


DE-POISSONIZATION VIA SINGULARITY ANALYSIS

Hsien-Kuei Hwang (joint with Vytas Zacharovas)

Institute of Statistical Science, Academia Sinica, Taiwan

Colloquium for
Philippe Flajolet's
60th Birthday
Paris, December 1-2, 2008

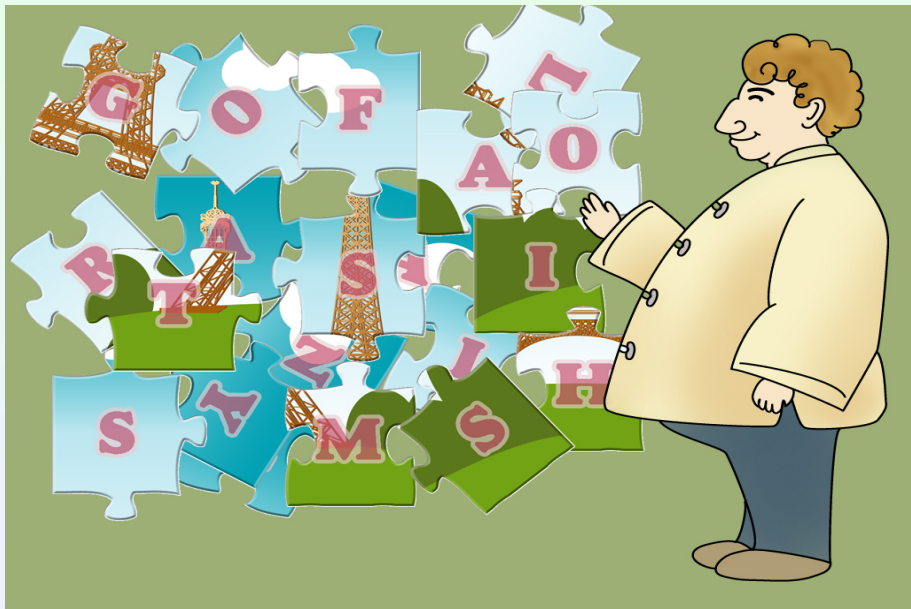
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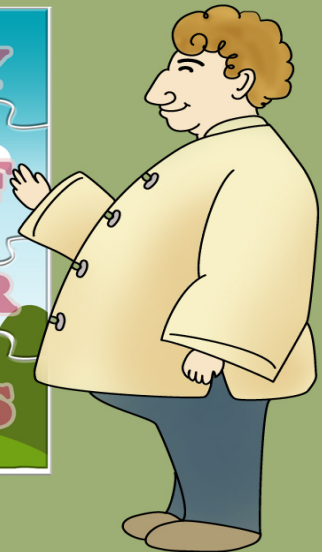
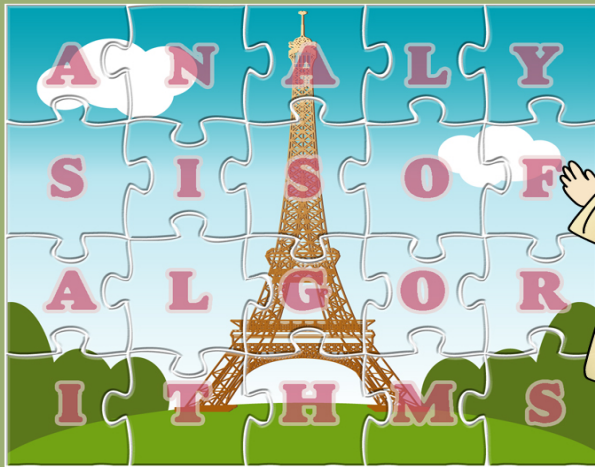
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Yuliy Baryshnikov
Berthe
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Philippe Chassaing
Brigitte Chauvin
Michael Demota
Guy Fayolle
Jean Flajolet
Hsien-Kuei Hwang
Philippe Jacquet
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Helmut Prodinger
Bruno Salvy
Vytas Zacharovas



ANALYSIS OF ALGORITHMS: OLD DAYS



ANALYSIS OF ALGORITHMS: NOW





Flajolet, Philippe

MR Author ID: **67375**

Earliest Indexed Publication: [1973](#)

Total Publications: **138**

Total Author/Related Publications: **144**

Total Citations: **1086**

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Top 50 Co-authors (by number of collaborations)

