DE-POISSONIZATION VIA SINGULARITY ANALYSIS

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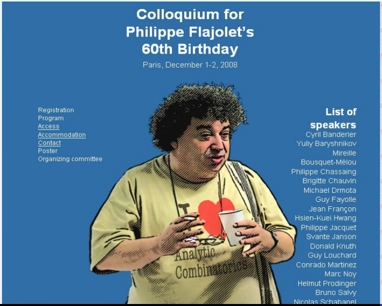
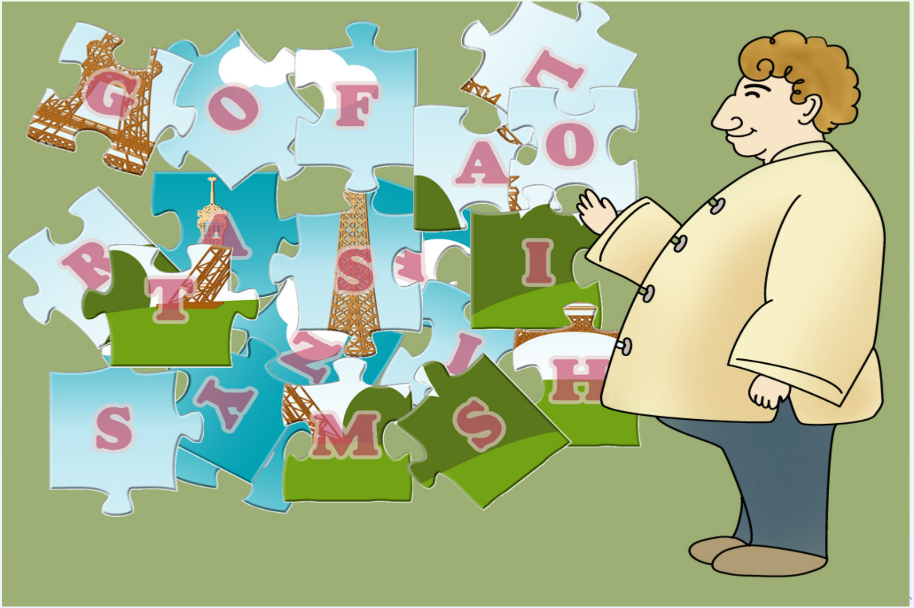
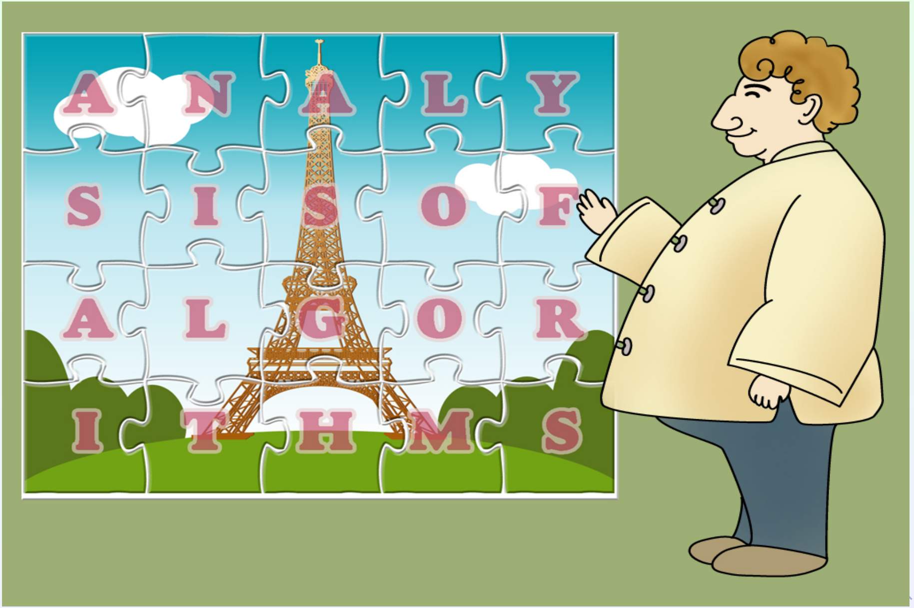


