What kind of laws come from urns?

Brigitte CHAUVIN

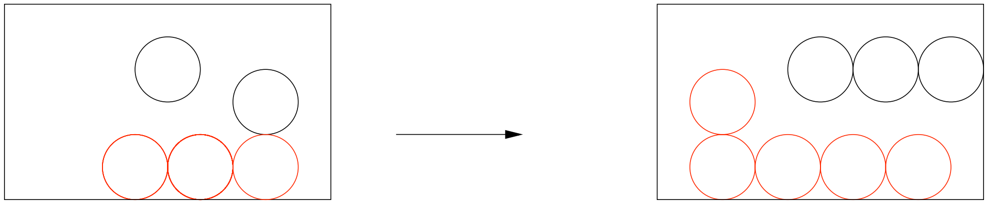
INRIA Rocquencourt, Projet Algorithms
and
Université de Versailles St-Quentin, LMV

Joint work with Nicolas POUYANNE

Colloquium for Philippe Flajolet's 60th birthday, December, 1-2, 2008

Pólya-Eggenberger urns with 2 colours

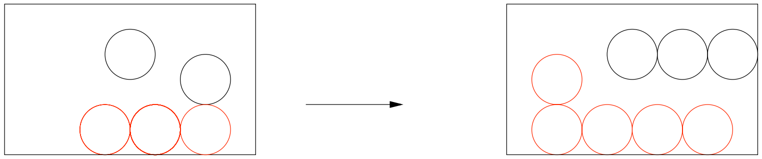
One urn, red and black balls



Replacement matrix $R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, a, b, c, d integers.

Composition vector $U^{DT}(n) = \begin{pmatrix} \# \text{ red at time n} \\ \# \text{ black at time n} \end{pmatrix}$; $\quad U^{DT}(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

Pólya-Eggenberger urns with 2 colours



Replacement matrix $R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, a, b, c, d integers.

Composition vector $U^{DT}(n) = \begin{pmatrix} \text{\# red at time n} \\ \text{\# black at time n} \end{pmatrix}$; $\quad U^{DT}(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

Question: Asymptotics of $U^{DT}(n)$?

References

[1] G. Pólya Sur quelques points de la théorie des probabilités. *Ann. Inst. Henri Poincaré* **1** (1931), 117–161.

[2] H.M. Mahmoud Pólya urn models. CRC Press, (2008)

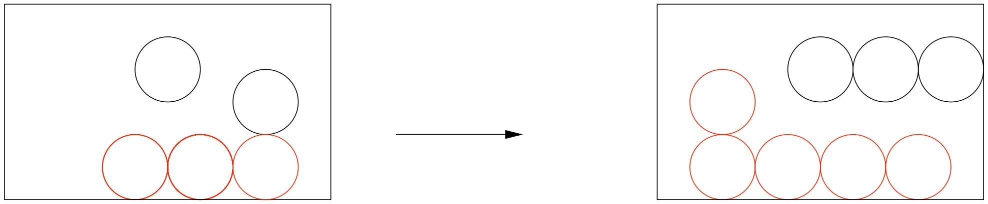
[3] S. Janson Functional limit theorem for multitype branching processes and generalized Pólya urns. *Stochastic Processes and Applications,* **110** (2004), 177–245.

[4] P. Flajolet, J. Gabarró, H. Pekari Analytic urns. *Annals of Probability,* **33** (3) (2005), 1200–1233.

[5] N. Pouyanne An algebraic approach of Pólya processes (2005). *Ann. Inst. Henri Poincaré*, Vol. 44, No. 2, 293–323.

Pólya-Eggenberger urns with 2 colours

One urn, red and black balls



Replacement matrix $R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, a, b, c, d integers.

Composition vector $U^{DT}(n) = \begin{pmatrix} \#\text{ red at time n} \\ \#\text{ black at time n} \end{pmatrix}$; $\quad U^{DT}(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

Assumption

Balance: $a + b = c + d := S$

$\#$ at time $n = x_0 + y_0 + Sn$.