Banderier, Cyril Clément, Julien Coffman, Edward G., Jr. Daudé, Hervé Denise, Alain Devroye, Luc P. Duchon, Philippe
 Dumas, Philippe Fayolle, Guy Flatto, Leopold Françon, Jean Gabarró, Joaquim **Gardy, Danièle** Golin, Mordecai J.
 Gonnet, Gaston H. **Gourdon, Xavier** Gouyou-Beauchamps, Dominique Grabner, Peter J. Hofri, Micha Hurtado, Ferran
 Jacquet, Philippe Kirschenhofer, Peter Lafforgue, T. Louchard, Guy Nebel, Markus E. Nicodème, Pierre
 Nikolettseas, Sotiris E. Noy, Marc **Odlyzko, Andrew M.** Panario, Daniel Pouyanne, Nicolas **Prodinger, Helmut**
Puech, Claude Raoult, Jean-Claude Régnier, Mireille Richmond, L. Bruce Robson, John Michael Saheb-Djahromi, Nasser
Salvy, Bruno Schaeffer, Gilles Sedgewick, Robert Soria, Michèle **Steyaert, Jean-Marc**
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P Flajolet, P Zimmerman, B Van Cutsem - *Theoretical Comp*

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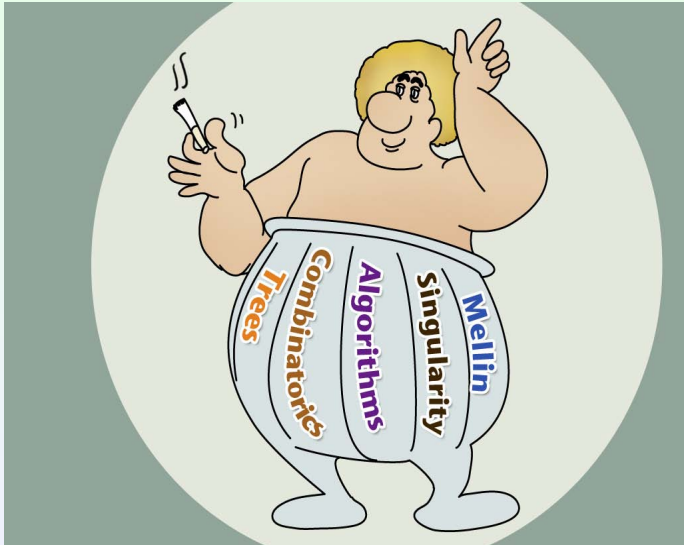
Most Cited Publications

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| 124 | MR1039294 (90m:05012) Flajolet, Philippe; Odlyzko, Andrew Singularity analysis of generating functions. <i>SIAM J. Discrete Math.</i> 3 (1990), no. 2, 216--240. (Reviewer: E. Rodney Canfield) 05A15 (30E20 40E05 41A60) |
| 59 | MR1337752 (96h:68093) Flajolet, Philippe; Gourdon, Xavier; Dumas, Philippe Mellin transforms and asymptotics: harmonic sums. Special volume on mathematical analysis of algorithms. <i>Theoret. Comput. Sci.</i> 144 (1995), no. 1-2, 3--58. (Reviewer: Peter Kirschenhofer) 68Q25 (44A15 68P05) |
| 53 | MR0592851 (82f:05002a) Flajolet, P. Combinatorial aspects of continued fractions. <i>Discrete Math.</i> 32 (1980), no. 2, 125--161. (Reviewer: L. Carlitz) 05A10 (05A15 30B70) |
| 41 | MR1884885 (2003c:05008) Banderier, Cyril; Bousquet-Mélou, Mireille; Denise, Alain; Flajolet, Philippe; Gardy, Danièle; Gouyou-Beauchamps, Dominique Generating functions for generating trees. Formal power series and algebraic combinatorics (Barcelona, 1999). <i>Discrete Math.</i> 246 (2002), no. 1-3, 29--55. (Reviewer: Mark Curtis Wilson) 05A15 (05C05) |
| 28 | MR1290534 (96f:05172) Flajolet, Philippe; Zimmerman, Paul; Van Cutsem, Bernard A calculus for the random generation of labelled combinatorial structures. <i>Theoret. Comput. Sci.</i> 132 (1994), no. 1-2, 1--35. (Reviewer: Norbert Blum) 05C80 (68R05) |
| 27 | MR1701625 (2000h:68056) Flajolet, P.; Poblete, P.; Viola, A. On the analysis of linear probing hashing. Average-case analysis of algorithms. <i>Algorithmica</i> 22 (1998), no. 4, 490--515. (Reviewer: E. M. Reingold) 68P10 (60F05 68W40) |

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depoissonization

DE-POISSONIZATION: IDEAS

Poisson GF: $\tilde{f}(z) := e^{-z} \sum_{n \geq 0} a_n z^n / n!$

If a_n doesn't grow too fast and doesn't vary too violently, then

Poisson heuristic: $a_n \approx \tilde{f}(n)$.