ANALYSIS OF ALGORITHMS: OLD DAYS



ANALYSIS OF ALGORITHMS: NOW



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Flajolet, Philippe

MR Author ID: **67375**
Earliest Indexed Publication: 1973
Total Publications: **138**
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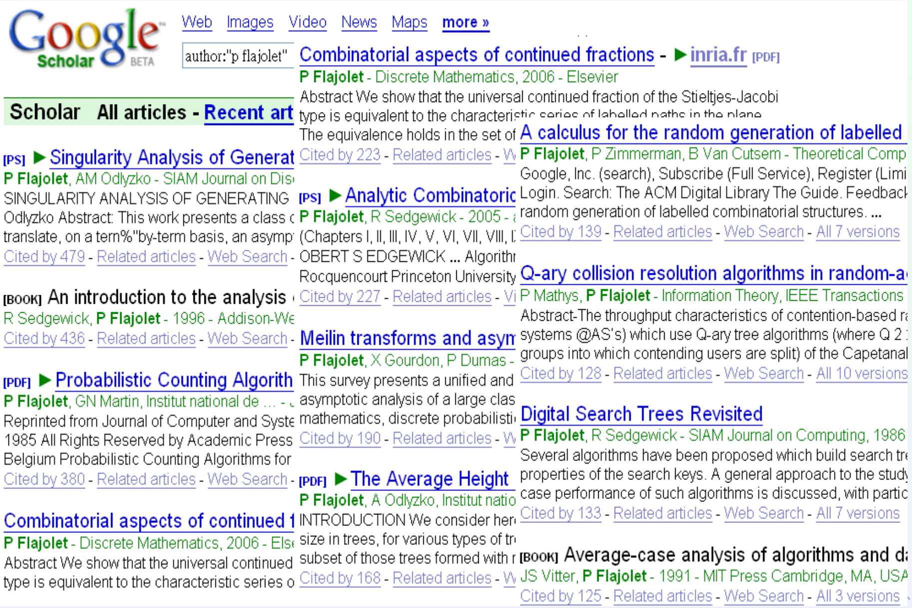

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59	**MR1337752 (96h:68093)** Flajolet, Philippe; Gourdon, Xavier; Dumas, Philippe Mellin transforms and asymptotics: harmonic sums. Special volume on mathematical analysis of algorithms. *Theoret. Comput. Sci.* 144 (1995), no. 1-2, 3--58. (Reviewer: Peter Kirschenhofer) 68Q25 (44A15 68P05)
53	**MR0592851 (82f:05002a)** Flajolet, P. Combinatorial aspects of continued fractions. *Discrete Math.* **32** (1980), no. 2, 125--161. (Reviewer: L. Carlitz) 05A10 (05A15 30B70)
41	**MR1884885 (2003c:05008)** Banderier, Cyril; Bousquet-Mélou, Mireille; Denise, Alain; Flajolet, Philippe; Gardy, Danièle; Gouyou-Beauchamps, Dominique Generating functions for generating trees. Formal power series and algebraic combinatorics (Barcelona, 1999). *Discrete Math.* 246 (2002), no. 1-3, 29--55. (Reviewer: Mark Curtis Wilson) 05A15 (05C05)
28	**MR1290534 (96f:05172)** Flajolet, Philippe; Zimmerman, Paul; Van Cutsem, Bernard A calculus for the random generation of labelled combinatorial structures. *Theoret. Comput. Sci.* 132 (1994), no. 1-2, 1--35. (Reviewer: Norbert Blum) 05C80 (68R05)
27	**MR1701625 (2000h:68056)** Flajolet, P.; Poblete, P.; Viola, A. On the analysis of linear probing hashing. Average-case analysis of algorithms. *Algorithmica* 22 (1998), no. 4, 490--515. (Reviewer: E. M. Reingold) 68P10 (60F05 68W40)

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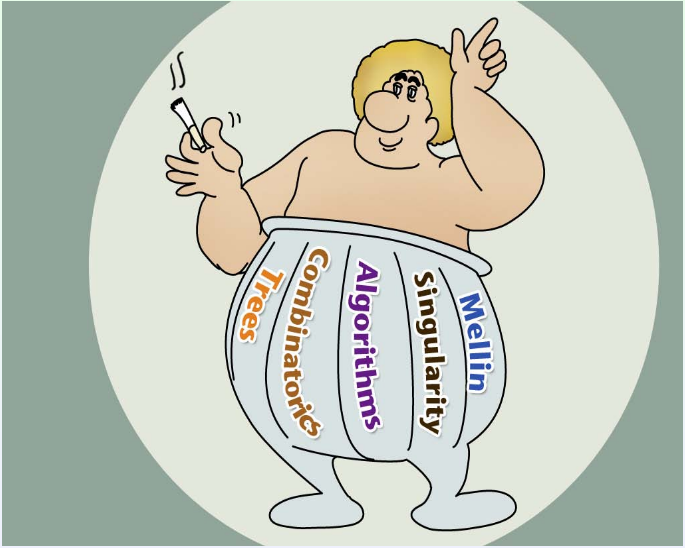
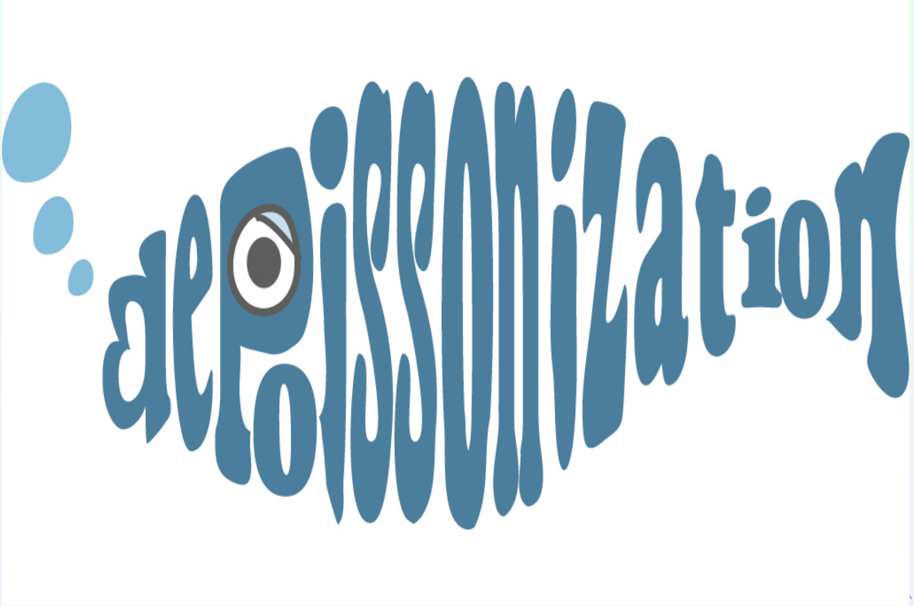