$R = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} S & 0 \\ 0 & m \end{pmatrix}$ $\quad m = \sigma S = a - c = d - b$

2 kinds of asymptotics:

- Gaussian: when $\sigma S < \frac{1}{2}S$ “small” urn

$$\frac{U^{DT}(n) - nv_1}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, \Sigma^2)$$

2 kinds of asymptotics:

- Gaussian: when $\sigma S \leq \frac{1}{2} S$ “small” urn

$$\frac{U^{DT}(n) - nv_1}{\sqrt{n}\sqrt{\log n}} \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, \Sigma^2)$$

2 kinds of asymptotics:

- Gaussian: when $\sigma S < \frac{1}{2}S$ "small" urn

$$\frac{U^{DT}(n) - nv_1}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, \Sigma^2)$$

- non Gaussian: when $\frac{1}{2}S < \sigma S < S$ "large" urn

$$\boxed{U^{DT}(n) = nv_1 + n^\sigma W^{DT} v_2 + o(n^\sigma)}$$

where

v_1, v_2 are deterministic vectors
W^{DT} is a martingale's limit
$o(\ )$ means a.s. and in any $L^p, p \geq 1$
the moments of W^{DT} can be recursively calculated.

2 kinds of asymptotics:

- Gaussian: when $\sigma S < \frac{1}{2}S$ "small" urn

$$\frac{U^{DT}(n) - nv_1}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, \Sigma^2)$$

- non Gaussian: when $\frac{1}{2}S < \sigma S < S$ "large" urn

$$\boxed{U^{DT}(n) = nv_1 + n^\sigma W^{DT} v_2 + o(n^\sigma)}$$

where

v_1, v_2 are deterministic vectors
W^{DT} is a martingale's limit
$o(\ )$ means a.s. and in any $L^p, p \geq 1$
the moments of W^{DT} can be recursively calculated.

Goal: distribution of W^{DT}

Embedding

Continuous-time Markov process $(U^{CT}(t))_{t\geq 0}$

$U^{CT}(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$.

$\mathcal{E}xp(1)$ clock on each ball; independence of the clocks.

Branching:

$\longrightarrow$ if a red ball rings, replace it by $a+1$ red balls and b black balls

$\longrightarrow$ if a black ball rings, replace it by c red balls and $d+1$ black balls

$0 < \tau_1 < \cdots < \tau_n < \ldots$ are the jump times

$$\boxed{(U^{CT}(\tau_n))_{n\geq 0} \overset{\mathcal{D}}{=} (U^{DT}(n))_{n\geq 0}}$$

Embedding

Continuous-time Markov process $(U^{CT}(t))_{t\geq 0}$

$U^{CT}(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$.

$\mathcal{E}xp(1)$ clock on each ball; independence of the clocks.

Branching:

⟶ if a red ball rings, replace it by $a+1$ red balls and b black balls

⟶ if a black ball rings, replace it by c red balls and $d+1$ black balls

$0 < \tau_1 < \dots < \tau_n < \dots$ are the jump times

$$\boxed{(U^{CT}(\tau_n))_{n\geq 0} \overset{\mathcal{D}}{=} (U^{DT}(n))_{n\geq 0}}$$

Benefit: independence

Outline

Discrete urn $U^{DT}(n)$ $\longrightarrow$ Continuous-time branching process $U^{CT}(t)$

W^{DT} W^{CT}

Outline

Discrete urn $U^{DT}(n)$ $\longrightarrow$ Continuous-time branching process $U^{CT}(t)$

W^{DT} W^{CT}

$\boxed{\text{connection: } W^{CT} = \xi^{\sigma} W^{DT}}$ where $\xi \overset{\mathcal{D}}{=} \Gamma(\frac{x_0+y_0}{S})$

Outline

Discrete urn $U^{DT}(n)$ $\longrightarrow$ Continuous-time branching process $U^{CT}(t)$

W^{DT} W^{CT}

$\boxed{\text{connection: } W^{CT} = \xi^{\sigma} W^{DT}}$ where $\xi \overset{\mathcal{D}}{=} \Gamma(\frac{x_0+y_0}{S})$

$\downarrow$

Independence properties

$\downarrow$

Dislocation equations on W^{CT}

Outline

Discrete urn $U^{DT}(n)$ $\longrightarrow$ Continuous-time branching process $U^{CT}(t)$
W^{DT} W^{CT}

