Asymptotics of the Stirling numbers of the first kind revisited: A saddle point approach

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A Philippe, pour tant d'agréables conversations Mellinesques (Melliniques? Mellinophiles? Melliniformes?)

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Introduction

Let $\left[\begin{array}{c} n \\ j \end{array} \right]$ be the Stirling number of the first kind (unsigned version). Their generating function is given by

$$\phi_n(z) = \prod_0^{n-1}(z+i) = \frac{\Gamma(z+n)}{\Gamma(z)}, \quad \phi_n(1) = n!.$$

An asymptotic expansion for $j = \mathcal{O}(1)$ is given in Wilf [14], which has been extended to the range $j = \mathcal{O}(\ln n)$ by Hwang [6]. The generalized Sirling numbers have been considered by Tsylova [13] and Chelluri et al. [2]. The $q-$Stirling numbers are studied in Kyriakoussis and Vamvakari [9].

In this talk, we revisit the asymptotic expansions in the central region and we analyse the non-central region $j = n - n^{\alpha}, \quad \alpha > 1/2$. We use Cauchy's integral formula and the saddle point method.

Central region

Consider

$$J_n(j) := \frac{\left[\begin{matrix} n \\ j \end{matrix} \right]}{n!}$$

as a random variable. The mean and variance are given by

$$M := \mathbb{E}(J_n) = \sum_{0}^{n-1} \frac{1}{1+i} = H_n = \psi(n+1) + \gamma,$$

$$\sigma^2 := \mathbb{V}(J_n) = \sum_{0}^{n-1} \frac{i}{(1+i)^2} = \psi(1, n+1) + \psi(n+1) - \frac{\pi^2}{6} + \gamma,$$

and

$$M \sim \ln(n) + \gamma + \frac{1}{2n} + \mathcal{O}\left(\frac{1}{n^2}\right),$$

$$\sigma^2 \sim \ln(n) - \frac{\pi^2}{6} + \gamma + \frac{3}{2n} + \mathcal{O}\left(\frac{1}{n^2}\right).$$

It is convenient to set

$$A := \ln(n) - \frac{\pi^2}{6} + \gamma = \ln\left(ne^{\gamma - \pi^2/6}\right),$$

and to consider all our next asymptotics ($n \to \infty$) as functions of A. Of course, all asymptotics can be reformulated in terms of $\ln(n)$.
We have

$$M \sim A + \frac{\pi^2}{6} + \mathcal{O}\left(\frac{1}{n}\right),$$
$$\sigma^2 \sim A + \mathcal{O}\left(\frac{1}{n}\right).$$

A celebretated theorem of Goncharov says that

$$J_n(j) \sim \mathcal{N}\left(\frac{j-M}{\sigma}\right),$$

where $\mathcal{N}$ is the Gaussian distribution, with a rate of convergence $\mathcal{O}(1/\sqrt{\ln(n)})$. This can also be deduced from the Quasi-Power theorem of Hwang [7],[8].

In this Section, we want to obtain a more precise local limit theorem for $J_n(j)$ in terms of $x := \frac{j-M}{\sigma}$ and A.

By Cauchy's theorem,

$$J_n(j) = \frac{1}{2\pi\mathbf{i}}\int_{\Omega}\frac{\phi_n(z)}{z^{j+1}}dz = \frac{1}{2\pi\mathbf{i}}\int_{\Omega}e^{S(z)}dz,$$

where Ω is inside the analyticity domain of the integrand and encircles the origin and

$$S(z) = S_1(z) + S_2(z), \quad S_1(z) = \sum_{i=0}^{n-1}\ln(z+i)) - \ln(n!),$$

$$S_2(z) = -(j+1)\ln(z).$$

Set

$$S^{(i)} := \frac{d^i S}{dz^i}.$$

These derivatives can be expressed in terms of $\psi(k, z+n)$ and $\psi(k, z)$.
We will use the Saddle point method (for a good introduction to this method, see Flajolet and Sedgewick [3], ch.*VIII*). First we must find the solution of

$$S^{(1)}(\tilde{z}) = 0 \tag{1}$$

with smallest module.
Set $\tilde{z} := z^* - \varepsilon$, where, here, it is easy to check that $z^* = 1$. Set $j = M + x\sigma$ and $B := \sqrt{A}$ to simplify the expressions.
This leads, to first order (keeping only the ε term in (1)), to

$$\varepsilon := \frac{-x}{B} + \frac{x^2-1}{B^2} + \mathcal{O}\left(\frac{1}{B^3}\right) + \frac{1}{n}\left(\frac{3x}{4B^3} + \mathcal{O}\left(\frac{1}{B^4}\right)\right) + \mathcal{O}\left(\frac{1}{n^2 B^4}\right).$$

