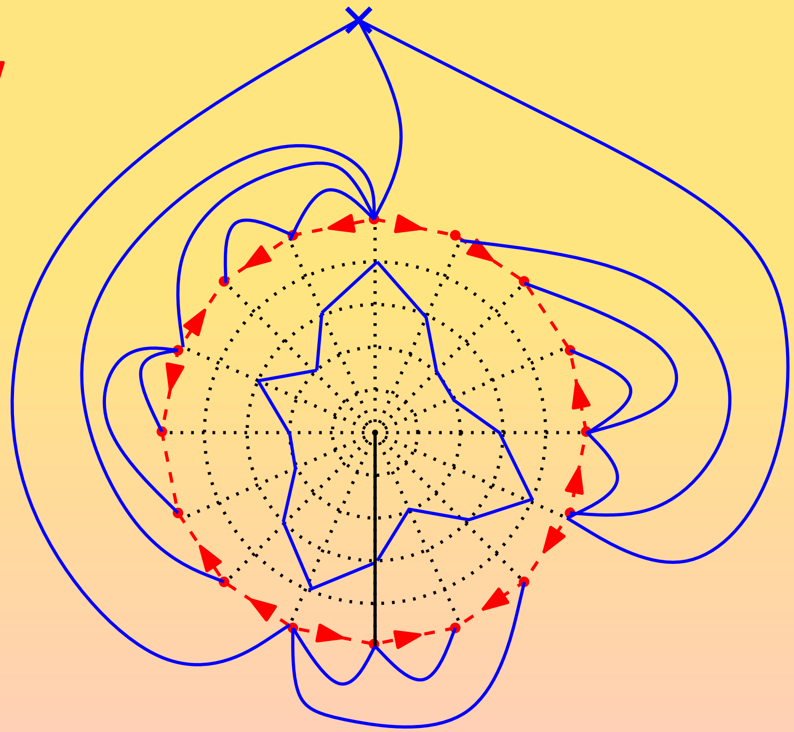

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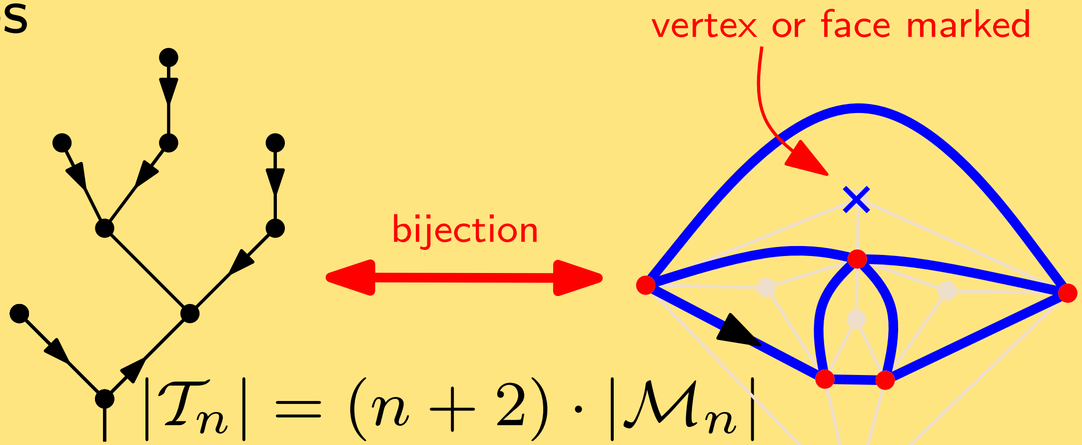

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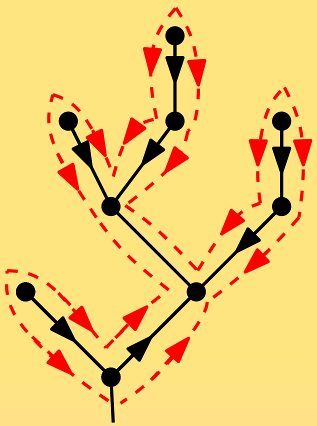

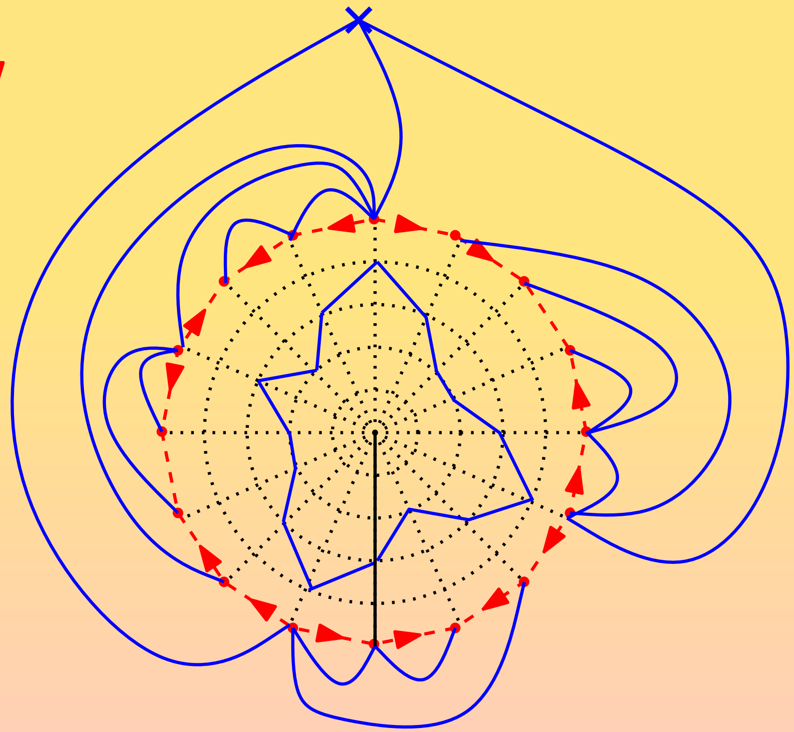

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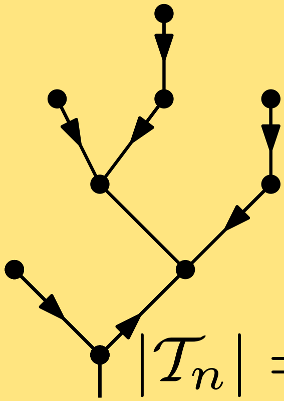

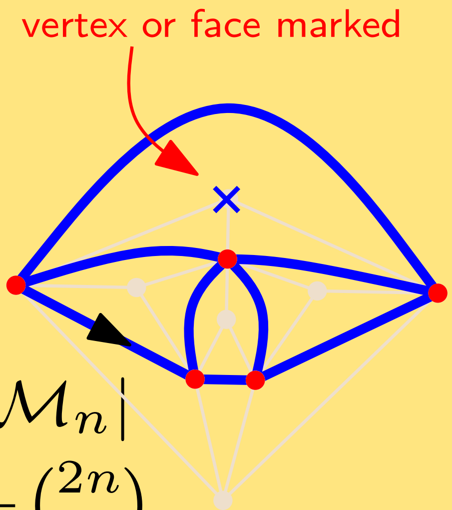

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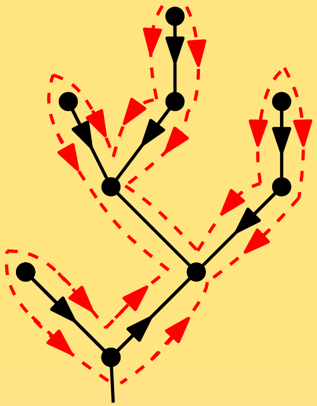