$\boxed{\text{connection: } W^{CT} = \xi^{\sigma} W^{DT}}$ where $\xi \overset{\mathcal{D}}{=} \Gamma(\frac{x_0+y_0}{S})$

↓
Independence properties
↓
Dislocation equations on W^{CT}
↓
Differential system on the Fourier transform of W^{CT}

Outline

Discrete urn $U^{DT}(n)$ $\longrightarrow$ Continuous-time branching process $U^{CT}(t)$
W^{DT} W^{CT}

$\boxed{\text{connection: } W^{CT} = \xi^{\sigma} W^{DT}}$ where $\xi \overset{\mathcal{D}}{=} \Gamma(\frac{x_0+y_0}{S})$

$\downarrow$

Independence properties

$\downarrow$

Dislocation equations on W^{CT}

$\downarrow$

Differential system on the Fourier transform of W^{CT}

$\downarrow$

Resolution $\longrightarrow$ distribution of W^{CT}

Outline

Discrete urn $U^{DT}(n)$ $\longrightarrow$ Continuous-time branching process $U^{CT}(t)$

W^{DT} W^{CT}

$\boxed{\text{connection: } W^{CT} = \xi^{\sigma} W^{DT}}$ where $\xi \overset{\mathcal{D}}{=} \Gamma(\frac{x_0+y_0}{S})$

$\downarrow$

Independence properties

$\downarrow$

Dislocation equations on W^{CT}

$\downarrow$

Differential system on the Fourier transform of W^{CT}

$\downarrow$

Resolution $\rightarrow$ distribution of W^{CT}

(connection) $\downarrow$

$\boxed{\text{Distribution of } W^{DT}}$

Dislocation equations

Simplification of the notations:

$$X_t := U^{CT}(t) \text{ starting from } (x_0, y_0) = (1, 0)$$
$$Y_t := U^{CT}(t) \text{ starting from } (x_0, y_0) = (0, 1)$$

Dislocation equations

Simplification of the notations:

$$\begin{aligned} X_t &:= U^{CT}(t) \text{ starting from } (x_0, y_0) = (1, 0) \\ Y_t &:= U^{CT}(t) \text{ starting from } (x_0, y_0) = (0, 1) \end{aligned}$$

$\forall t >$ first splitting time τ_1,

$$\begin{cases} X_t \overset{\mathcal{D}}{=} [a+1]X_{t-\tau_1} + [b]Y_{t-\tau_1} \\ Y_t \overset{\mathcal{D}}{=} [c]X_{t-\tau_1} + [d+1]Y_{t-\tau_1}, \end{cases}$$

$[\;]$ *is a notation*

Dislocation equations

Simplification of the notations:

$$X_t := U^{CT}(t) \text{ starting from } (x_0, y_0) = (1,0)$$
$$Y_t := U^{CT}(t) \text{ starting from } (x_0, y_0) = (0,1)$$

$\forall t >$ first splitting time τ_1,

$$\begin{cases} X_t \overset{\mathcal{D}}{=} [a+1]X_{t-\tau_1} + [b]Y_{t-\tau_1} \\ Y_t \overset{\mathcal{D}}{=} [c]X_{t-\tau_1} + [d+1]Y_{t-\tau_1}, \end{cases}$$

$[\ ]$ *is a notation*

Take projections, renormalize, take the limit, $t \longrightarrow +\infty$:

$$\begin{cases} W^{CT}_{(1,0)} \overset{not}{=} X \overset{\mathcal{D}}{=} e^{-m\tau_1}\Big([a+1]X + [b]Y\Big) \\ W^{CT}_{(0,1)} \overset{not}{=} Y \overset{\mathcal{D}}{=} e^{-m\tau_1}\Big([c]X + [d+1]Y\Big) \end{cases}$$

Dislocation equations

Simplification of the notations:

$$X_t := U^{CT}(t) \text{ starting from } (x_0, y_0) = (1, 0)$$
$$Y_t := U^{CT}(t) \text{ starting from } (x_0, y_0) = (0, 1)$$

$\forall t >$ first splitting time τ_1,

$$\begin{cases} X_t \overset{\mathcal{D}}{=} [a+1]X_{t-\tau_1} + [b]Y_{t-\tau_1} \\ Y_t \overset{\mathcal{D}}{=} [c]X_{t-\tau_1} + [d+1]Y_{t-\tau_1}, \end{cases}$$