This shows that, asymptotically, ε is given by a Laurent series of powers of n^{-1}, where each coeficient is given by a Laurent series of powers of B^{-1}. To obtain more precision, we set again $j = M + x\sigma$, expand in powers of n^{-1}, and equate each coefficient to 0. Note that we will need the $1/n$ term of ε later on. This leads to

$$\varepsilon = \frac{-x}{B} - \frac{1}{B^2} + \frac{0}{B^3} + \mathcal{O}\left(\frac{1}{B^4}\right) + \frac{1}{n}\left(\frac{3x}{4B^3} + \frac{x^2+3/2}{B^4} + \mathcal{O}\left(\frac{1}{B^5}\right)\right) + \mathcal{O}\left(\frac{1}{n^2 B^4}\right).$$

We have, with $\tilde{z} := z^* - \varepsilon = 1 - \varepsilon$,

$$J_n(j) = \frac{1}{2\pi\mathbf{i}} \int_{\Omega} \exp\left[S(\tilde{z}) + S^{(2)}(\tilde{z})(z-\tilde{z})^2/2! + \sum_{l=3}^{\infty} S^{(l)}(\tilde{z})(z-\tilde{z})^l/l!\right] dz.$$

Note that the linear term vanishes. Set $z = \tilde{z} + \mathbf{i}\tau$. This gives

$$J_n(j) = \frac{1}{2\pi}\exp[S(\tilde{z})]\int_{-\infty}^{\infty}\exp\left[S^{(2)}(\tilde{z})(\mathbf{i}\tau)^2/2! + \sum_{l=3}^{\infty} S^{(l)}(\tilde{z})(\mathbf{i}\tau)^l/l!\right]d\tau. \quad (2)$$

Let us first analyze $S(\tilde{z})$. We obtain (we see now why we need the $1/n$ term of ε: there is a summation $\sum_{i=0}^{n-1}$ in $S_1(z)$)

$$\begin{aligned} S(\tilde{z}) = {} & -x^2/2 + \frac{x^3/6 - x}{B} + \frac{-x^4/12 + x^2/2 - 1/2}{B^2} \\ & + \frac{-x^3/3 + x^5/20 + x/2 - \pi^2 x^3/18 + \zeta(3)x^3/3}{B^3} \\ & + \mathcal{O}\left(\frac{1}{B^4}\right) + \mathcal{O}\left(\frac{1}{nB^3}\right). \end{aligned}$$

Now we compute the derivatives, (here and in the following, we provide only a few terms but Maple knows more).

$$S^{(2)}(\tilde{z}) = B^2 - Bx - 1 + x^2 + \dots,$$
$$S^{(3)}(\tilde{z}) = -2B^2 + 4Bx - \pi^2/3 + 2\zeta(3) - 6x^2 + 4 + \dots,$$
$$S^{(4)}(\tilde{z}) = 6B^2 - 18Bx + 36x^2 - 18 + \pi^2 - \pi^4/15 + \dots,$$
$$S^{(l)}(\tilde{z}) = \mathcal{O}\left(B^2\right), l \geq 5.$$

We need these many terms in the following.

We can now compute (2), for instance by using the classical trick of setting

$$S^{(2)}(\tilde{z})(i\tau)^2/2! + \sum_{l=3}^{\infty} S^{(l)}(\tilde{z})(i\tau)^l/l! = -u^2/2.$$

Computing τ as a truncated series in u, this gives, by inversion,

$$\tau = \frac{u(1 + x/(2B) + \ldots) + u^2(\mathbf{i}/(3B) + \ldots) + u^3(-1/(36B^2) + \ldots)}{B} + \ldots$$

Setting $d\tau = \frac{d\tau}{du}du$, expanding w.r.t. B and integrating on $[u = -\infty..\infty]$, this gives

$$\frac{1}{\sqrt{2\pi}B}\left[1 + \frac{x}{2B} + \frac{5/12 - x^2/8}{B^2} + \frac{x(8\pi^2 - 10 - 93x^2 - 48\zeta(3))}{48B^3} + \ldots\right].$$