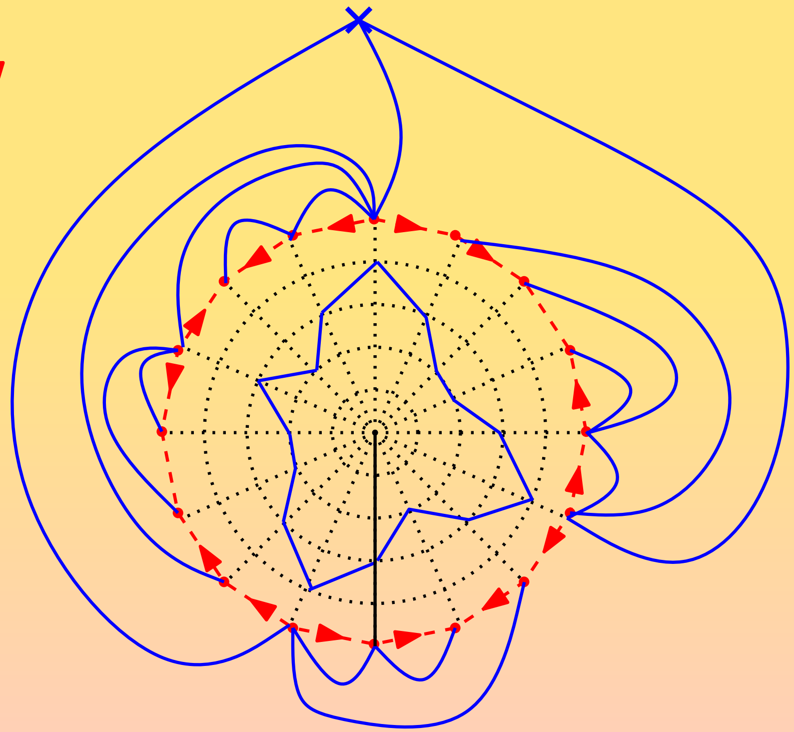

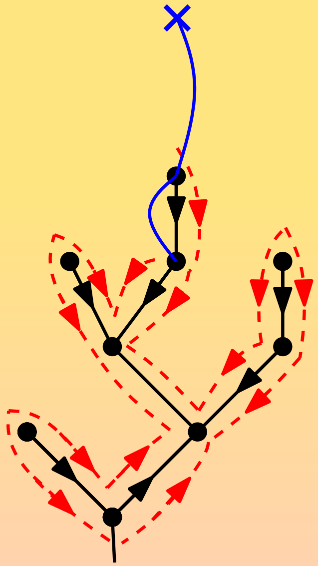

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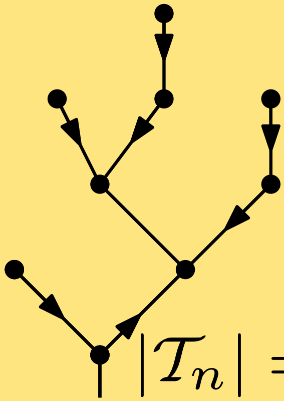

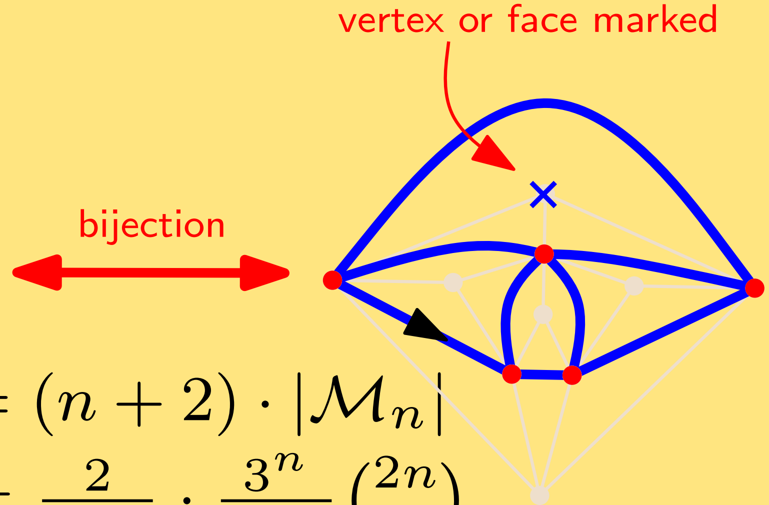

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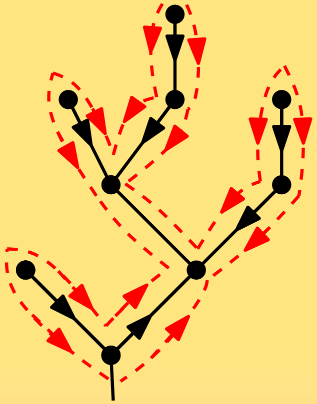

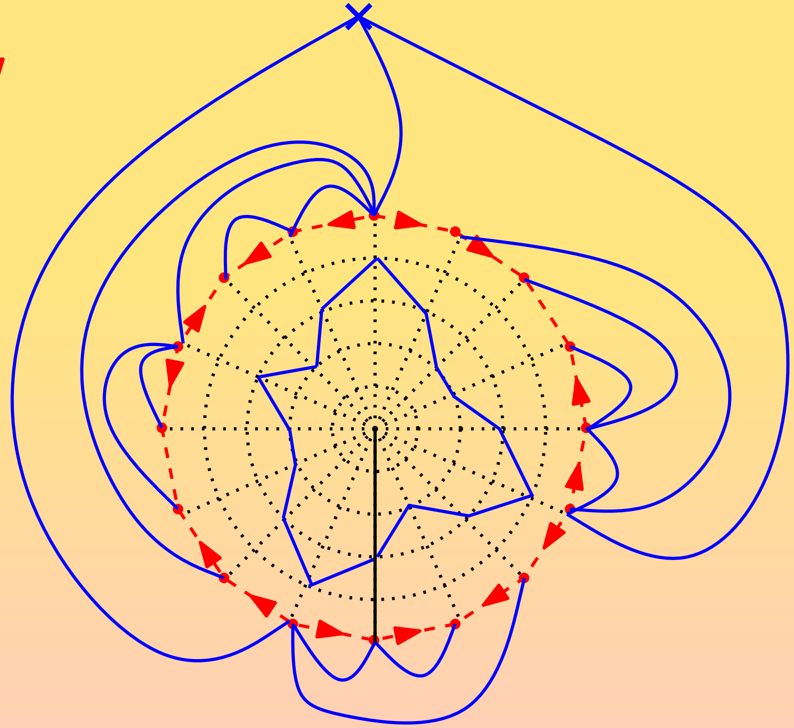

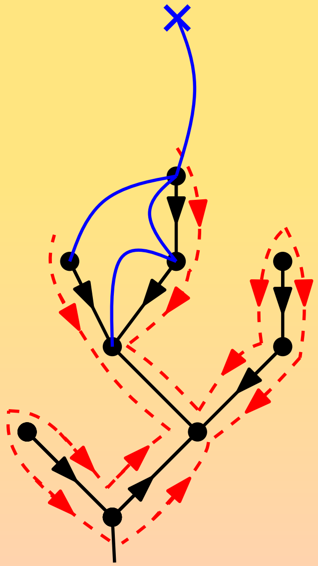

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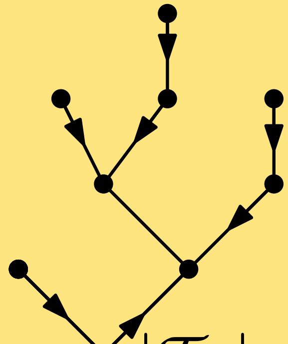

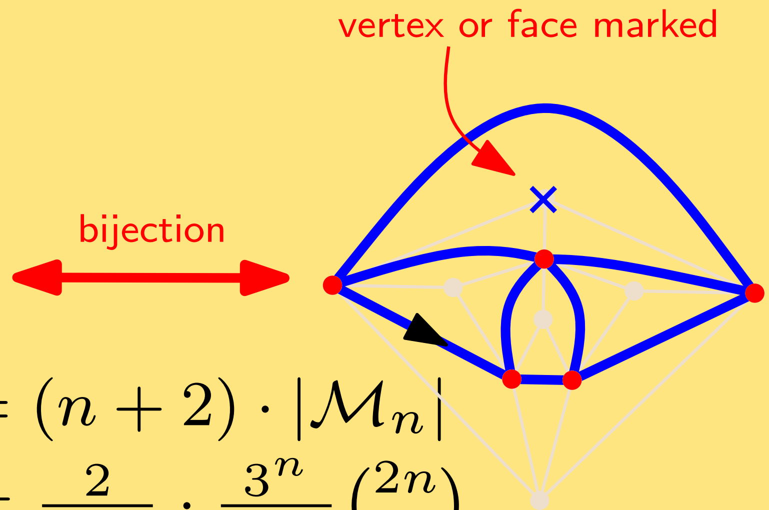

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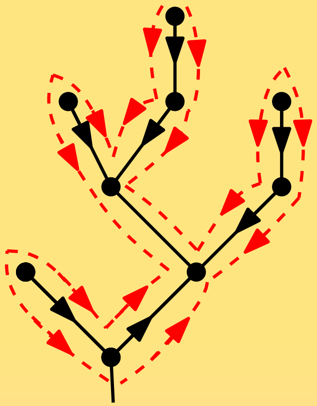