$[\ ]$ *is a notation*

Take projections, renormalize, take the limit, $t \longrightarrow +\infty$:

$$\begin{cases} W^{CT}_{(1,0)} \overset{not}{=} X \overset{\mathcal{D}}{=} e^{-m\tau_1}\Big([a+1]X + [b]Y\Big) \\ W^{CT}_{(0,1)} \overset{not}{=} Y \overset{\mathcal{D}}{=} e^{-m\tau_1}\Big([c]X + [d+1]Y\Big) \end{cases}$$

Goal: ? distribution of X and Y;

1. If you love Analytic Combinatorics:

Let $a_n = \mathbb{E}(X^n) \qquad b_n = \mathbb{E}(Y^n)$

Let $F(T) := \sum_{n \geq 0} \frac{a_n}{n!} T^n \qquad G(T) := \sum_{n \geq 0} \frac{b_n}{n!} T^n$ Laplace series

1. If you love Analytic Combinatorics:

Let $a_n = \mathbb{E}(X^n) \qquad b_n = \mathbb{E}(Y^n)$

Let $F(T) := \sum_{n \geq 0} \frac{a_n}{n!} T^n \qquad G(T) := \sum_{n \geq 0} \frac{b_n}{n!} T^n$ Laplace series

Dislocation equations

$$\downarrow$$

Recursion on the moments

1. If you love Analytic Combinatorics:

Let $a_n = \mathbb{E}(X^n) \qquad b_n = \mathbb{E}(Y^n)$

Let $F(T) := \sum_{n\geq 0} \frac{a_n}{n!} T^n \qquad G(T) := \sum_{n\geq 0} \frac{b_n}{n!} T^n$ Laplace series

Dislocation equations

↓

Recursion on the moments

↓ (multinomial formula)

$$\begin{cases} F(T) + mTF'(T) = F(T)^{a+1} G(T)^b \\ G(T) + mTG'(T) = F(T)^c G(T)^{d+1} \\ + \text{ initial conditions} \end{cases}$$

2. If you like Probability, too:

Let $\mathcal{F}(x) = \mathbb{E}(e^{ixX})$ $\mathcal{G}(x) = \mathbb{E}(e^{ixY})$

2. If you like Probability, too:

Let $\quad \mathcal{F}(x) = \mathbb{E}(e^{ixX}) \qquad \mathcal{G}(x) = \mathbb{E}(e^{ixY})$

Insert in the dislocation equation:

$$\mathcal{F}(x) = \frac{x}{mx^{1+\frac{1}{m}}} \int_0^x \mathcal{F}^{a+1}(t)\mathcal{G}^b(t)\frac{dt}{t^{1-1/m}}, \qquad (x > 0)$$

2. If you like Probability, too:

Let $\quad \mathcal{F}(x) = \mathbb{E}(e^{ixX}) \qquad \mathcal{G}(x) = \mathbb{E}(e^{ixY})$

Insert in the dislocation equation:

$$\mathcal{F}(x) = \frac{x}{mx^{1+\frac{1}{m}}} \int_0^x \mathcal{F}^{a+1}(t)\mathcal{G}^b(t)\frac{dt}{t^{1-1/m}}, \qquad (x > 0)$$

Derive:

$$\begin{cases} \mathcal{F}(x) + mx\mathcal{F}'(x) = \mathcal{F}(x)^{a+1}\mathcal{G}(x)^b \\ \mathcal{G}(x) + mx\mathcal{G}'(x) = \mathcal{F}(x)^c\mathcal{G}(x)^{d+1} \\ + \text{ initial conditions} \end{cases}$$

2. If you like Probability, too:

Let $\quad \mathcal{F}(x) = \mathbb{E}(e^{ixX}) \qquad \mathcal{G}(x) = \mathbb{E}(e^{ixY})$

Insert in the dislocation equation:

$$\mathcal{F}(x) = \frac{x}{mx^{1+\frac{1}{m}}} \int_0^x \mathcal{F}^{a+1}(t)\mathcal{G}^b(t)\frac{dt}{t^{1-1/m}}, \qquad (x > 0)$$