Finally (2) leads to

$$J_n(j) \sim \frac{1}{\sqrt{2\pi}B} e^{-x^2/2}.$$

$$\cdot \exp\left[\frac{\frac{x^3}{6} - x}{B} + \frac{-\frac{x^4}{12} + \frac{x^2}{2} - \frac{1}{2}}{B^2} + \frac{-\frac{x^3}{3} + \frac{x^5}{20} + \frac{x}{2} - \frac{\pi^2 x^3}{18} + \frac{\zeta(3)x^3}{3}}{B^3} + \ldots\right].$$

$$\cdot\left[1 + \frac{x}{2B} + \frac{5/12 - x^2/8}{B^2} + \frac{x(8\pi^2 - 10 - 93x^2 - 48\zeta(3))}{48B^3} + \ldots\right],$$

or

$$J_n(j) \sim R_1,$$

$$R_1 = \frac{1}{\sqrt{2\pi}B} e^{-x^2/2}.$$

$$\cdot\left[1 + \frac{x^3/6 - x/2}{B} + \frac{3x^2/8 - x^4/6 - 1/12 + x^6/72}{B^2}\right.$$

$$\left. + \frac{-\frac{\pi^2 x^3}{18} + \frac{37x^5}{240} - \frac{355x^3}{144} + \frac{x}{8} - \frac{x^7}{48} + \frac{x^9}{1296} + \frac{\pi^2 x}{6} - \zeta(3)x + \frac{\zeta(3)x^3}{3}}{B^3} + \ldots\right].$$

For $n = 3000$, a comparison between $J_n(j)$ and $\frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\left(\frac{j-M}{\sigma}\right)^2 \Big/ 2\right]$ is given in Figure 1.

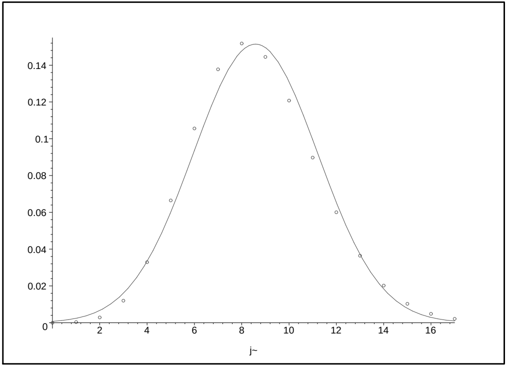

Figure 1: Comparison between $J_n(j)$ and $\frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\left(\frac{j-M}{\sigma}\right)^2 \Big/ 2\right], n = 3000$

Of course, only few values of j are significant and also the quality of the Gaussian is low, all asymptotic expressions depend actually on powers of A^{-1}, but A is not large.

A comparison of $J_n(j) \Big/ \left[\frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\left(\frac{j-M}{\sigma}\right)^2 \Big/ 2\right]\right]$ with $J_n(j)/R_1$, with 2 terms in R_1, is given in Figure 2.

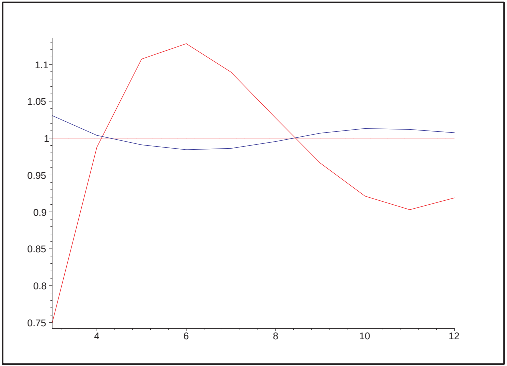

Figure 2: $J_n(j) \Big/ \left[\frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\left(\frac{j-M}{\sigma}\right)^2 \Big/ 2\right]\right]$, color=red, $J_n(j)/R_1$, color=blue, $n = 3000$

The precision of R_1 is of order 10^{-2}. Using 3 terms in R_1 leads to a less good result: A is not large enough to take advantage of the $A^{-3/2}$ term: $A = 6.94$ here, we deal with asymptotic series, not necessarily convergent ones. More terms can be computed in R_1 (which is almost automatic with Maple).