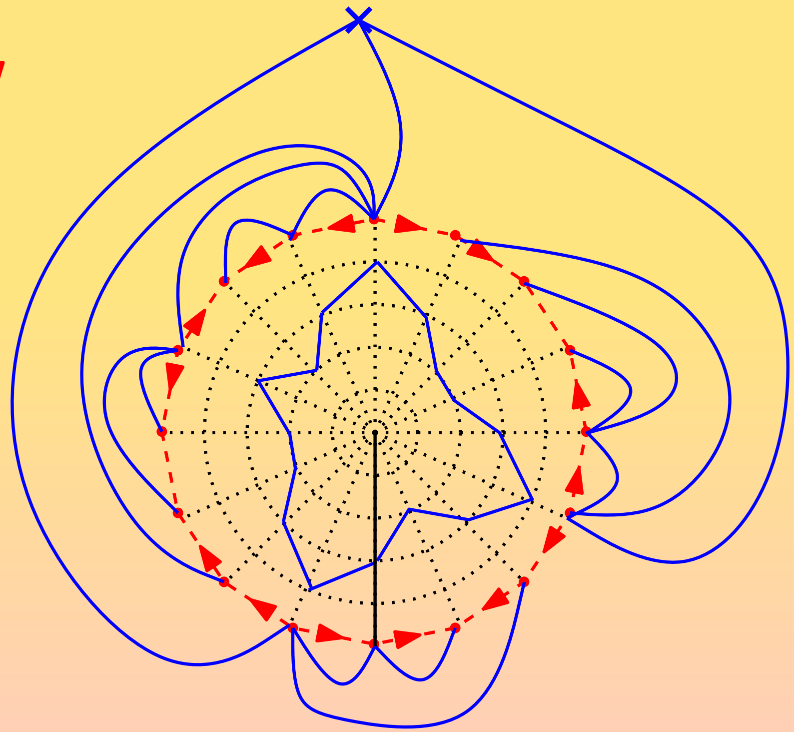

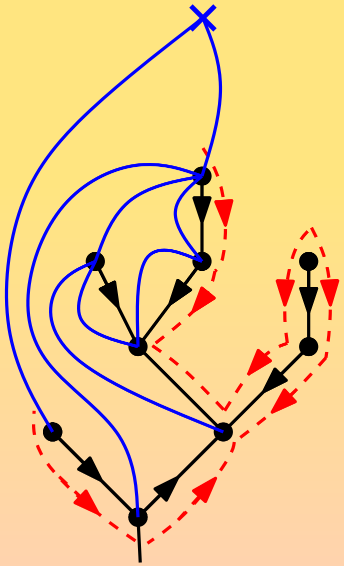

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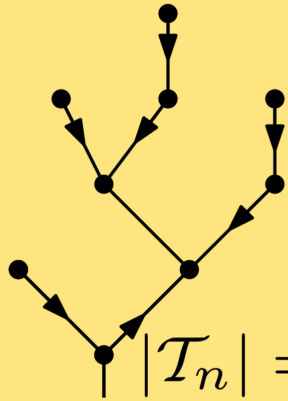

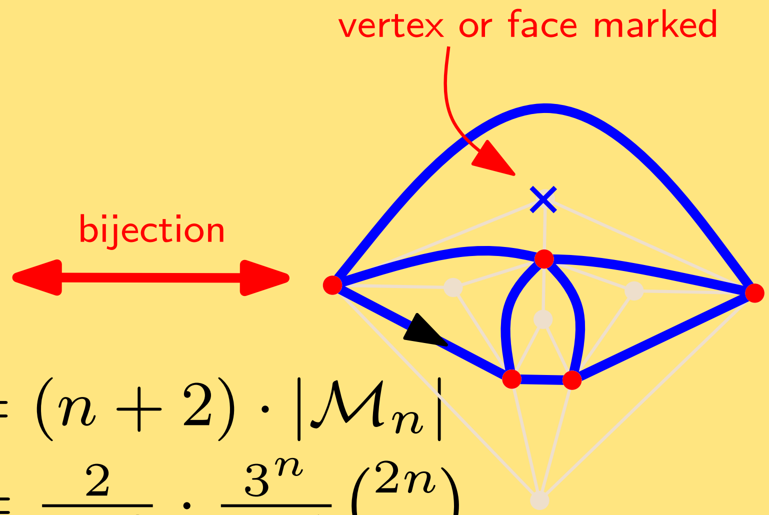

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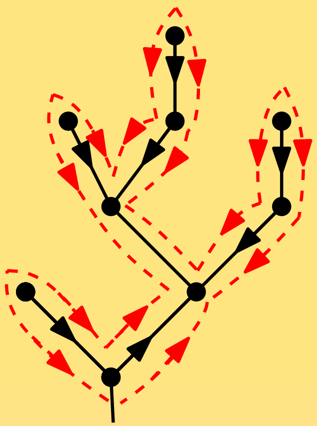

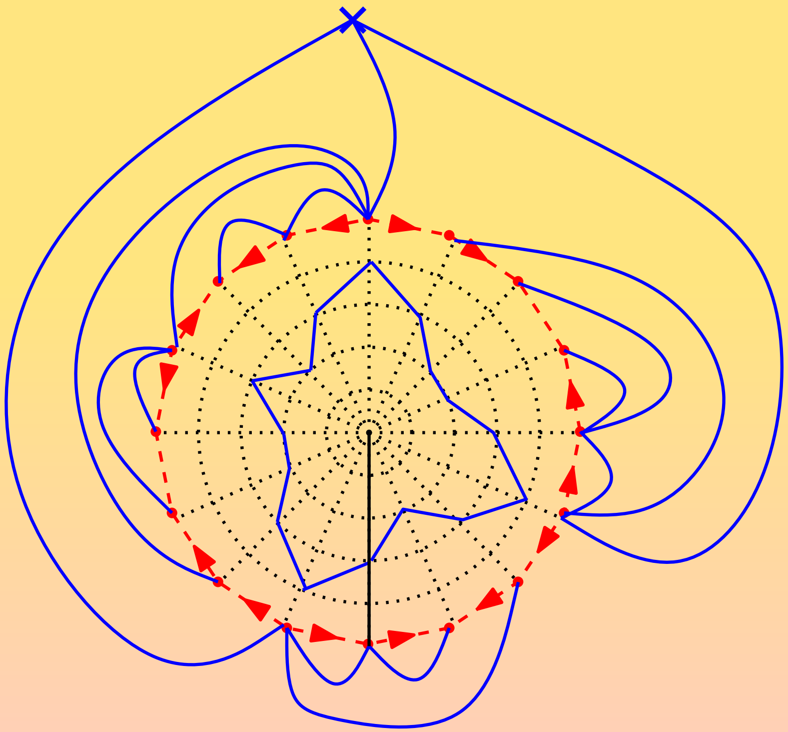

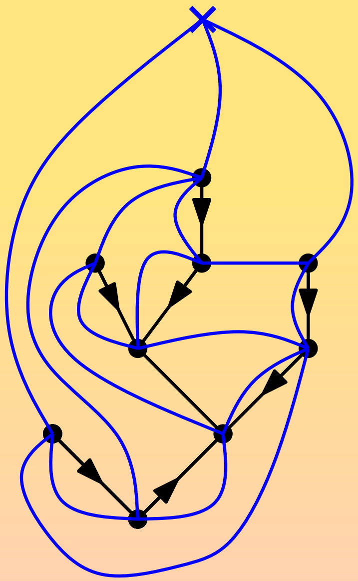

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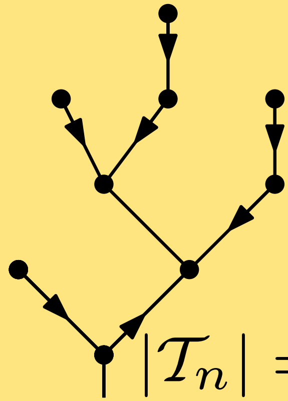