DE-POISSONIZATION



ANALYTIC DE-POISSONIZATION

Poisson GF: $\tilde{f}(z) := e^{-z} \sum_{n \geqslant 0} a_n z^n / n!$

If a_n doesn't grow too fast and doesn't vary too violently, then

$$\textbf{\textit{Poisson heuristic}}: a_n \approx \tilde{f}(n).$$

Widely used in diverse problems

Useful in Borel summability, Tauberian theorems, stochastic processes, statistics, statistical physics, analysis of algorithms, . . .

Dated back to at least Ramanujan's Notebooks (P1)

DE-POISSONIZATION: IDEAS

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A USEFUL OBSERVATION: A CHARLIER EXPANSION

Series expansion + term-by-term coeff

$$
\begin{aligned}
a_n &= n![z^n]e^z\tilde{f}(z) \\
&= \sum_{j\geqslant 0} \frac{\tilde{f}^{(j)}(n)}{j!} \underbrace{n![z^n](z-n)^j e^z}_{=:\tau_j(n)} \\
&= \sum_{j\geqslant 0} \frac{\tilde{f}^{(j)}(n)}{j!} \tau_j(n) .
\end{aligned}
$$

$$
\textbf{Charlier}_j(\lambda, n) := n!\lambda^{-j}[z^n](z-\lambda)^j e^z
$$

deg $\tau_j(n) = \lfloor j/2 \rfloor$ **and** $\{\tau_j(n)\}_{j\geqslant 0} =$

$\{1, 0, -n, 2n, 3n(n-2), -4n(5n-6), -5n(3n^2-26n+24), \dots\}$

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CHARLIER EXPANSION

An identity: If $\tilde{f}$ is *entire*, then

$$a_n = \sum_{j\geqslant 0} \frac{\tilde{f}^{(j)}(n)}{j!}\,\tau_j(n).$$

Nothing to do with *growth order* or *variation* or *smoothness* of $\tilde{f}(z)$ at infinity.

An example: $\tilde{f}(z) = e^{-2z}$

$$(-1)^n = e^{-2n} \sum_{j\geqslant 0} \frac{(-2)^j}{j!} \tau_j(n).$$

But $(-1)^n \not\sim e^{-2n}$.

Another example: $\tilde{f}(z) = e^{z}$

$$2^n = e^n \sum_{j\geqslant 0} \frac{\tau_j(n)}{j!}.$$

But $2^n \not\sim e^n$.