Widely used in diverse problems

Useful in Borel summability, Tauberian theorems, stochastic processes, statistics, statistical physics, analysis of algorithms, ...

Dated back to at least Ramanujan's Notebooks (P1)

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A USEFUL OBSERVATION: A CHARLIER EXPANSION

Series expansion + term-by-term coeff

$$\begin{aligned} a_n &= n! [z^n] e^z \tilde{f}(z) \\ &= \sum_{j \geq 0} \frac{\tilde{f}^{(j)}(n)}{j!} \underbrace{n! [z^n] (z-n)^j e^z}_{=: \tau_j(n)} \\ &= \sum_{j \geq 0} \frac{\tilde{f}^{(j)}(n)}{j!} \tau_j(n). \end{aligned}$$

$$\text{Charlier}_j(\lambda, n) := n! \lambda^{-j} [z^n] (z - \lambda)^j e^z$$

$\deg \tau_j(n) = \lfloor j/2 \rfloor$ and $\{\tau_j(n)\}_{j \geq 0} =$

$\{1, 0, -n, 2n, 3n(n-2), -4n(5n-6), -5n(3n^2-26n+24), \dots\}$

A USEFUL OBSERVATION: A CHARLIER EXPANSION

Series expansion + term-by-term coeff

$$\begin{aligned} a_n &= n! [z^n] e^z \tilde{f}(z) \\ &= \sum_{j \geq 0} \frac{\tilde{f}^{(j)}(n)}{j!} \underbrace{n! [z^n] (z-n)^j e^z}_{=: \tau_j(n)} \\ &= \sum_{j \geq 0} \frac{\tilde{f}^{(j)}(n)}{j!} \tau_j(n). \end{aligned}$$

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$$\{1, 0, -n, 2n, 3n(n-2), -4n(5n-6), -5n(3n^2-26n+24), \dots\}$$

CHARLIER EXPANSION

An identity: If \tilde{f} is *entire*, then

$$a_n = \sum_{j \geq 0} \frac{\tilde{f}^{(j)}(n)}{j!} \tau_j(n).$$

Nothing to do with *growth order* or *variation* or *smoothness* of $\tilde{f}(z)$ at infinity.

CHARLIER EXPANSION

An example: $\tilde{f}(z) = e^{-2z}$

$$(-1)^n = e^{-2n} \sum_{j \geq 0} \frac{(-2)^j}{j!} \tau_j(n).$$

But $(-1)^n \not\sim e^{-2n}$.

Another example: $\tilde{f}(z) = e^z$

$$2^n = e^n \sum_{j \geq 0} \frac{\tau_j(n)}{j!}.$$

But $2^n \not\sim e^n$.

Major difficulty: prove the asymptotic nature

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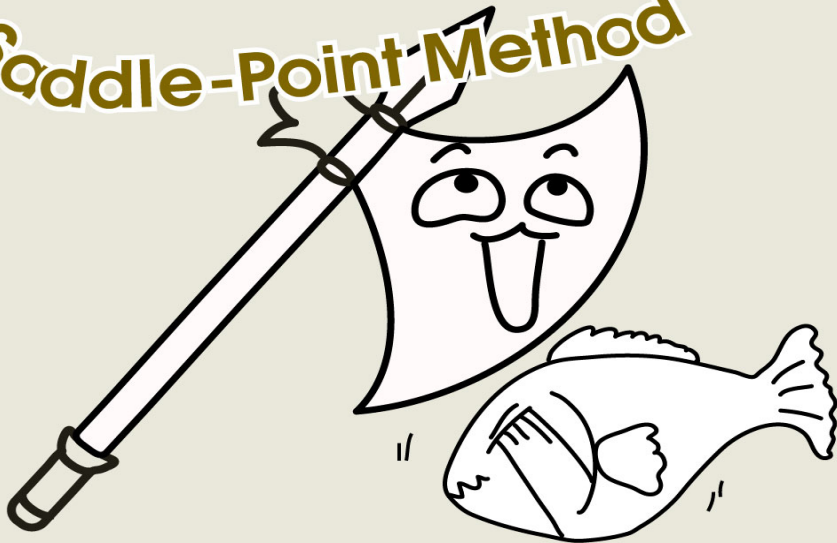
Another example: $\tilde{f}(z) = e^z$