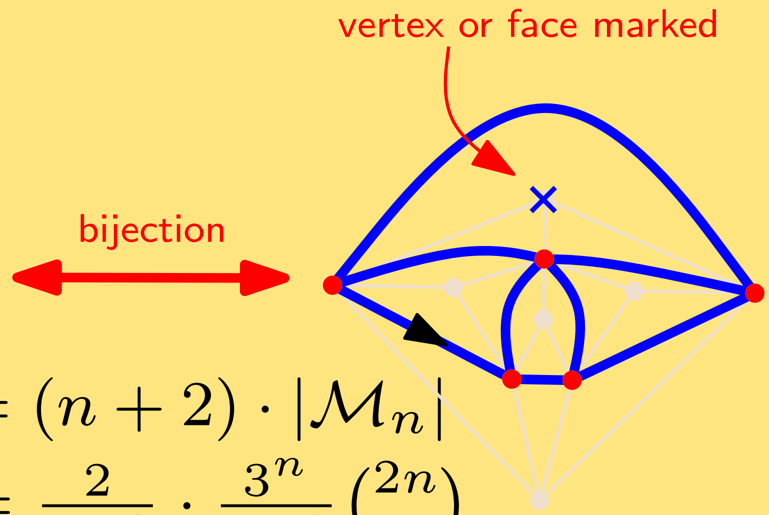

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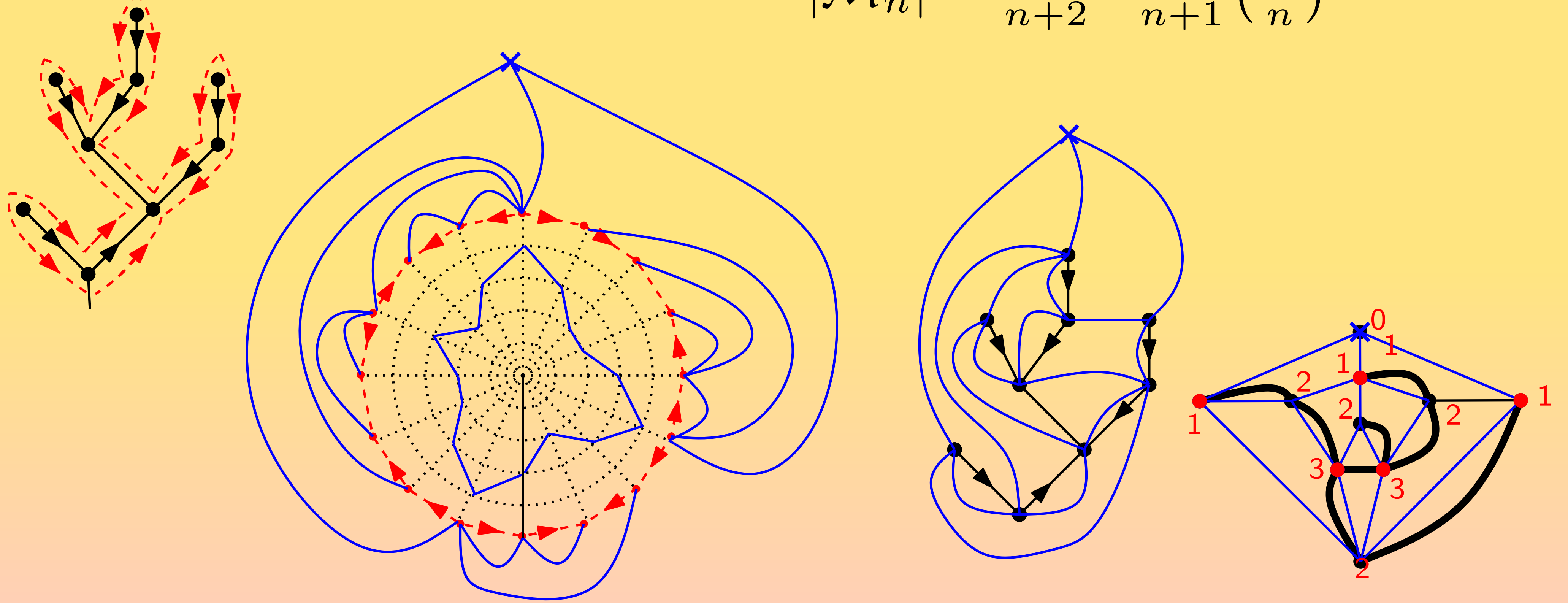

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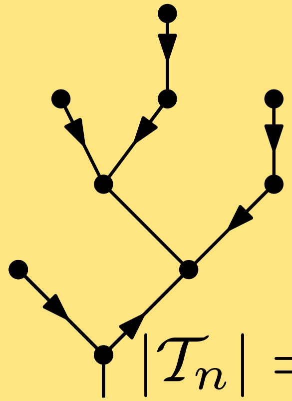

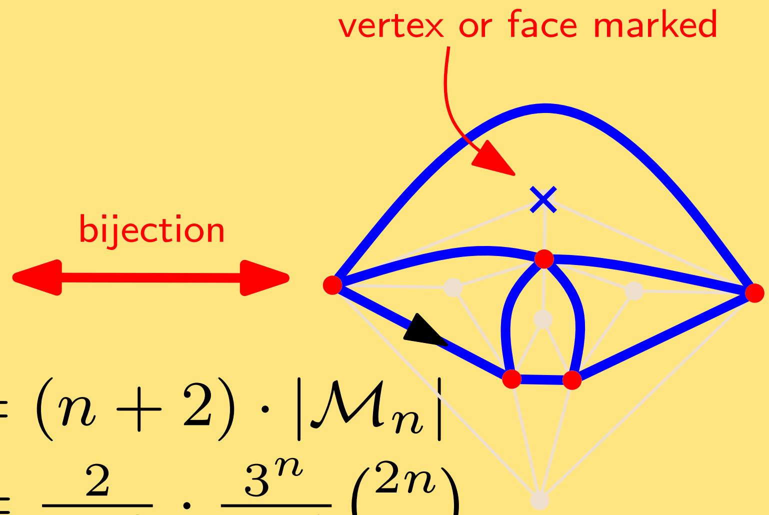

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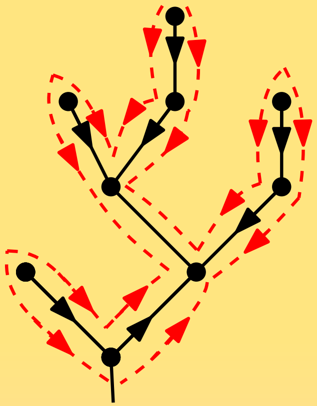

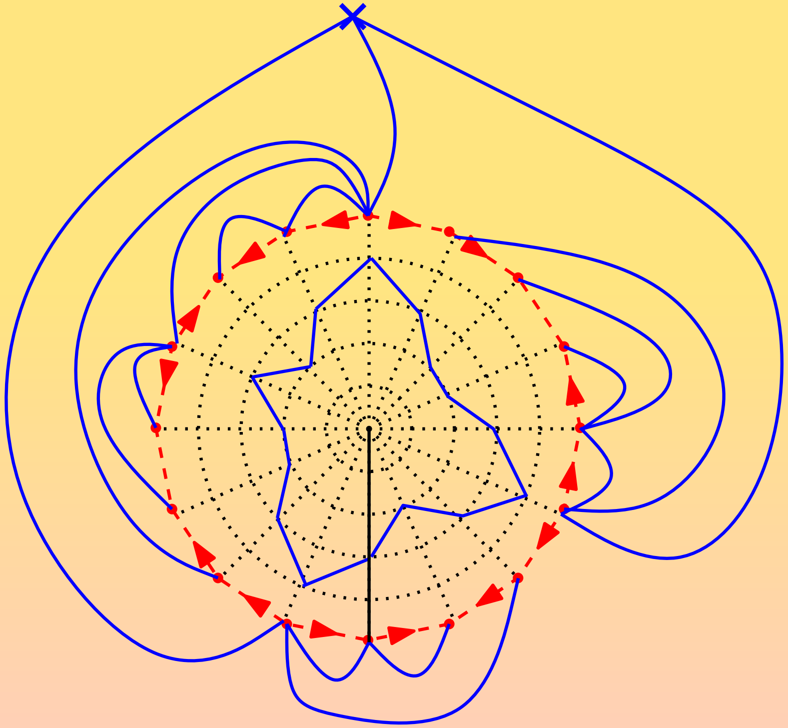

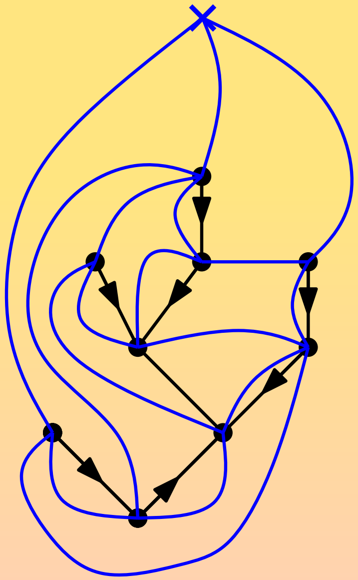

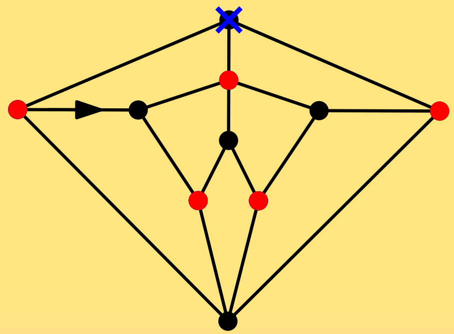

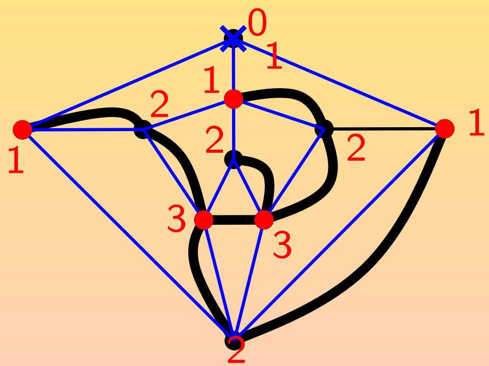

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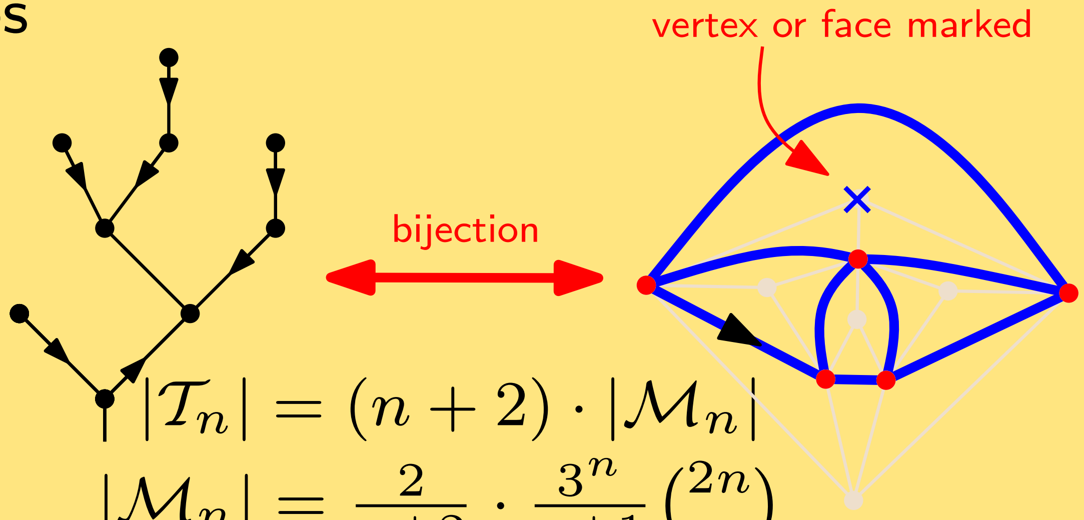