Derive:

$$\begin{cases} \mathcal{F}(x) + mx\mathcal{F}'(x) = \mathcal{F}(x)^{a+1}\mathcal{G}(x)^b \\ \mathcal{G}(x) + mx\mathcal{G}'(x) = \mathcal{F}(x)^c\mathcal{G}(x)^{d+1} \\ + \text{ initial conditions} \end{cases}$$

Singular in 0

Resolution of the differential system

$$\begin{cases} \mathcal{F}(x) + mx\mathcal{F}'(x) = \mathcal{F}(x)^{a+1}\mathcal{G}(x)^{b} \\ \\ \mathcal{G}(x) + mx\mathcal{G}'(x) = \mathcal{F}(x)^{c}\mathcal{G}(x)^{d+1} \end{cases}$$

Change of function, with $x^S(-Sw)^m = 1$.

$$\begin{cases} f(w) = x^{\frac{1}{m}}\mathcal{F}(x) \\ \\ g(w) = x^{\frac{1}{m}}\mathcal{G}(x), \end{cases}$$

Non singular differential system

$$(pf60) \qquad \begin{cases} f' = f^{a+1}g^{b} \\ \\ g' = f^{c}g^{d+1} \end{cases}$$

Resolution of the differential system

$$(pf60) \qquad \begin{cases} f' = f^{a+1} g^{b} \\ \\ g' = f^{c} g^{d+1} \end{cases}$$

First integral:

$$\frac{1}{g^m} - \frac{1}{f^m} = Constant = \frac{1}{\kappa^m}$$

Resolution of the differential system

$$(pf60) \qquad \begin{cases} f' = f^{a+1} g^b \\ \\ g' = f^c g^{d+1} \end{cases}$$

First integral:

$$\frac{1}{g^m} - \frac{1}{f^m} = Constant = \frac{1}{\kappa^m}$$

g is a function of f:

$$g = \frac{f}{\left(1 + \left(\frac{f}{\kappa}\right)^m\right)^{1/m}}$$

Resolution of the differential system

$$(pf60) \qquad \begin{cases} f' = f^{a+1} g^{b} \\ \\ g' = f^{c} g^{d+1} \end{cases}$$

First integral:

$$\frac{1}{g^m} - \frac{1}{f^m} = Constant = \frac{1}{\kappa^m}$$

g is a function of f:

$$g = \frac{f}{\left(1 + \left(\frac{f}{\kappa}\right)^m\right)^{1/m}}$$

Insert in (pf60):

$$\frac{f'}{f^{S+1}} \times \left(1 + \left(\frac{f}{\kappa}\right)^m\right)^{\frac{b}{m}} = 1 \qquad (+\text{ boundary condition})$$

Resolution of the differential system $(\dots)\ \frac{f'}{f^{S+1}} \times \left(1 + \left(\frac{f}{\kappa}\right)^m\right)^{\frac{b}{m}} = 1$ can be written

$$\frac{d}{dw}\left(I \circ (\frac{f}{\kappa})\right) = -\kappa^S \qquad with \qquad I'(z) = -\frac{(1+z^m)^{\frac{b}{m}}}{z^{S+1}}.$$

Resolution of the differential system $(\dots)\ \frac{f'}{f^{S+1}} \times \left(1 + \left(\frac{f}{\kappa}\right)^m\right)^{\frac{b}{m}} = 1$ can be written

$$\frac{d}{dw}\left(I \circ (\frac{f}{\kappa})\right) = -\kappa^S \qquad with \qquad I'(z) = -\frac{(1+z^m)^{\frac{b}{m}}}{z^{S+1}}.$$

$$\boxed{I(z) = \int_{[z,z\infty)} (1+u^m)^{\frac{b}{m}} \frac{du}{u^{S+1}}}$$

Resolution of the differential system $(\dots)$ $\frac{f'}{f^{S+1}} \times \left(1+\left(\frac{f}{\kappa}\right)^m\right)^{\frac{b}{m}} = 1$
can be written

$$\frac{d}{dw}\left(I \circ (\frac{f}{\kappa})\right) = -\kappa^S \qquad \textit{with} \qquad I'(z) = -\frac{(1+z^m)^{\frac{b}{m}}}{z^{S+1}}.$$

and

$$\downarrow \qquad\qquad \boxed{I(z) = \int_{[z,z\infty)} (1+u^m)^{\frac{b}{m}} \frac{du}{u^{S+1}}}$$

$$I\left(\frac{f(w)}{\kappa}\right) = C_0 - \kappa^S w,$$ with C_0 a negative real constant.