Large deviation, $j = n - n^\alpha$, $\quad \alpha > 1/2$

The case $j = \mathcal{O}(n)$ is analyzed in Timashev [12] and the case $j = n - c$, c constant, in Grünberg [5]. As previous work for the case $j = n - n^\alpha$, let us mention Bender [1], Temme [11], Moser and Wyman [10] (see also the comments by Odlyzko in [4], p.1182). They all use, explicitly or not, the Saddle point method. For $\alpha < 1/2$, Moser and Wyman (6.9) give an explicit asymptotic expression. For $\alpha > 1/2$, they first compute in (4.52) the numerical solution zn of $S'(zn) = 0$ and give in (4.51) an asymptotic expression. This is rather precise: for $n = 50$, this gives a precision of order 10^{-4}. [1] and [11] also compute numerically zn. However, all theses results do not shed light on the dependence of $[z^j]\phi(z)$ on n^α. This what we want to explicit in this Section. It appears that the range $\alpha > 1/2$ is more delicate than the other range.

Recall that

$$\phi_n(z) = \prod_{0}^{n-1}(z+i) = \frac{\Gamma(z+n)}{\Gamma(z)}.$$

We have

$$G_n(z) := \frac{\Gamma(z+n)}{\Gamma(z)z^{j+1}} = \exp[S(z)],$$

with

$$S(z) = S_1(z) + S_2(z), S_1(z) = \sum_{0}^{n-1} \ln(z+i), S_2(z) = -(j+1)\ln(z).$$

We first compute $\tilde{z}$ such that

$$S'(\tilde{z}) = 0. \qquad (3)$$

We have

$$S'(z) = \psi(z+n) - \psi(z) - \frac{j+1}{z}.$$

Similarly (we need these expressions later on)

$$S^{(2)}(z) = \psi(1, z+n) - \psi(1, z) + \frac{j+1}{z^2},$$

$$S^{(k)}(z) = \psi(k-1, z+n) - \psi(k-1, z) + (-1)^k (k-1)! \frac{j+1}{z^k}.$$

Some experiments with some values for α ($\alpha = 5/8$ is a good choice) show that $\tilde{z}$ must be a combili of $x = n^{\alpha}$ and $y = n^{1-\alpha}$ and $x \gg y$. Note that both x and y are large. The first terms in the asymptotics of $\tilde{z}$ are easy to compute: set $\tilde{z} = n\beta$. Equation (3) leads to

$$\psi(n(1+\beta)) - \psi(n\beta) = \frac{1}{\beta} - \frac{1}{y\beta} + \frac{1}{n\beta}.$$

But $\psi(n) \sim \ln(n)$. So we have

$$\ln\left(1 + \frac{1}{\beta}\right) \sim \frac{1}{\beta} - \frac{1}{y\beta} + \frac{1}{n\beta}, \text{ or } \frac{1}{\beta} - \frac{1}{2\beta^2} \sim \frac{1}{\beta} - \frac{1}{y\beta},$$

or $\beta \sim \frac{y}{2}$.

More generally, we have

$$\beta = \frac{y}{2}\left[1 + \frac{a_1}{y} + \frac{a_2}{y^2} + \mathcal{O}\left(\frac{1}{y^3}\right) + \frac{1}{x}\left(1 + \frac{b_1}{y} + \frac{b_2}{y^2} + \mathcal{O}\left(\frac{1}{y^3}\right)\right)\right.$$
$$\left. + \frac{1}{x^2}\left(1 + \frac{c_1}{y} + \frac{c_2}{y^2} + \mathcal{O}\left(\frac{1}{y^3}\right)\right) + \mathcal{O}\left(\frac{1}{x^3}\right)\right].$$

By bootstrapping, we obtain (we give the first terms)

$$\tilde{z} = \frac{ny}{2}\left[1 - \frac{4}{3y} + \frac{2}{9y^2} + \frac{8}{135y^3} + \frac{8}{405y^4}\right.$$
$$+ \frac{16}{1701y^5} + \frac{232}{45525y^6} + \frac{64}{18225y^7} + \mathcal{O}\left(\frac{1}{y^8}\right)$$
$$+ \frac{1}{x}\left[1 - \frac{1}{y} + \frac{4}{9y^2} - \frac{16}{135y^3} + \mathcal{O}\left(\frac{1}{y^4}\right)\right]$$
$$\left. + \frac{1}{x^2}\left[1 - \frac{1}{y} + \frac{0}{y^2} + \mathcal{O}\left(\frac{1}{y^3}\right)\right] + \frac{1}{x^3}\left[1 + \mathcal{O}\left(\frac{1}{y}\right)\right] + \mathcal{O}\left(\frac{1}{x^4}\right)\right]. \tag{4}$$