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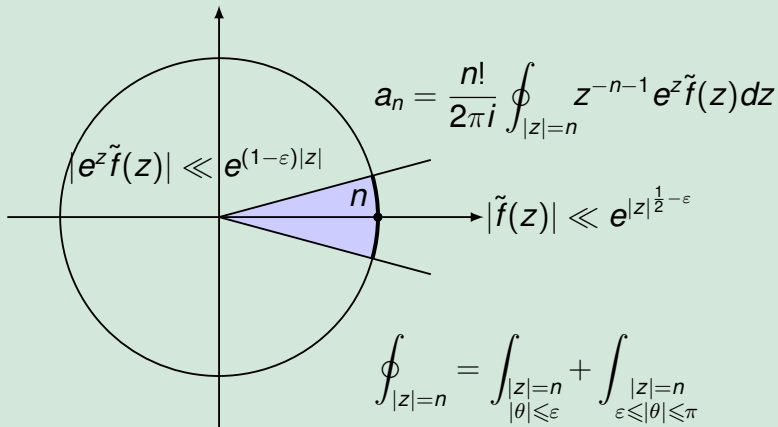
Major difficulty: prove the asymptotic nature

Saddle-Point Method



ANALYTIC DE-POISSONIZATION: IDEAS

Analytic justification: Cauchy + saddle-point method



Then $a_n \sim \tilde{f}(n)$.

HAYMAN ADMISSIBLE FUNCTIONS

Hayman (1956): A generalization of Stirling's formula

Analytic functions $f(z) = \sum_n a_n z^n / n!$ **for which the saddle-point method applies and** $(rf'(r)/f(r) = n)$

$$\frac{a_n}{n!} \sim \frac{r^{-n} f(r)}{\sqrt{2\pi\sigma^2(r)}}, \quad \sigma^2(r) := \frac{rf'(r)}{f(r)} + \frac{r^2 f''(r)}{f(r)} - \left(\frac{rf'(r)}{f(r)} \right)^2.$$

Poisson admissible functions

Specialized to $f(z) = e^z \tilde{f}(z)$ so that $(r = n)$

$$a_n \sim n! \frac{n^{-n} e^n \tilde{f}(n)}{\sqrt{2\pi n}} \sim \tilde{f}(n).$$

HAYMAN ADMISSIBLE FUNCTIONS

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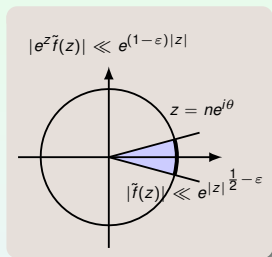
Poisson admissible functions

Specialized to $f(z) = e^z \tilde{f}(z)$ **so that** $(r = n)$

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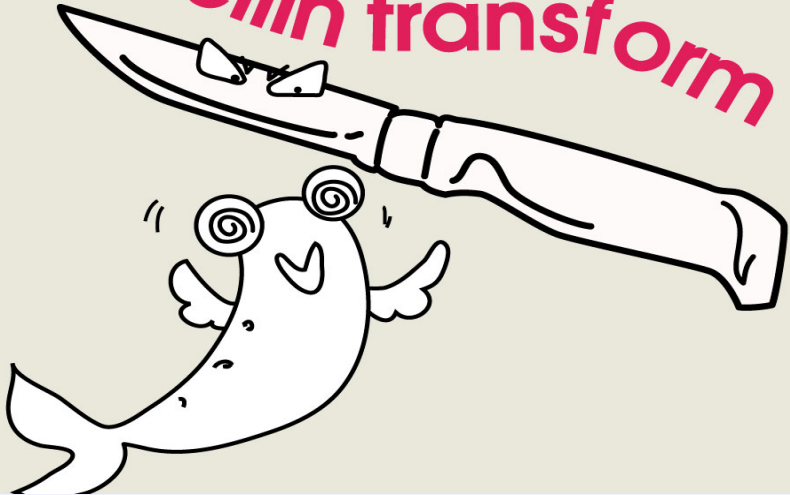
Similar to Wyman- and Harris-Schoenfeld-admissibility