Major difficulty: prove the asymptotic nature

CHARLIER EXPANSION

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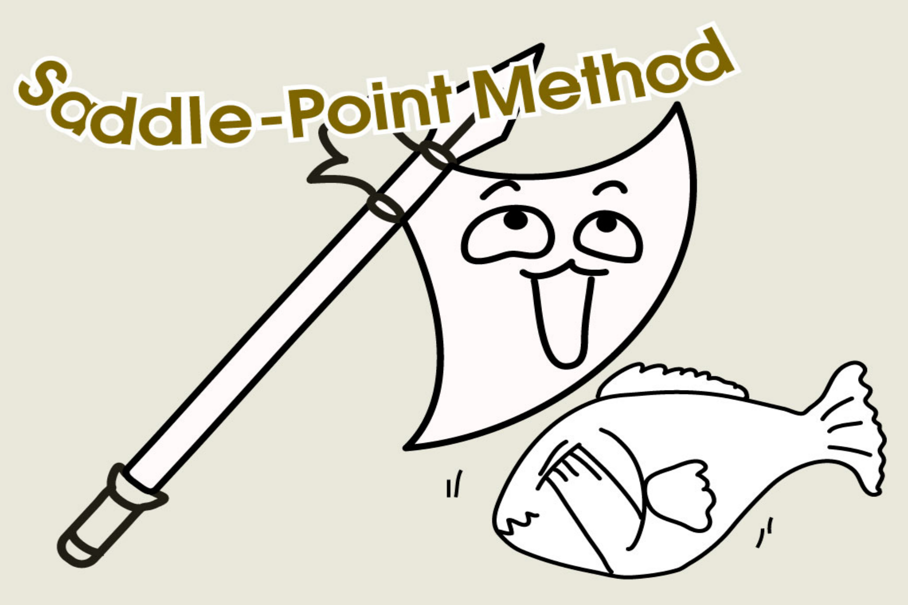
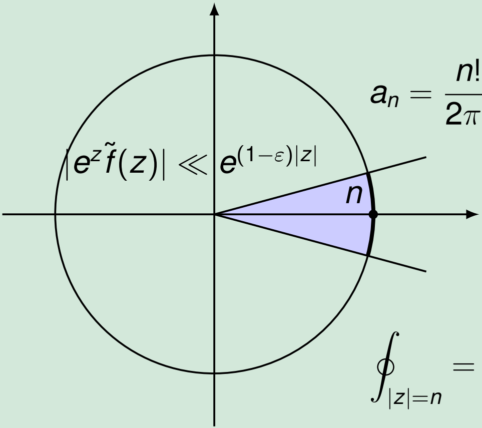

Saddle-Point Method

Analytic justification: Cauchy + saddle-point method



$$|e^z\tilde{f}(z)| \ll e^{(1-\varepsilon)|z|}$$

n

$$a_n = \frac{n!}{2\pi i}\oint_{|z|=n} z^{-n-1}e^z\tilde{f}(z)\,dz$$

$$|\tilde{f}(z)| \ll e^{|z|^{\frac{1}{2}-\varepsilon}}$$

$$\oint_{|z|=n} = \int_{\substack{|z|=n\\|\theta|\leqslant\varepsilon}} + \int_{\substack{|z|=n\\\varepsilon\leqslant|\theta|\leqslant\pi}}$$

Then $a_n \sim \tilde{f}(n)$.

HAYMAN ADMISSIBLE FUNCTIONS

Hayman (1956): A generalization of Stirling's formula

Analytic functions $f(z) = \sum_n a_n z^n/n!$ **for which the saddle-point method applies and** $(rf'(r)/f(r) = n)$

$$\frac{a_n}{n!} \sim \frac{r^{-n} f(r)}{\sqrt{2\pi\sigma^2(r)}}, \qquad \sigma^2(r) := \frac{rf'(r)}{f(r)} + \frac{r^2 f''(r)}{f(r)} - \left(\frac{rf'(r)}{f(r)}\right)^2.$$

Poisson admissible functions

Specialized to $f(z) = e^z \tilde{f}(z)$ **so that** $(r = n)$

$$a_n \sim n! \frac{n^{-n} e^n \tilde{f}(n)}{\sqrt{2\pi n}} \sim \tilde{f}(n).$$

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$$a_n \sim n! \frac{n^{-n} e^n \tilde{f}(n)}{\sqrt{2\pi n}} \sim \tilde{f}(n).$$

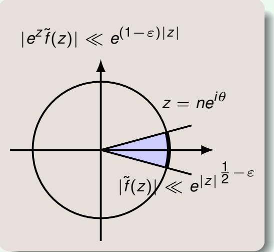
Similar to Wyman- and Harris-Schoenfeld-admissibility

$$a_n = \frac{n!}{2\pi i}\oint_{|z|=n} z^{-n-1}e^z \underbrace{\tilde{f}(z)}_{\textbf{Expand at } z=n} dz$$