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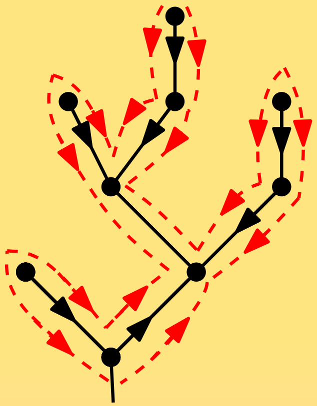

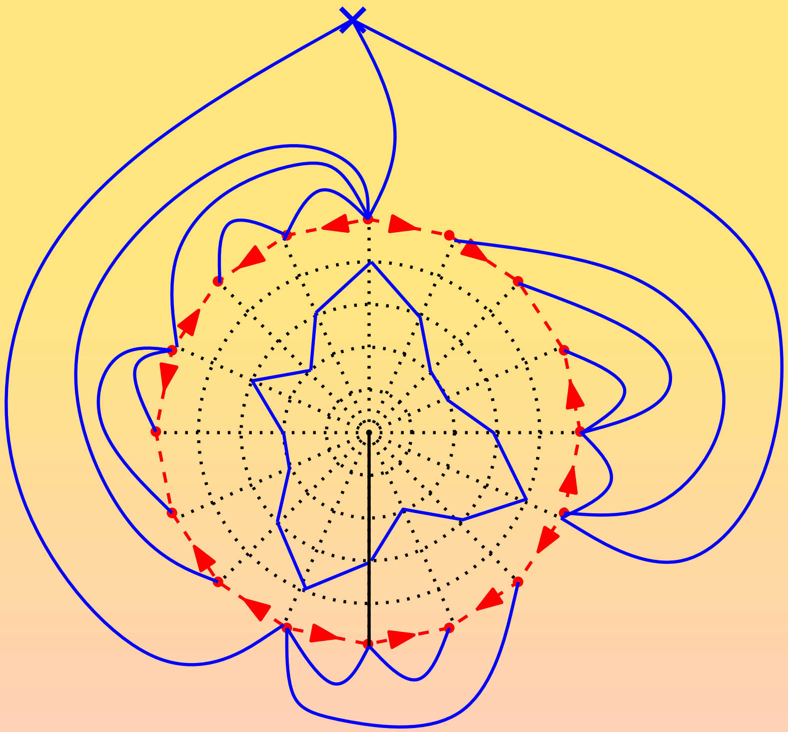

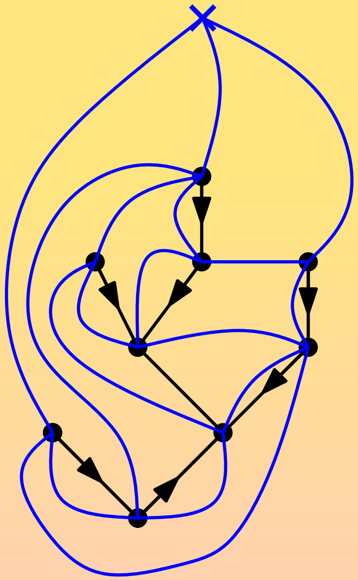

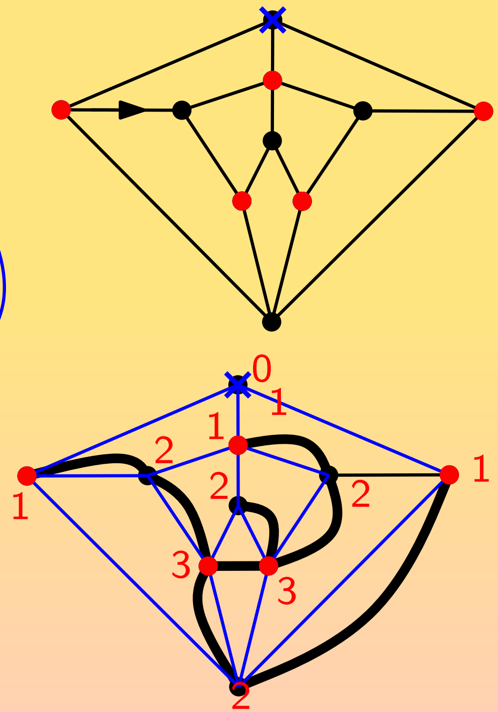

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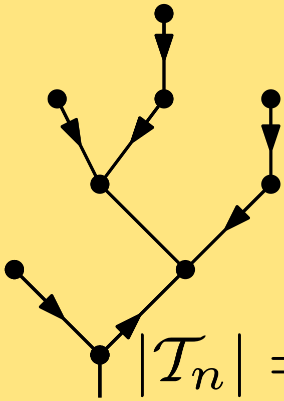

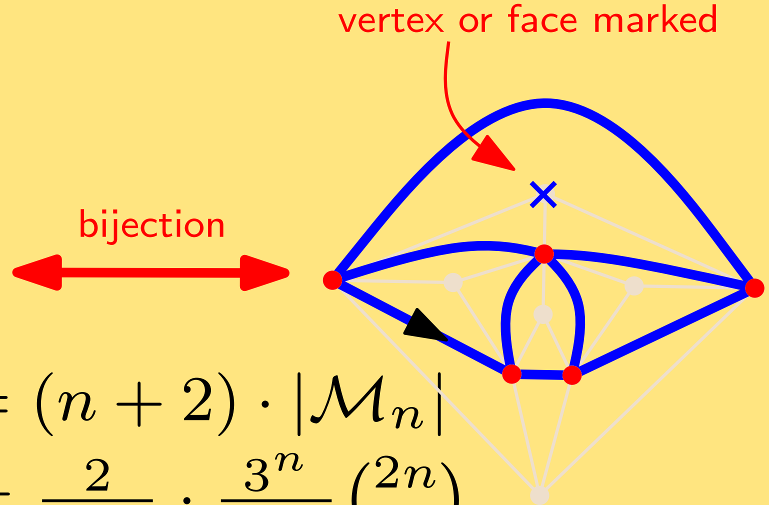

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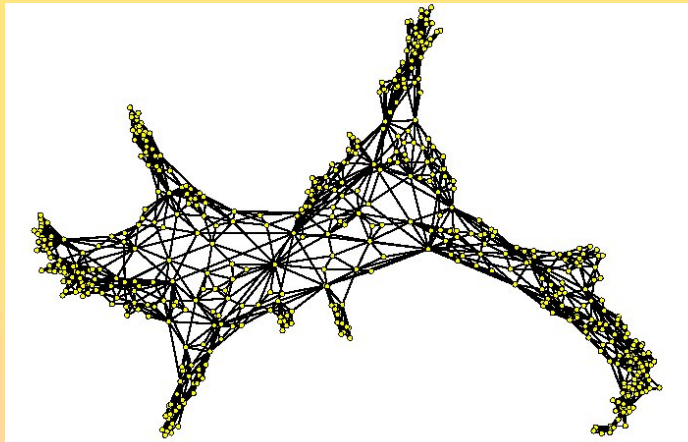
Theorem: Uniform random planar maps with n edges can be generated in linear time from the *closure* of uniform random ordered trees.