$$\begin{aligned}
 a_n &= \frac{n!}{2\pi i} \oint_{|z|=n} z^{-n-1} e^z \underbrace{\tilde{f}(z)}_{\text{Expand at } z=n} dz \\
 &= \sum_{0 \leq j \leq m} \frac{\tilde{f}^{(j)}(n)}{j!} \underbrace{\frac{n!}{2\pi i} \oint_{|z|=n} z^{-n-1} e^z (z-n)^j dz}_{=\tau_j(n)} \\
 &+ \frac{n!}{2\pi i} \oint_{|z|=n} z^{-n-1} e^z (z-n)^{m+1} \mathbf{Taylor\ remainder}(z) dz \\
 &= \sum_{0 \leq j \leq m} \frac{\tilde{f}^{(j)}(n)}{j!} \tau_j(n) + O\left(n^{(m+1)/2} \tilde{f}^{(m+1)}(n)\right).
 \end{aligned}$$



Widely used

Mellin transform



Represent a_n as a Mellin inversion integral

$$\begin{aligned}
 a_n &= \frac{n!}{2\pi i} \oint_{|z|=n} z^{-n-1} e^z \tilde{f}(z) dz \\
 &= \frac{n!}{2\pi i} \oint_{|z|=n} z^{-n-1} e^z \underbrace{\frac{1}{2\pi i} \int_{(\uparrow)} \mathcal{M}(s) z^{-s} ds}_{=\tilde{f}(z)} dz \\
 &= \frac{1}{2\pi i} \int_{(\uparrow)} \mathcal{M}(s) \frac{\Gamma(n+1)}{\Gamma(n+1+s)} ds.
 \end{aligned}$$

More general than Rice, but less explored so far

THE POISSON-MELLIN-NEWTON CYCLE

Ph. Flajolet, R. Sedgewick / Theoretical Computer Science 144 (1995) 101-124

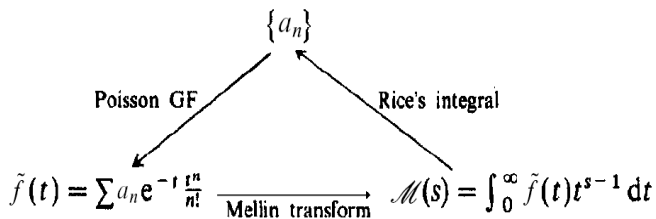


Fig. 3. The Poisson-Mellin-Newton cycle.

Poisson-Mellin-Newton Cycle. The coefficients of a **Poisson generating function** are expressible as a **Rice integral** applied to the **Mellin transform** of the Poisson generating function.

THE POISSON-MELLIN-NEWTON CYCLE

The Mellin transform is itself a Newton series

$$\begin{aligned}\mathcal{M}(s) &:= \int_0^{\infty} \tilde{f}(x) x^{s-1} dx \\ &= \sum_{j \geq 0} \frac{a_j}{j!} \Gamma(j+s) \\ &= \Gamma(s) \sum_{j \geq 0} a_j \binom{j+s-1}{j}.\end{aligned}$$

a_n as a finite difference

By taking successive difference

$$a_n = \sum_{0 \leq j \leq n} \binom{n}{j} (-1)^j \frac{\mathcal{M}(-j)}{\Gamma(-j)}.$$

THE POISSON-MELLIN-NEWTON CYCLE

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By taking successive difference

$$a_n = \sum_{0 \leq j \leq n} \binom{n}{j} (-1)^j \frac{\mathcal{M}(-j)}{\Gamma(-j)}.$$