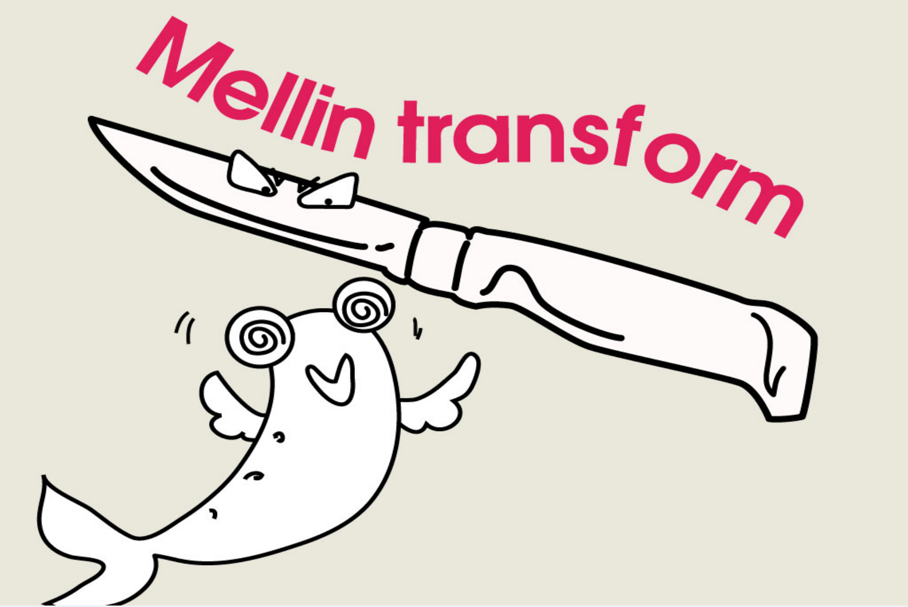
$$= \sum_{0\leqslant j\leqslant m} \frac{\tilde{f}^{(j)}(n)}{j!} \underbrace{\frac{n!}{2\pi i}\oint_{|z|=n} z^{-n-1}e^z(z-n)^j dz}_{=\tau_j(n)}$$

$$+ \frac{n!}{2\pi i}\oint_{|z|=n} z^{-n-1}e^z(z-n)^{m+1}\textbf{Taylor remainder}(z)dz$$

$$= \sum_{0\leqslant j\leqslant m} \frac{\tilde{f}^{(j)}(n)}{j!}\tau_j(n) + O\left(n^{(m+1)/2}\tilde{f}^{(m+1)}(n)\right).$$



Widely used


Mellin transform

Represent a_n as a Mellin inversion integral

$$a_n = \frac{n!}{2\pi i}\oint_{|z|=n} z^{-n-1} e^z \tilde{f}(z)\,dz$$

$$= \frac{n!}{2\pi i}\oint_{|z|=n} z^{-n-1} e^z \underbrace{\frac{1}{2\pi i}\int_{(\uparrow)} \mathscr{M}(s) z^{-s}\,ds}_{=\tilde{f}(z)}\,dz$$

$$= \frac{1}{2\pi i}\int_{(\uparrow)} \mathscr{M}(s)\frac{\Gamma(n+1)}{\Gamma(n+1+s)}\,ds.$$

More general than Rice, but less explored so far

THE POISSON-MELLIN-NEWTON CYCLE

Ph. Flajolet, R. Sedgewick / Theoretical Computer Science 144 (1995) 101–124

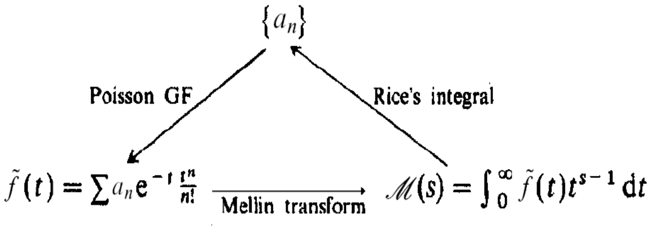


Fig. 3. The Poisson–Mellin–Newton cycle.

Poisson–Mellin–Newton Cycle. *The coefficients of a Poisson generating function are expressible as a Rice integral applied to the Mellin transform of the Poisson generating function.*

The Mellin transform is itself a Newton series

$$\begin{aligned}\mathscr{M}(s) &:= \int_0^\infty \tilde{f}(x) x^{s-1} dx \\ &= \sum_{j\geqslant 0} \frac{a_j}{j!}\Gamma(j+s) \\ &= \Gamma(s)\sum_{j\geqslant 0} a_j \binom{j+s-1}{j}.\end{aligned}$$

a_n as a finite difference

By taking successive difference

$$a_n = \sum_{0\leqslant j\leqslant n}\binom{n}{j}(-1)^j \frac{\mathscr{M}(-j)}{\Gamma(-j)}.$$

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a_n as a finite difference

By taking successive difference

$$a_n = \sum_{0\leqslant j\leqslant n}\binom{n}{j}(-1)^j\frac{\mathscr{M}(-j)}{\Gamma(-j)}.$$