Note that the choice of dominant terms in the bracket of (4) depends on α. For instance, for $\alpha = 3/4$, the dominant terms (in decreasing order) are

$$1, \frac{1}{y}, \frac{1}{y^2}, \left\{\frac{1}{x}, \frac{1}{y^3}\right\}, \left\{\frac{1}{xy}, \frac{1}{y^4}\right\}, \left\{\frac{1}{xy^2}, \frac{1}{y^5}\right\}, \left\{\frac{1}{x^2}, \frac{1}{xy^3}, \frac{1}{y^6}\right\}, \ldots$$

The quality of asymptotic (4) is given in Figure 3 and 4, for $n = 500$, and $x \in [\sqrt{n}, n^{0.9}]$ (first range) so that $y \in [n^{0.1}, \sqrt{n}]$. For some values of $j = n - x$, we show $\tilde{z}/zn$, where, as mentioned, zn is the numerical solution of $S'(zn) = 0$. In the full range $j \in [n - n^{0.9}, n - \sqrt{n}]$, the precision is of order 10^{-5}, in a restricted range, the precision is of order 10^{-6}.
Also a comparison of $G_n(\tilde{z})$ and $G_n(zn)$ is given in Figure 5, showing again a precision of order 10^{-6}.

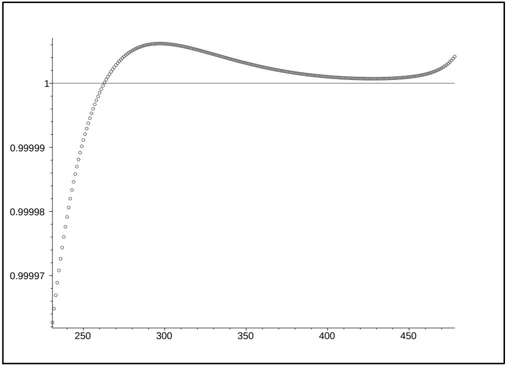

Figure 3: $zn/\tilde{z}, n = 500$, as function of j, full range

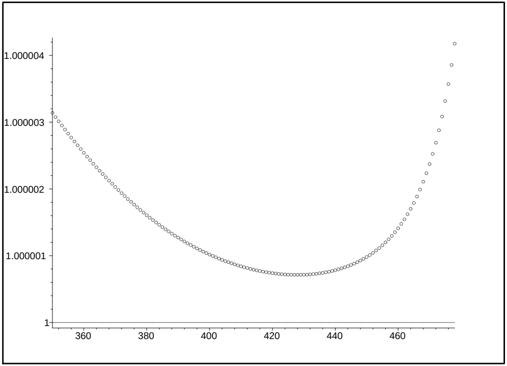

Figure 4: $zn/\tilde{z}, n = 500$, as function of j, restricted range

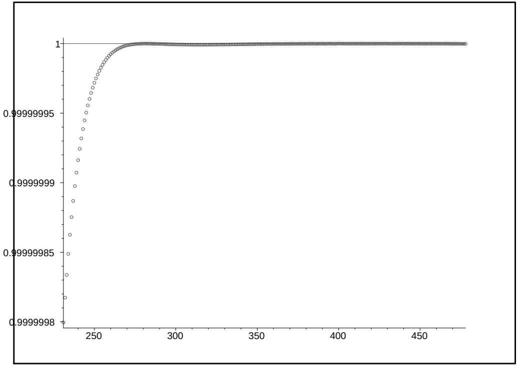

Figure 5: $G_n(zn)/G_n(\tilde{z})$, $n = 500$, as function of j

Now we must compute $S(\tilde{z})$ and its asymptotics. First we compute $\ln(\tilde{z} + i)$, take the asymptotics wrt x, sum on i, and again take the asymptotics wrt x (recall that $n = xy$). this leads to

$$S_1(\tilde{z}) = x\left[(-\ln(2) + 2\ln(y) + \ln(x))y - \frac{1}{3} + \frac{4}{405y^2} + \frac{2}{405y^3} + \ldots\right]$$
$$+ y - \frac{2}{3} - \frac{2}{3y} - \frac{49}{135y^2} + \ldots$$
$$+ \frac{1}{x}\left(\frac{y}{2} + \frac{1}{6y} + \ldots\right) + \frac{1}{x^2}\left(\frac{y}{3} + \ldots\right) + \mathcal{O}\left(\frac{y}{x^3}\right).$$