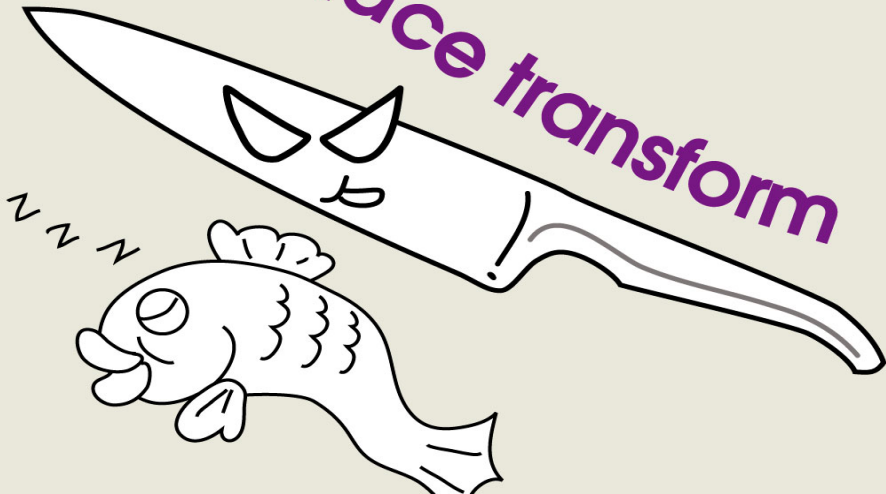
The formal expansion

$$\begin{aligned}
 a_n &= \frac{1}{2\pi i} \int_{(\uparrow)} \mathcal{M}(s) \underbrace{\frac{\Gamma(n+1)}{\Gamma(n+1+s)}}_{\text{}} ds \\
 &\stackrel{?}{=} \sum_{j \geq 0} \frac{\tau_j(n)}{j!} \frac{(-1)^j}{2\pi i} \int_{(\uparrow)} \mathcal{M}(s) s \cdots (s+j-1) n^{-s-j} ds \\
 &\stackrel{?}{=} \sum_{j \geq 0} \frac{\tilde{f}^{(j)}(n)}{j!} \tau_j(n)
 \end{aligned}$$

since $\tilde{f}^{(j)}(n) = \frac{(-1)^j}{2\pi i} \int_{(\uparrow)} \mathcal{M}(s) s \cdots (s+j-1) n^{-s-j} ds.$

An asymptotic theory can be developed

Laplace transform



The Laplace transform of $\tilde{f}(z)$

$$\begin{aligned}\mathcal{L}(s) &:= \int_0^{\infty} \tilde{f}(x) e^{-xs} dx \\ &= \sum_{j \geq 0} a_j (1+s)^{-j-1} \\ &= \frac{1}{1+s} F\left(\frac{1}{1+s}\right),\end{aligned}$$

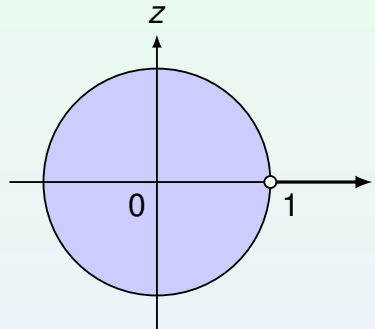
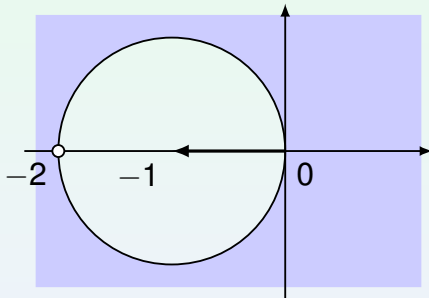
where $F(z) := \sum_{j \geq 0} a_j z^j$ denotes the OGF.

Laplace = OGF + Euler transform (Flajolet-Richmond, 1992)

THE MAPPING $z = 1/(1+s)$

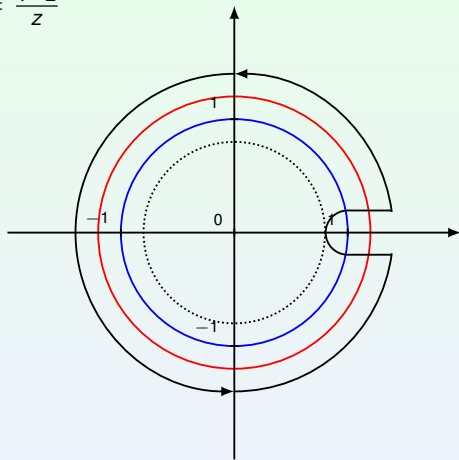
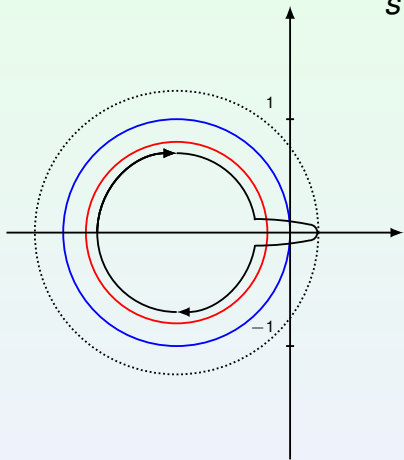
$$z = \frac{1}{1+s}$$

s



THE MAPPING $z = 1/(1 + s)$

$$s = \frac{1-z}{z}$$



a_n represented as a Laplace inversion integral

The change of variables $z \mapsto 1/(1+s)$

$$\begin{aligned} a_n &= \frac{1}{2\pi i} \oint_{|z|=r<1} F(z) z^{-n-1} dz \\ &= \frac{1}{2\pi i} \oint_{|z|=R>1} F\left(\frac{1}{1+s}\right) (1+s)^{n-1} ds \\ &= \frac{1}{2\pi i} \oint_{|s|=R} \mathcal{L}(s) (1+s)^n ds. \end{aligned}$$