The formal expansion

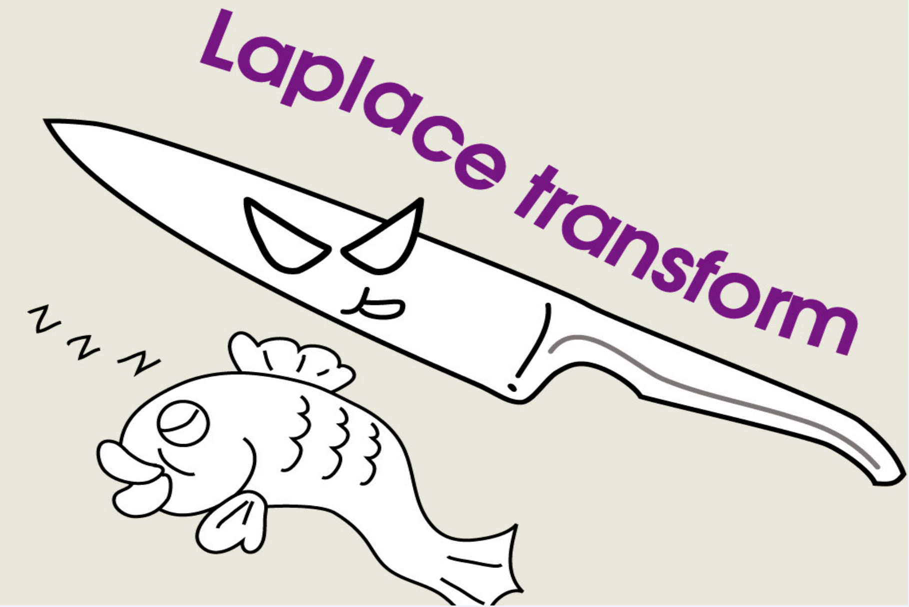
$$a_n = \frac{1}{2\pi i}\int_{(\uparrow)} \mathscr{M}(s)\, \underbrace{\frac{\Gamma(n+1)}{\Gamma(n+1+s)}}\, ds$$

$$\stackrel{?}{=} \sum_{j\geqslant 0} \frac{\tau_j(n)}{j!}\frac{(-1)^j}{2\pi i}\int_{(\uparrow)} \mathscr{M}(s) s\cdots(s+j-1) n^{-s-j}\, ds$$

$$\stackrel{?}{=} \sum_{j\geqslant 0} \frac{\tilde{f}^{(j)}(n)}{j!}\tau_j(n)$$

since $\tilde{f}^{(j)}(n) = \frac{(-1)^j}{2\pi i}\int_{(\uparrow)} \mathscr{M}(s) s\cdots(s+j-1) n^{-s-j}\, ds$**.**

An asymptotic theory can be developed


Laplace transform

DE-POISSONIZATION VIA LAPLACE TRANSFORM

The Laplace transform of $\tilde{f}(z)$

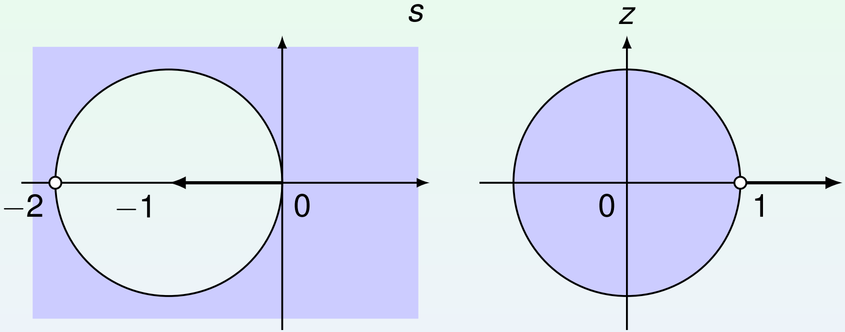
$$
\begin{aligned}
\mathscr{L}(s) &:= \int_0^\infty \tilde{f}(x) e^{-xs} dx \\
&= \sum_{j \geqslant 0} a_j (1+s)^{-j-1} \\
&= \frac{1}{1+s} F\left(\frac{1}{1+s}\right),
\end{aligned}
$$

where $F(z) := \sum_{j \geqslant 0} a_j z^j$ **denotes the OGF.**

Laplace = OGF + Euler transform (Flajolet-Richmond, 1992)