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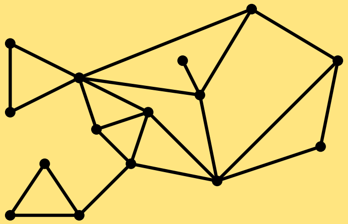

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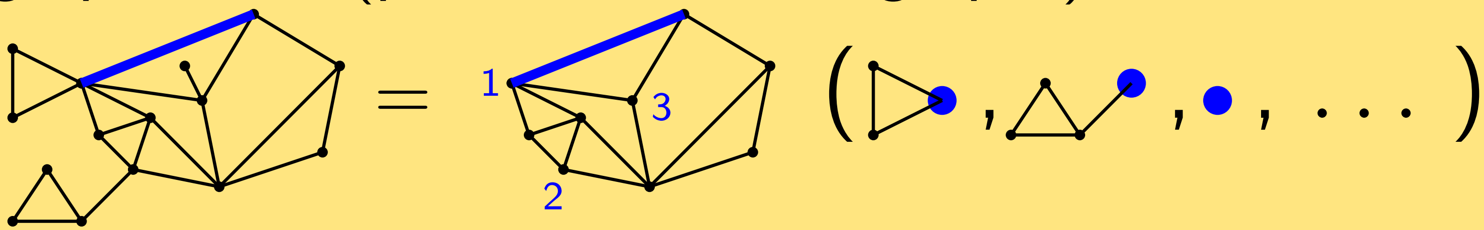

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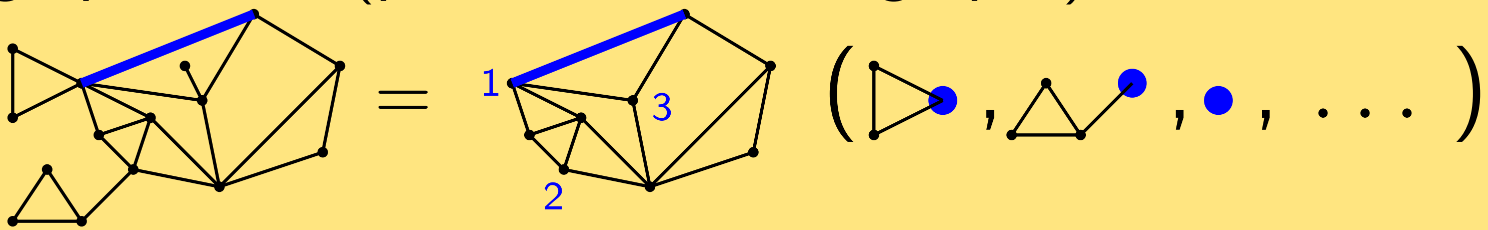

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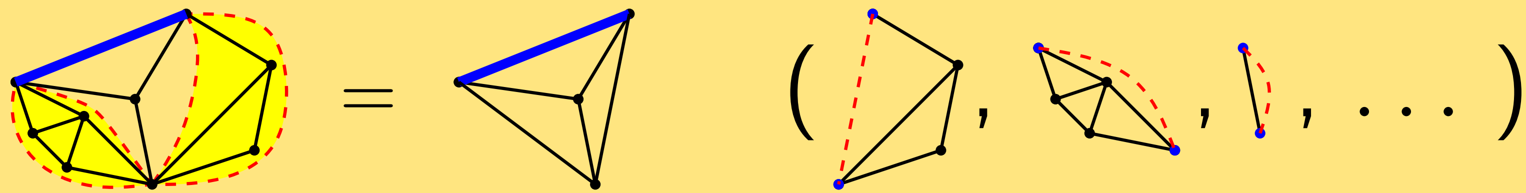

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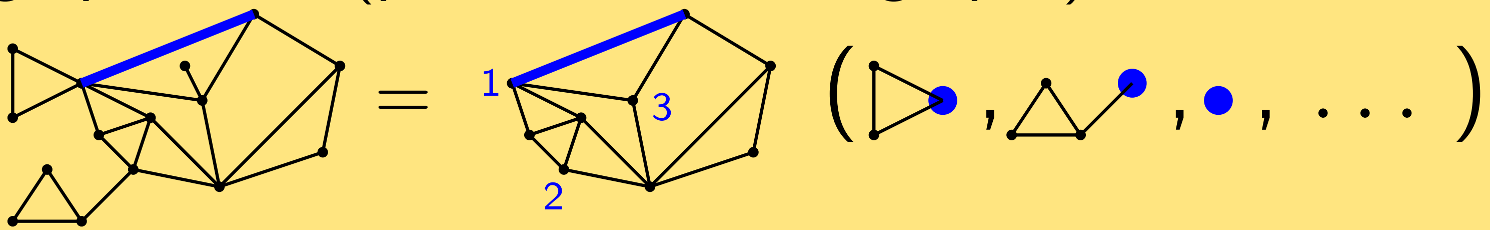

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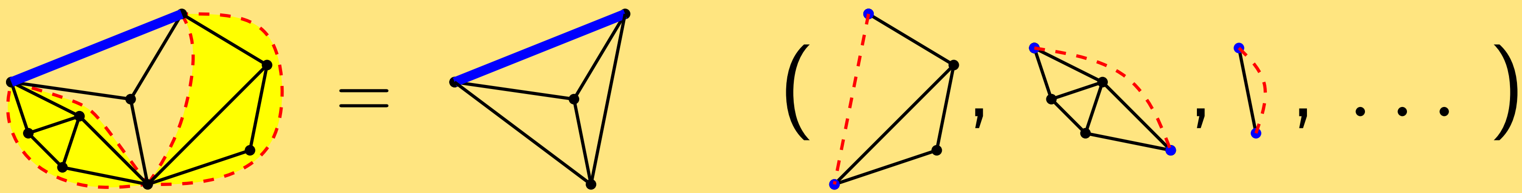

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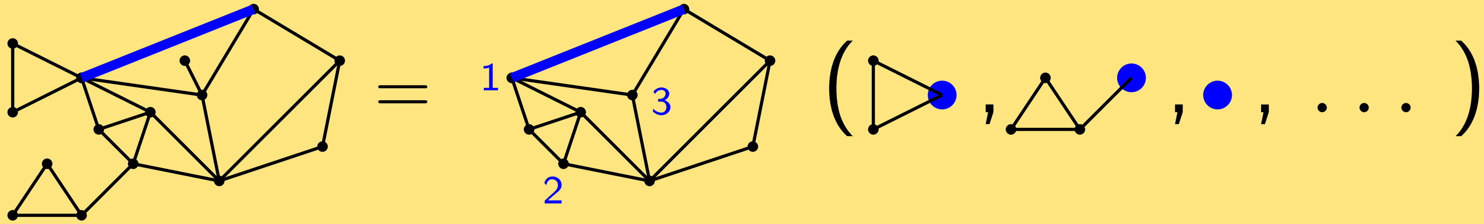

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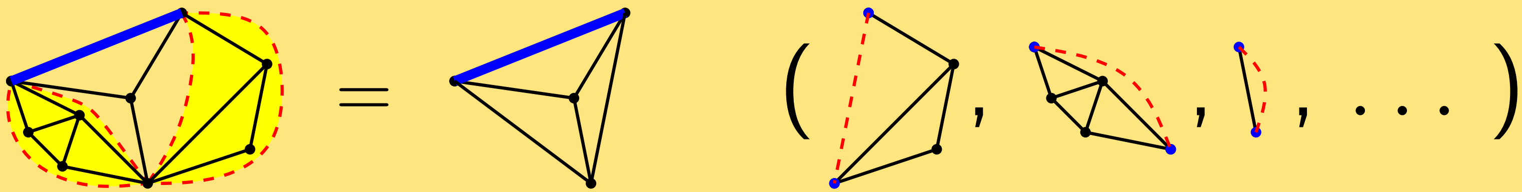

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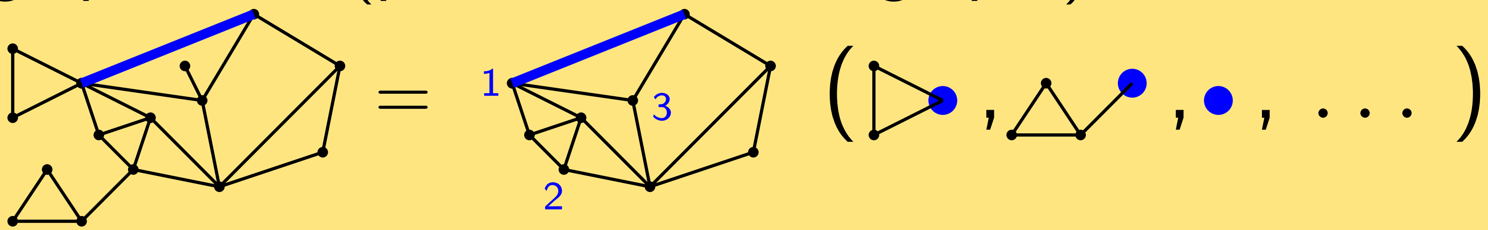

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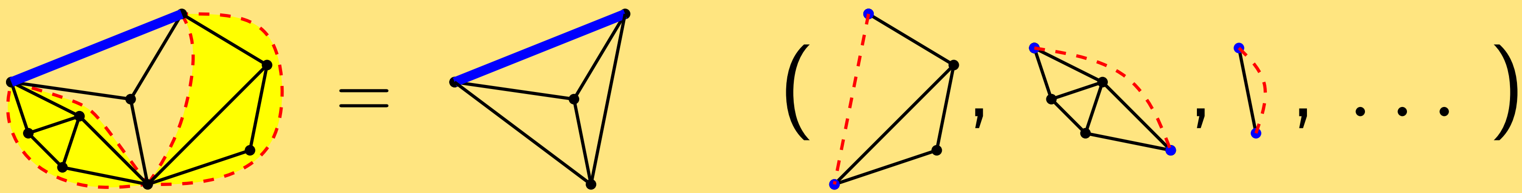