Laplace inversion for $\tilde{f}(n)$

$$\tilde{f}(n) = \frac{1}{2\pi i} \int_{(\uparrow)} \mathcal{L}(s) e^{ns} ds = \frac{1}{2\pi i} \oint_{|s|=R>1} \mathcal{L}(s) e^{ns} ds.$$

a_n represented as a Laplace inversion integral

The change of variables $z \mapsto 1/(1+s)$

$$\begin{aligned} a_n &= \frac{1}{2\pi i} \oint_{|z|=r<1} F(z) z^{-n-1} dz \\ &= \frac{1}{2\pi i} \oint_{|z|=R>1} F\left(\frac{1}{1+s}\right) (1+s)^{n-1} ds \\ &= \frac{1}{2\pi i} \oint_{|s|=R} \mathcal{L}(s) (1+s)^n ds. \end{aligned}$$

Laplace inversion for $\tilde{f}(n)$

$$\tilde{f}(n) = \frac{1}{2\pi i} \int_{(\uparrow)} \mathcal{L}(s) e^{ns} ds = \frac{1}{2\pi i} \oint_{|s|=R>1} \mathcal{L}(s) e^{ns} ds.$$

The formal expansion

An identity:

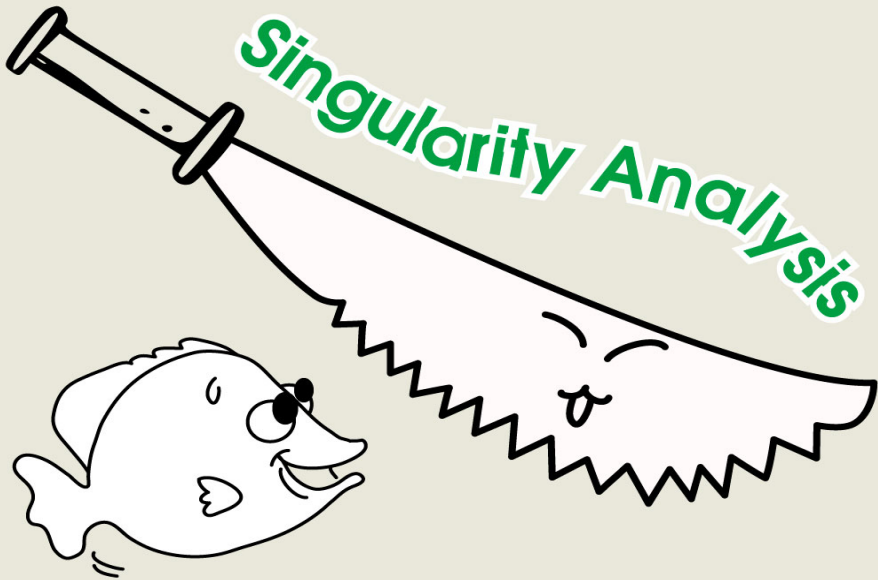
$$j![s^j](1+s)^n e^{ns} = n![z^n](z-n)^j e^z = \tau_j(n).$$

Thus $(1+s)^n = e^{ns} \sum_{j \geq 0} \tau_j(n) s^j / j!$ and formally

$$a_n = \sum_{j \geq 0} \frac{\tau_j(n)}{j!} \frac{1}{2\pi i} \oint_{|s|=R} \mathcal{L}(s) s^j e^{ns} ds = \sum_{j \geq 0} \frac{\tilde{f}^{(j)}(n)}{j!} \tau_j(n)$$

since $\tilde{f}^{(j)}(n) = \frac{1}{2\pi i} \oint_{|s|=R} \mathcal{L}(s) s^j e^{ns} ds.$ **An asymptotic theory can be developed**