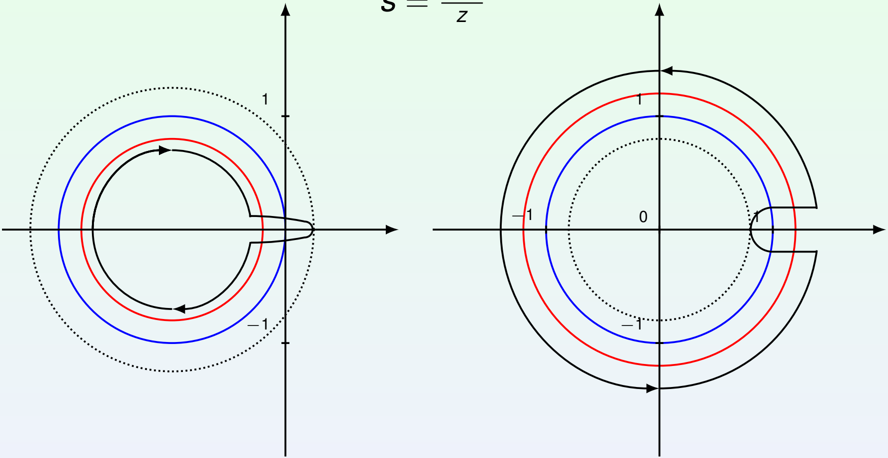
THE MAPPING $z = 1/(1+s)$

$$z = \frac{1}{1+s}$$



THE MAPPING $z = 1/(1+s)$

$$s = \frac{1-z}{z}$$



DE-POISSONIZATION VIA LAPLACE TRANSFORM

a_n represented as a Laplace inversion integral

The change of variables $z \mapsto 1/(1+s)$

$$
\begin{aligned}
a_n &= \frac{1}{2\pi i} \oint_{|z|=r<1} F(z) z^{-n-1} dz \\
&= \frac{1}{2\pi i} \oint_{|z|=R>1} F\left(\frac{1}{1+s}\right) (1+s)^{n-1} ds \\
&= \frac{1}{2\pi i} \oint_{|s|=R} \mathscr{L}(s)\, (1+s)^n\, ds.
\end{aligned}
$$

Laplace inversion for $\tilde{f}(n)$

$$
\tilde{f}(n) = \frac{1}{2\pi i} \int_{(\uparrow)} \mathscr{L}(s) e^{ns} ds = \frac{1}{2\pi i} \oint_{|s|=R>1} \mathscr{L}(s)\, e^{ns}\, ds.
$$

DE-POISSONIZATION VIA LAPLACE TRANSFORM

a_n represented as a Laplace inversion integral

The change of variables $z \mapsto 1/(1+s)$

$$\begin{aligned} a_n &= \frac{1}{2\pi i} \oint_{|z|=r<1} F(z) z^{-n-1} dz \\ &= \frac{1}{2\pi i} \oint_{|z|=R>1} F\left(\frac{1}{1+s}\right) (1+s)^{n-1} ds \\ &= \frac{1}{2\pi i} \oint_{|s|=R} \mathscr{L}(s) \, (1+s)^n \, ds. \end{aligned}$$

Laplace inversion for $\tilde{f}(n)$

$$\tilde{f}(n) = \frac{1}{2\pi i} \int_{(\uparrow)} \mathscr{L}(s) e^{ns} ds = \frac{1}{2\pi i} \oint_{|s|=R>1} \mathscr{L}(s) \, e^{ns} \, ds.$$

The formal expansion

An identity:

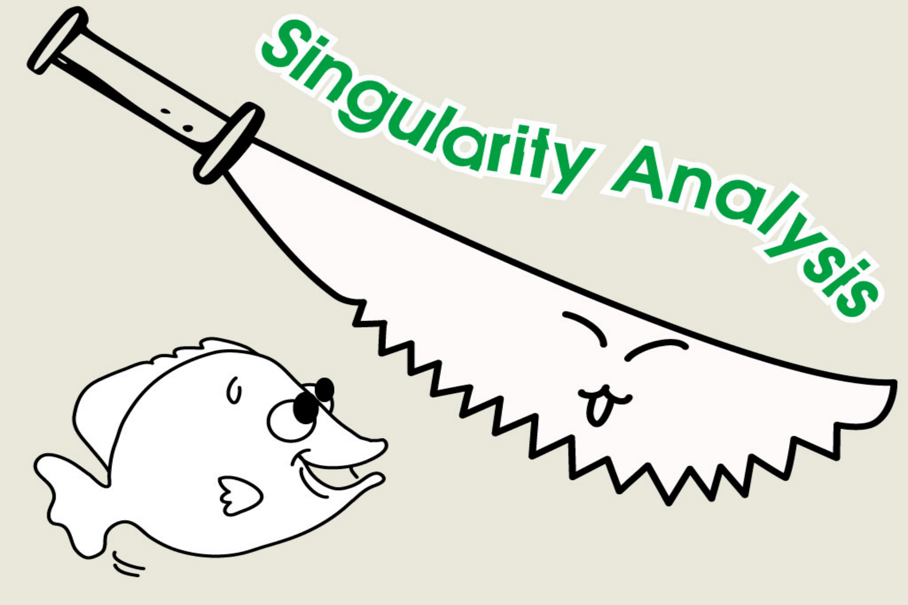
$$j![s^j](1+s)^n e^{ns} = n![z^n](z-n)^j e^z = \tau_j(n).$$

Thus $(1+s)^n = e^{ns}\sum_{j\geqslant 0}\tau_j(n)s^j/j!$ **and formally**

$$a_n = \sum_{j\geqslant 0}\frac{\tau_j(n)}{j!}\frac{1}{2\pi i}\oint_{|s|=R}\mathscr{L}(s)s^j e^{ns}\,ds = \sum_{j\geqslant 0}\frac{\tilde{f}^{(j)}(n)}{j!}\tau_j(n)$$

since $\tilde{f}^{(j)}(n) = \dfrac{1}{2\pi i}\oint_{|s|=R}\mathscr{L}(s)s^j e^{ns}\,ds$**.**