Here we provide only a few terms but Maple knows more. Next

$$
\begin{aligned}
S_2(\tilde{z}) = x \Bigg[& (\ln(2) - 2\ln(y) - \ln(x))y + \frac{4}{3} \\
& - \ln(2) + 2\ln(y) + \ln(x) - \frac{2}{3y} - \frac{94}{405y^2} + \dots \Bigg] \\
& - y + \frac{2}{3} + \ln(2) - 2\ln(y) - \ln(x) + \frac{1}{y} + \frac{94}{135y^2} + \dots \\
& + \frac{1}{x}\left(\frac{y}{2} + \frac{1}{6y} + \dots\right) + \frac{1}{x^2}\left(\frac{y}{3} + \dots\right) + \mathcal{O}\left(\frac{y}{x^3}\right).
\end{aligned}
$$

So, finally

$$
\begin{aligned}
S(\tilde{z}) \sim x & \left[1 - \ln(2) + 2\ln(y) + \ln(x) - \frac{2}{3y} + \dots\right] \\
& + \ln(2) - 2\ln(y) - \ln(x) + \frac{1}{3y} + \frac{1}{3y^2} + \dots \\
& + \frac{1}{x}\left(-\frac{1}{2} + \frac{1}{3y} - \frac{1}{2y^2} + \dots\right) \\
& + \frac{1}{x^2}\left(-\frac{1}{6} + \frac{19}{18y^2} \dots\right) \\
& + \mathcal{O}\left(\frac{1}{x^3}\right).
\end{aligned}
$$

Now we split $S(\tilde{z})$ into two parts:

$$T_1 = x\left[1 - \ln(2) + 2\ln(y) + \ln(x) - \frac{2}{3y} + \dots\right] + \ln(2) - 2\ln(y) - \ln(x),$$

$$T_2 = \frac{1}{3y} + \frac{1}{3y^2} + \dots + \frac{1}{x}\left(-\frac{1}{2} + \frac{1}{3y} - \frac{1}{2y^2} + \dots\right) + \frac{1}{x^2}\left(-\frac{1}{6} - \frac{17}{18y^2}\dots\right) + \mathcal{O}\left(\frac{1}{x^3}\right).$$

This leads to

$$\exp(S(\tilde{z})) = e^{T_1} e^{T_2} = e^{T_1} T_3,$$

with

$$\begin{aligned} T_3 = e^{T_2} = 1 &+ \frac{1}{3y} + \frac{7}{18y^2} + \frac{89}{270y^3} + \frac{18263}{3240y^4} \\ &+ \frac{98009}{3240y^5} + \frac{9517337}{97200y^6} + \frac{491504273}{2041200y^7} + \dots \\ &+ \frac{1}{x}\left(-\frac{1}{2} + \frac{1}{6y} - \frac{7}{12y^2} + \frac{2311}{540y^3} + \frac{112469}{6480y^4} + \frac{5137}{144y^5} + \dots\right) \\ &+ \frac{1}{x^2}\left(-\frac{1}{24} - \frac{13}{72y} - \frac{557}{932y^2} + \dots\right) \\ &+ \mathcal{O}\left(\frac{1}{x^3}\right). \end{aligned}$$

Here we have given all terms compatible with the expansion (4). Also, with more precision,

$$T_1 = x\left[1 - \ln(2) + 2\ln(y) + \ln(x) - \frac{2}{3y} - \frac{2}{9y^2} - \frac{44}{405y^3} - \frac{26}{405y^4} + \frac{40}{27y^5} + \frac{179968}{18225y^6} + \frac{4727552}{127575y^7} + \frac{3436796}{32805y^8} + \frac{5492621728}{22143375y^9} + \dots\right] + \ln(2) - 2\ln(y) - \ln(x).$$

Now we must consider $S^{(k)}(\tilde{z})$. By direct expansion, this gives the following expressions (again we provide only the first few terms). We must use up to six derivatives to get a sufficient precision (of order x^{-2}) in the Saddle integrals.