Singularity Analysis



Connection between PGF and OGF

$$\tilde{f}(n) = \frac{1}{2\pi i} \oint_{|z|=r} \frac{F(z)}{z} e^{n(1-z)/z} dz.$$

Now use the expansion

$$z^{-n} = \left(1 + \frac{1-z}{z}\right)^n = e^{n(1-z)/z} \sum_{j \geq 0} \frac{\tau_j(n)}{j!} \left(\frac{1-z}{z}\right)^j$$

and obtain

$$\begin{aligned} a_n &= \frac{1}{2\pi i} \oint_{|z|=r} \frac{F(z)}{z} z^{-n} dz \\ &= \sum_{j \geq 0} \frac{\tilde{f}^{(j)}(n)}{j!} \tau_j(n). \end{aligned}$$

A simple application to “large functions”

If $F(z) = O(|1 - z|^{-\alpha})$ for $|z| < 1$, where $\alpha > 1$, then

$$a_n = \tilde{f}(n) + \begin{cases} O(n^{(\alpha-1)/2}), & \alpha \in (1, 3) \\ O(n \log n), & \alpha = 3 \\ O(n^{\alpha-2}), & \alpha > 3. \end{cases}$$

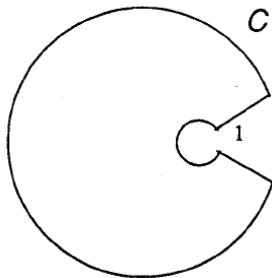
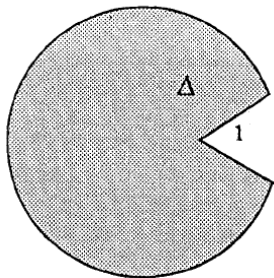
Flajolet-Odlyzko (1990): $a_n = O(n^{\alpha-1})$.

Flajolet-Odlyzko-admissible functions:

THEOREM 1. Assume that, with the sole exception of the singularity $z = 1$, $F(z)$ is analytic in the domain $\Delta = \Delta(\phi, \eta)$, where $\eta > 0$ and $0 < \phi < (\pi/2)$. Assume further that as z tends to 1 in Δ ,

(2.6a)

$$F(z) = O(|1 - z|^\alpha),$$



FO-admissibility \implies JS-admissibility

A SUMMARY AND COMPARISONS

\int -representations for a_n

$$\begin{aligned} a_n &= \frac{1}{2\pi i} \oint_{|z|=n} e^z \frac{n!}{z^{n+1}} \tilde{f}(z) dz \\ &= \frac{1}{2\pi i} \int_{\uparrow} \mathcal{M}(s) \frac{\Gamma(n+1)}{\Gamma(n+1+s)} ds \\ &= \frac{1}{2\pi i} \oint_{|s+1|=R>1} \mathcal{L}(s) (1+s)^n ds \\ &= \frac{1}{2\pi i} \oint_{|z|=r<1} \frac{F(z)}{z} z^{-n} dz \end{aligned}$$

\int -representations for $\tilde{f}(n)$

$$\begin{aligned} \tilde{f}(n) &= \sum_{j \geq 0} e^{-n} \frac{n^j}{j!} a_j \\ &= \frac{1}{2\pi i} \int_{\uparrow} \mathcal{M}(s) n^{-s} ds \\ &= \frac{1}{2\pi i} \oint_{|s+1|=R>1} \mathcal{L}(s) e^{ns} ds \\ &= \frac{1}{2\pi i} \oint_{|z|=r<1} \frac{F(z)}{z} e^{n(1-z)/z} dz \end{aligned}$$

Finite domain vs infinite domain; polynomial growth vs exponential growth; saddle-point method vs singularity analysis, identity vs asymptotic, connections between admissibilities, closure properties, ...

TWO APPLICATIONS

Cost of exhaustive search for MIS in $G_{n,p}$

Banderier, H., Ravelomanana and Zacharovas (2008+):

$$\tilde{f}'(z) = \tilde{f}(qz) + e^{-z}.$$

Use Laplace or singularity analysis (Mellin fails since $\log \tilde{f}(x) \asymp (\log x)^2$).

Variance of TPL of random unbiased b-DST

Fuchs, H., Zacharovas (2008+): Much simpler expressions for the average value of the periodic function

$$\sum_{0 \leq j \leq b} \binom{b}{j} \tilde{f}^{(j)}(z) = 2\tilde{f}(z/2) + \tilde{g}(z).$$

BON ANNIVERSAIRE À PHILIPPE