An asymptotic theory can be developed


Singularity Analysis

Connection between PGF and OGF

$$\tilde{f}(n) = \frac{1}{2\pi i} \oint_{|z|=r} \frac{F(z)}{z} e^{n(1-z)/z} dz.$$

Now use the expansion

$$z^{-n} = \left(1 + \frac{1-z}{z}\right)^n = e^{n(1-z)/z} \sum_{j \geqslant 0} \frac{\tau_j(n)}{j!} \left(\frac{1-z}{z}\right)^j$$

and obtain

$$a_n = \frac{1}{2\pi i} \oint_{|z|=r} \frac{F(z)}{z} z^{-n} dz$$
$$= \sum_{j \geqslant 0} \frac{\tilde{f}^{(j)}(n)}{j!} \tau_j(n).$$

A simple application to “large functions”

If $F(z) = O(|1-z|^{-\alpha})$ **for** $|z| < 1$**, where** $\alpha > 1$**, then**

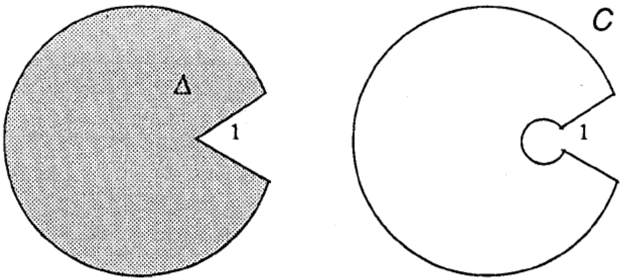
$$a_n = \tilde{f}(n) + \begin{cases} O(n^{(\alpha-1)/2}), & \alpha \in (1,3) \\ O(n \log n), & \alpha = 3 \\ O(n^{\alpha-2}), & \alpha > 3. \end{cases}$$

Flajolet-Odlyzko (1990): $a_n = O(n^{\alpha-1})$**.**

Flajolet-Odlyzko-admissible functions:

THEOREM 1. *Assume that, with the sole exception of the singularity $z = 1$, $F(z)$ is analytic in the domain $\Delta = \Delta(\phi, \eta)$, where $\eta > 0$ and $0 < \phi < (\pi/2)$. Assume further that as z tends to 1 in Δ,*

(2.6a) $$F(z) = O(|1-z|^{\alpha}),$$



FO-admissibility* $\Longrightarrow$ *JS-admissibility

A SUMMARY AND COMPARISONS

$\int$-representations for a_n

$$a_n = \frac{1}{2\pi i}\oint_{|z|=n} e^z \frac{n!}{z^{n+1}} \tilde{f}(z)\,dz$$

$$= \frac{1}{2\pi i}\int_{\uparrow} \mathscr{M}(s) \frac{\Gamma(n+1)}{\Gamma(n+1+s)}\,ds$$

$$= \frac{1}{2\pi i}\oint_{|s+1|=R>1} \mathscr{L}(s)\,(1+s)^n\,ds$$

$$= \frac{1}{2\pi i}\oint_{|z|=r<1} \frac{F(z)}{z}\,z^{-n}\,dz$$

$\int$-representations for $\tilde{f}(n)$

$$\tilde{f}(n) = \sum_{j\geqslant 0} e^{-n}\frac{n^j}{j!}\,a_j$$

$$= \frac{1}{2\pi i}\int_{\uparrow} \mathscr{M}(s)\,n^{-s}\,ds$$

$$= \frac{1}{2\pi i}\oint_{|s+1|=R>1} \mathscr{L}(s)\,e^{ns}\,ds$$

$$= \frac{1}{2\pi i}\oint_{|z|=r<1} \frac{F(z)}{z}\,e^{n(1-z)/z}\,dz$$

Finite **domain vs** ***infinite*** **domain;** ***polynomial*** **growth vs** ***exponential*** **growth;** ***saddle-point method*** **vs** ***singularity analysis*****,** ***identity*** **vs** ***asymptotic*****, connections between admissibilities, closure properties, . . .**

TWO APPLICATIONS

Cost of exhaustive search for MIS in $G_{n,p}$

Banderier, H., Ravelomanana and Zacharovas (2008+):

$$\tilde{f}'(z) = \tilde{f}(qz) + e^{-z}.$$

Use Laplace or singularity analysis (Mellin fails since $\log \tilde{f}(x) \asymp (\log x)^2$**).**

Variance of TPL of random unbiased b-DST

Fuchs, H., Zacharovas (2008+): Much simpler expressions for the average value of the periodic function

$$\sum_{0 \leqslant j \leqslant b} \binom{b}{j} \tilde{f}^{(j)}(z) = 2\tilde{f}(z/2) + \tilde{g}(z).$$

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