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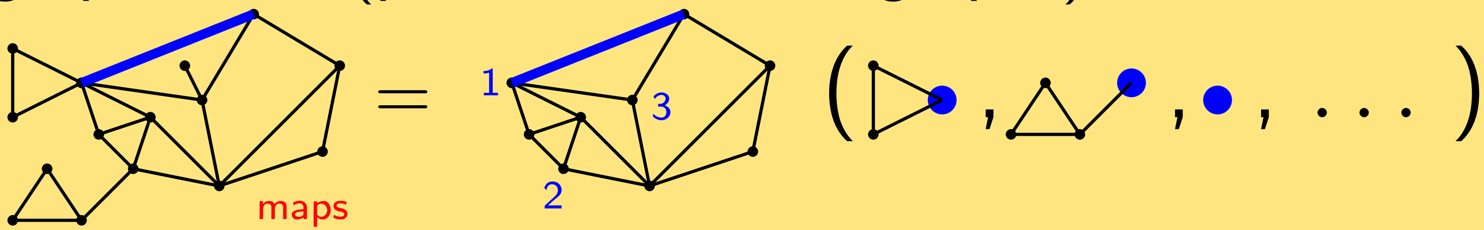

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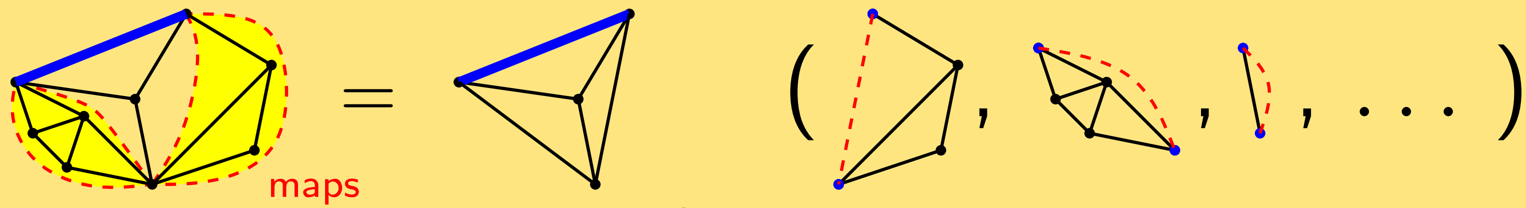

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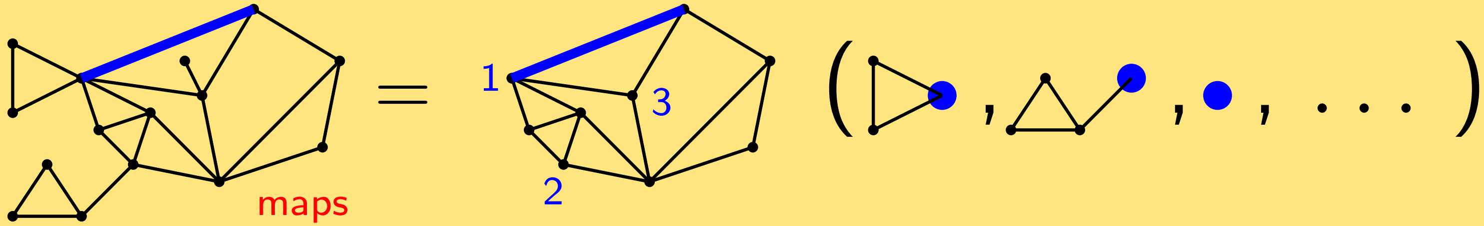

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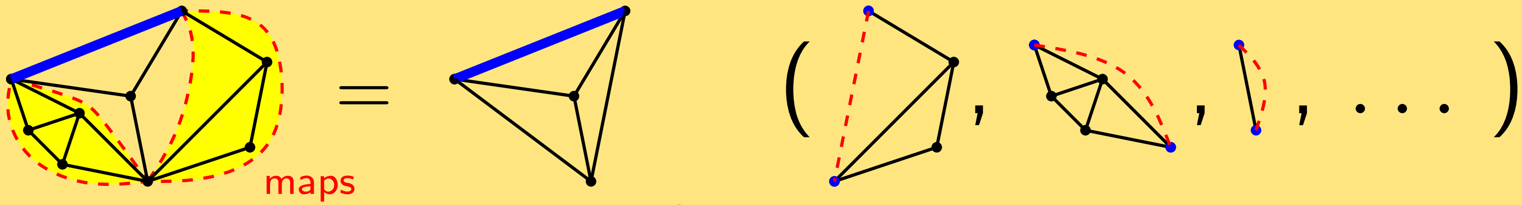

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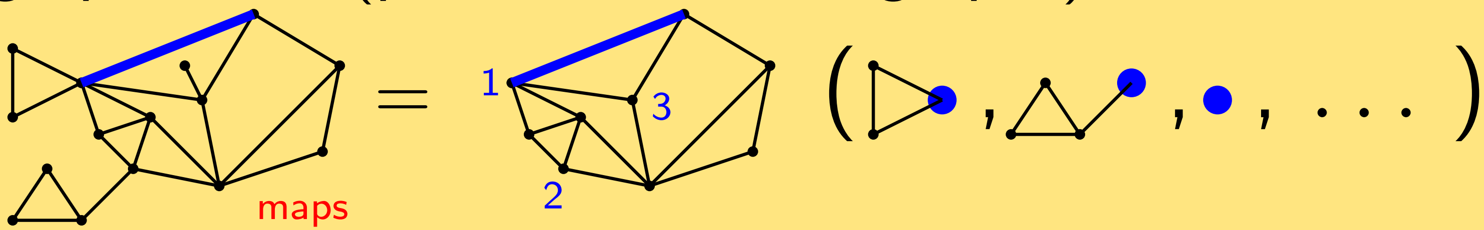

connected planar graph = edge rooted 2-connected $\circ_v$ vertex rooted connected planar graphs

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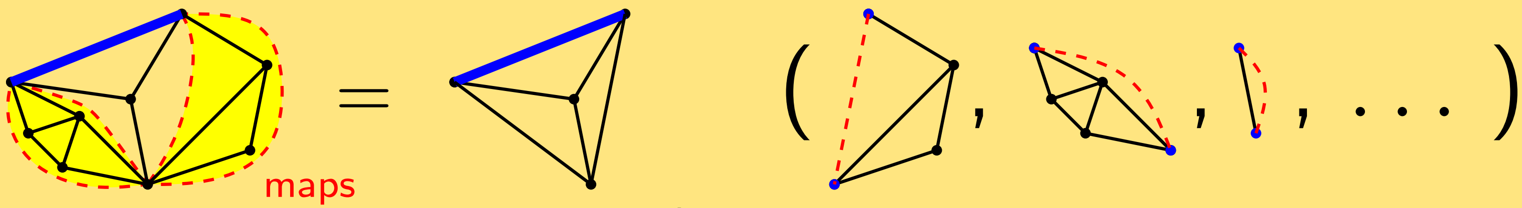

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Theorem (Whitney). A 3-c planar graph has a unique embedding

In other terms: 4) 3-c planar graphs and in bijection with 3-c planar maps

Then, essentially the same decomposition allows to relate planar 3-c maps to 2-c and connected planar maps the other way round.

Recall Recurrent Claim: planar maps can be generated from trees.

One needs an easy way to perform sampling for the composition of two combinatorial structures

Decomposing planar graphs

A decomposition for planar graphs was essentially given by Tutte.

cf Marc Noy's talk for its application to asymptotics.

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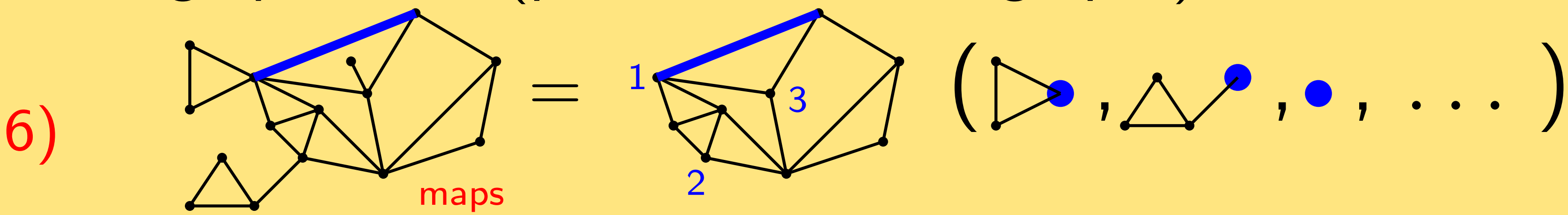

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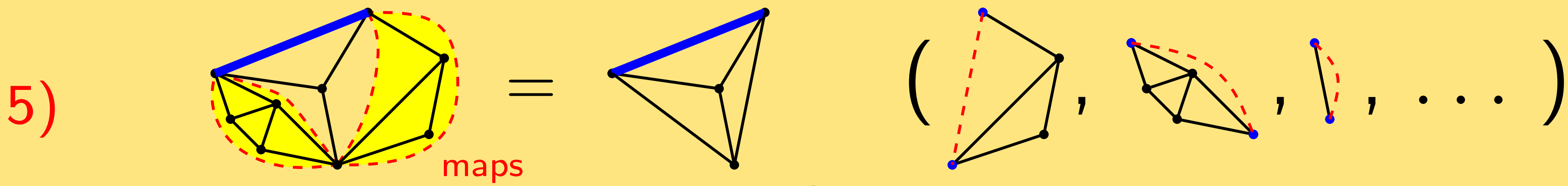

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Boltzmann models, Boltzmann sampling

A combinatorial class $\mathcal{A} = (\mathcal{A}_n)_{n \geq 0}$

Its generating function $A(x) = \sum_{a \in A} x^{|a|} = \sum_n |\mathcal{A}_n| x^n$.