$$S^{(2)}(\tilde{z}) = \frac{1}{x}\left[\frac{4}{y^4} + \frac{16}{3y^5} + \ldots\right] + \frac{1}{x^2}\left[-\frac{12}{y^4} - \frac{40}{3y^5} + \ldots\right] + \frac{1}{x^3}\left[\frac{12}{y^4} + \frac{8}{y^5} + \ldots\right] + \frac{1}{x^4}\left[\frac{-4}{y^4} + \ldots\right] + \mathcal{O}\left(\frac{1}{x^5 y^4}\right), \quad (5)$$

$$
\begin{aligned}
S^{(3)}(\tilde{z}) &= \frac{1}{x^2}\left[-\frac{32}{y^6}+\dots\right] + \frac{1}{x^3}\left[\frac{128}{y^6}+\dots\right] + \frac{1}{x^4}\left[-\frac{192}{y^6}+\dots\right] \\
&\quad + \frac{1}{x^5}\left[\frac{128}{y^6}+\dots\right] + \mathcal{O}\left(\frac{1}{x^6 y^6}\right), \\
S^{(4)}(\tilde{z}) &= \frac{1}{x^3}\left[\frac{288}{y^8}+\dots\right] + \frac{1}{x^4}\left[-\frac{1440}{y^8}+\dots\right] + \frac{1}{x^5}\left[\frac{2880}{y^8}+\dots\right] \\
&\quad + \frac{1}{x^6}\left[-\frac{2880}{y^8}+\dots\right] + \mathcal{O}\left(\frac{1}{x^7 y^8}\right),
\end{aligned}
$$

$$S^{(5)}(\tilde{z}) = \frac{1}{x^4}\left[-\frac{3072}{y^{10}} + \dots\right] + \frac{1}{x^5}\left[\frac{18432}{y^{10}} + \dots\right] + \mathcal{O}\left(\frac{1}{x^6 y^{10}}\right),$$

$$S^{(6)}(\tilde{z}) = \frac{1}{x^5}\left[\frac{38400}{y^{12}} + \dots\right] + \frac{1}{x^6}\left[\frac{268800}{y^{12}} + \dots\right] + \mathcal{O}\left(\frac{1}{x^7 y^{12}}\right).$$

To check the quality of asymptotic (4), we give in Figure 6 the comparison between the expression (5) and $S^{(2)}(\tilde{z})$. The precision is of of order 10^{-2}. In a restricted range, given in Figure 7, the precision is of order 10^{-4}. $\alpha \leq 0.84$ in this range.

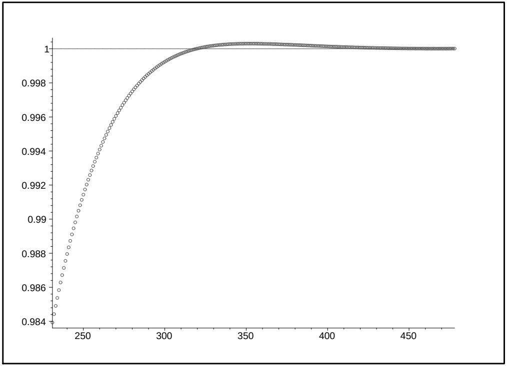

Figure 6: The quotient of the expression (5) and $S^{(2)}(\tilde{z})$, as function of j, $n = 500$

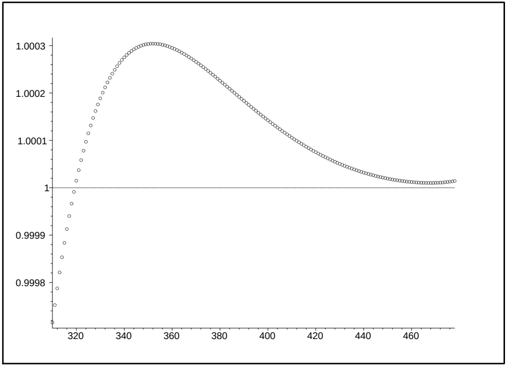

Figure 7: The quotient of the expression (5) and $S^{(2)}(\tilde{z})$, as function of j, $n = 500$. Restricted range, $\alpha < .84$

We proceed now as in Section 2. This leads to

$$\tau = \frac{y^2\sqrt{x}}{2}\left[ua_1 + \frac{u^2 a_2}{x^{1/2}} + \frac{u^3 a_3}{x} + \frac{u^4 a_4}{x^{3/2}} + \frac{u^5 a_5}{x^2} + \mathcal{O}\left(\frac{u^6}{x^{5/2}}\right)\right].$$