Boltzmann models, Boltzmann sampling

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Let $x_0 > 0$ be such that $A(x_0)$ is finite (e.g. $x_0 < \rho_A$)
$\Gamma[\mathcal{A}](x_0)$ is a Boltzmann generator of parameter x_0 for $\mathcal{A}$ if

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- Composite Boltzmann generators can be *assembled* for the sum, product and composition of combinatorial classes.

  Suppose we have Boltzmann generators $\Gamma[\mathcal{A}](x)$ and $\Gamma[\mathcal{B}](x)$. Then
  $\Gamma[\mathcal{A} + \mathcal{B}](x) := \texttt{if } \mathrm{Bern}(\frac{A(x)}{A(x)+B(x)}) \texttt{ then } \Gamma[\mathcal{A}](x) \texttt{ else } \Gamma[\mathcal{B}](x)$
  $\Gamma[\mathcal{A} \times \mathcal{B}](x) := (\Gamma[\mathcal{A}(x)], \Gamma[\mathcal{B}(x)])$
  $\Gamma[\mathcal{A} \circ \mathcal{B}](x) := \texttt{let } a = \Gamma[\mathcal{A}](B(x)) \texttt{ in } (a; (\Gamma[\mathcal{B}](x))^{|a|})$

Composition in Boltzmann sampling

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Proof: Let $\gamma \in A \circ B$ with $\gamma = (a; b_1, \ldots, b_k)$ where $a \in \mathcal{A}$, $k = |a|$, $b_i \in \mathcal{B}$ for $i = 1, \ldots, k$, and $|\gamma| = |b_1| + \ldots + |b_k|$.

Then $\Pr\left(\Gamma[\mathcal{A} \circ \mathcal{B}](x) = \gamma\right)$

$$= \Pr\left(\Gamma[\mathcal{A}] = a\right) \cdot \prod_{i=1}^{|a|} \Pr\left(\Gamma[\mathcal{B}](x) = b_i\right)$$

$$= \frac{B(x)^{|a|}}{A(B(x))} \cdot \frac{\prod_i x^{|b_i|}}{B(x)^{|a|}} = \frac{x^{|b_1|+\cdots+|b_k|}}{A(B(x))} = \frac{x^{|\gamma|}}{(A \circ B)(x)}.$$

□

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Theorem: if $\Gamma[\mathcal{A} \circ \mathcal{B}]$ is Boltzmann then so are Core$(\Gamma[\mathcal{A} \circ \mathcal{B}])$ and First$(\Gamma[\mathcal{A} \circ \mathcal{B}])$, where Core$(\gamma) = a$ and First$(\gamma) = b_1$.

Uniform sampling from Boltzmann sampling

- Rejection yields uniform sampling (elements of same size have same proba)

  $\mathsf{U}[\mathcal{A}](n) := \texttt{do let } a = \Gamma[\mathcal{A}](x) \texttt{ until } |a| = n;$ return a;

Complexity depends on $|\mathcal{A}_n| \frac{x^n}{A(x)}$: good choice of $x = x_n$ and pointing.

Exact size uniform sampling can be often done in quadratic expected time and approximate size uniform sampling can be done in linear time.

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- Boltzmann in progress...

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Applications: plane partitions, colored structures, deterministic automata, XML documents, Appolonian structures...

Bodini, Fusy, Pivoteau (2006), Bodini, Jacquot (2008), Bassino, Nicaud (2006), Bassino, David, Nicaud (2008), Darasse, Soria (2007), Darasse (2008),Bernasconi, Panagiotou, Steger, Weißt (2006)

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Warning: I skipt a "lot" of details (rerootings, bivariate compositions...)

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Many thanks again to Philippe, and to the audience

Random graphs on surfaces: a conjecture (S. 2007)

Take a uniform random labelled graph X_n in the set of graphs of genus $\leq g$ with n vertices.

Then X_n a.s. has a unique 3-connected component of linear size $C(X_n)$, and:

- $C(X_n)$ is a.s. a random 3-connected graphs with minimum genus g,
- $C(X_n)$ a.s. has a unique embedding on $\mathcal{S}_g$,
- all other components are planar and of size $O(n^{2/3})$,

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Let $\mathcal{A}$ is the familly of ordered trees: a tree decomposes into a root and a sequence of subtrees attached by edges:

$$\mathcal{A} = \{\mathsf{r}\} \times \mathrm{Seq}(\{e\} \times \mathcal{A})$$

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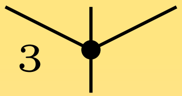
$\Gamma[\mathsf{Seq}] = 3$ 

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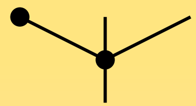

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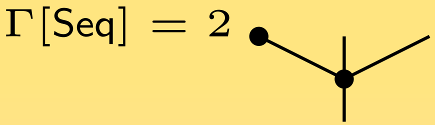

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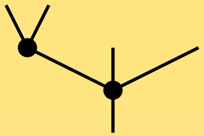

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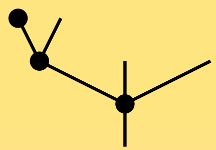

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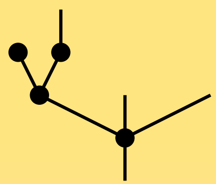

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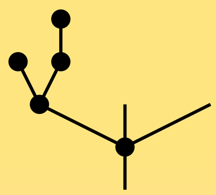

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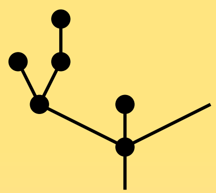

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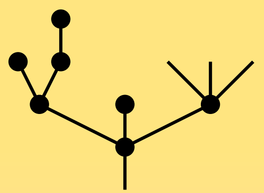

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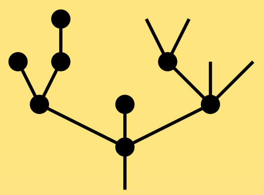

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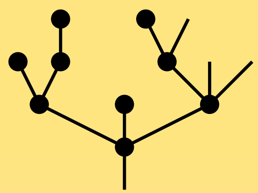

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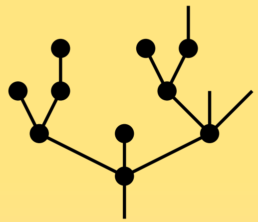

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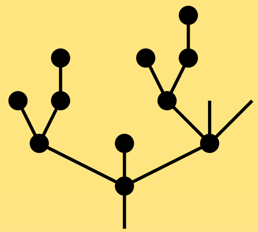

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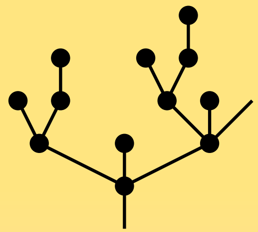

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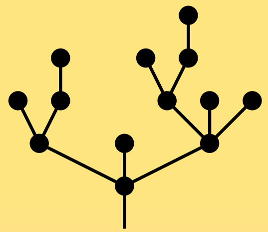

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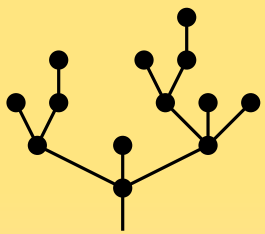

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The probability to get size n depends on the choice of x, increasing near the singularity: if $x_n = \frac{1}{4}(1 - \frac{1}{n})$

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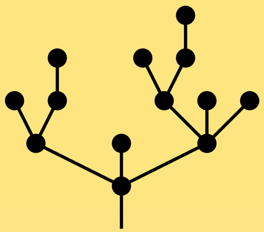

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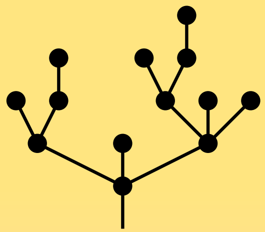

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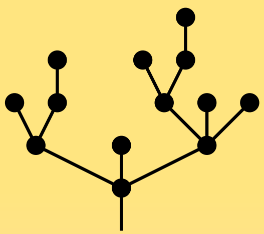

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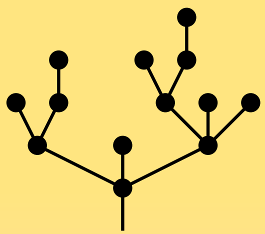

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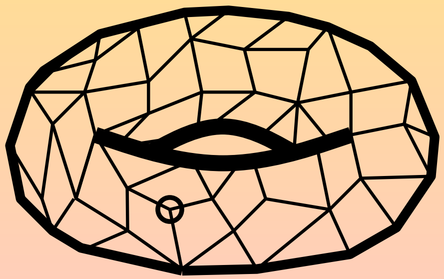

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Conjectures.
There is no non-contractible cycles with size $\ll n^{1/4}$.
The rescaled continuum limit exists and has genus g.