We give only a_1:

$$a_1 = 1 - \frac{2}{3y} - \frac{2}{9y^2} + \ldots + \frac{1}{x}\left(\frac{3}{2} - \frac{4}{3y} + \ldots\right) + \frac{1}{x^2}\left(\frac{15}{8} - \frac{7}{4y} + \ldots\right) + \mathcal{O}\left(\frac{1}{x^3}\right).$$

This leads to

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-u^2/2}\tau'(u)du = \frac{y^2\sqrt{x}}{2}T_4,$$

with

$$T_4 = 1 - \frac{2}{3y} - \frac{2}{9y^2} + \ldots + \frac{1}{x}\left(\frac{5}{12} - \frac{11}{18y} + \ldots\right) + \frac{1}{x^2}\left(\frac{73}{288} - \frac{133}{432y} + \ldots\right) + \frac{1}{x^3}\left(\frac{721}{576} + \ldots\right) + \mathcal{O}\left(\frac{1}{x^4}\right).$$

Set

$$T_5 := \frac{1}{\sqrt{2\pi}} \frac{y^2\sqrt{x}}{2} e^{T_1}.$$

This leads to

$$[z^j]\phi_n(z) \sim T_5 T_3 T_4. \tag{6}$$

We can of course combine T_3 and T_4:

$$
\begin{aligned}
T_6 := T_3 T_4 = 1 - \frac{3}{3y} - \frac{1}{18y^2} - \frac{1}{30y^3} + \frac{17207}{3240y^4} + \dots \\
+ \frac{1}{x}\left(-\frac{1}{12} + \frac{1}{36y} - \frac{35}{216y^2} + \frac{15029}{3240y^3} + \dots\right) \\
+ \frac{1}{x^2}\left(\frac{1}{288} - \frac{1}{864y} + \frac{3527}{5184y^2} + \dots\right) + \mathcal{O}\left(\frac{1}{x^3}\right).
\end{aligned}
$$

We have made several experiments with (6), with n up to 500. The result is unsatisfactory, only values of x of order $\sqrt{n}$ give reasonable results. Also using e^{T_2} instead of T_3 does not improve the precision. Actually, only very large values of n lead to good precision.

So we turn to another formulation: instead of using $e^{T_1} T_3$ for $e^{S(\tilde{z})}$, we plug directly $\tilde{z}$ into $G_n(z)$, ie we set

$$T_7 = G_n(\tilde{z}),$$

leading to

$$[z^j]\phi_n(z) \sim \frac{1}{\sqrt{2\pi}} \frac{y^2\sqrt{x}}{2} T_7 T_4 =: T_8 \text{ say }.$$

For $n = 500$, using two and three terms in $T4$, we give in Figures 8 and 9, the quotient $[z^j]\phi_n(z)/T_8$. The precision is of order 10^{-5}.

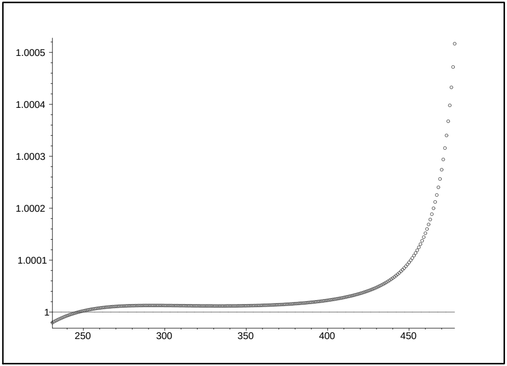

Figure 8: The quotient $[z^j]\phi_n(z)/T_8$, two terms in T_4, as function of j, $n = 500$

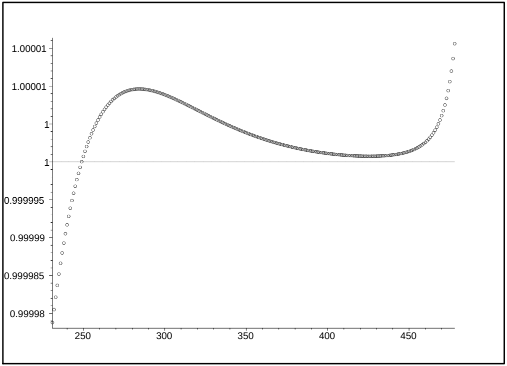

Figure 9: The quotient $[z^j]\phi_n(z)/T_8$, three terms in T_4, as function of j, $n = 500$

Conclusion

Using an almost mechanized program in Maple, we have obtained some asymptotic expressions for Stirling numbers in central and non-central regions. We intend to use these techniques in other non-central ranges.

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