Resolution of the differential system $(\dots)\ \frac{f'}{f^{S+1}} \times \left(1+\left(\frac{f}{\kappa}\right)^m\right)^{\frac{b}{m}} = 1$ can be written

$$\frac{d}{dw}\left(I \circ (\frac{f}{\kappa})\right) = -\kappa^S \quad \textit{with} \quad I'(z) = -\frac{(1+z^m)^{\frac{b}{m}}}{z^{S+1}}.$$

and

$$\downarrow \qquad \boxed{I(z) = \int_{[z,z\infty)} (1+u^m)^{\frac{b}{m}} \frac{du}{u^{S+1}}}$$

$I\left(\frac{f(w)}{\kappa}\right) = C_0 - \kappa^S w$, with C_0 a negative real constant.

$\downarrow$ Let J be the inverse function of I

$$f(w) = \kappa J\left(C_0 - \kappa^S w\right)$$

Resolution of the differential system $(\dots)$ $\frac{f'}{f^{S+1}} \times \left(1+\left(\frac{f}{\kappa}\right)^{m}\right)^{\frac{b}{m}} = 1$ can be written

$$\frac{d}{dw}\left(I \circ (\frac{f}{\kappa})\right) = -\kappa^{S} \quad with \quad I'(z) = -\frac{(1+z^{m})^{\frac{b}{m}}}{z^{S+1}}.$$

and

$$\downarrow \qquad \boxed{I(z) = \int_{[z,z\infty)} (1+u^{m})^{\frac{b}{m}} \frac{du}{u^{S+1}}}$$

$$I\left(\frac{f(w)}{\kappa}\right) = C_0 - \kappa^{S} w,$$ with C_0 a negative real constant.

$$\downarrow$$

Let J be the inverse function of I

$$f(w) = \kappa J\left(C_0 - \kappa^{S} w\right)$$

$$\boxed{\mathcal{F}(x) = \kappa x^{-\frac{1}{m}} J\left(C_0 + \frac{\kappa^{S}}{S} x^{-\frac{S}{m}}\right)}$$

Consequences of

$$\boxed{\mathcal{F}(x) = \kappa x^{-\frac{1}{m}} J\left(C_0 + \frac{\kappa^S}{S} x^{-\frac{S}{m}}\right)}$$

many...

Consequences of

$$\boxed{\mathcal{F}(x) = \kappa x^{-\frac{1}{m}} J\left(C_0 + \frac{\kappa^S}{S} x^{-\frac{S}{m}}\right)}$$

many...

Ex 1:

$$\mathcal{F}(x) \underset{x\to+\infty}{\sim} \kappa J(C_0) x^{-\frac{1}{m}}$$

The Fourier transform of X is not integrable. The existence of a density does not come from there.

Consequences of

$$\boxed{\mathcal{F}(x) = \kappa x^{-\frac{1}{m}} J\left(C_0 + \frac{\kappa^S}{S} x^{-\frac{S}{m}}\right)}$$

many...

Ex 1:

$$\mathcal{F}(x) \underset{x\to+\infty}{\sim} \kappa J(C_0) x^{-\frac{1}{m}}$$

The Fourier transform of X is not integrable. The existence of a density does not come from there.

Ex 2:

X is not stable

PROOF. Insert the Fourier transform of a stable distribution in the differential equation and see the contradiction.

More on these distributions:

Define more general continuous-time $\mathbb{R}^2$-valued Markov processes.

Initial condition: $(x_0, y_0) \in \mathbb{R}_+ \times \mathbb{R}_+$

Activities α and jumps governed by matrix $R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

State vector $U_{x_0,y_0,\alpha,R}(t)$,

renormalized → martingale

$\xrightarrow[t\to\infty]{} W_{x_0,y_0,\alpha,R}$ with the property:

for every $n \in \mathbb{N}$,

$$W_{x_0,y_0,\alpha,R} = [n] W_{\frac{x_0}{n},\frac{y_0}{n},\alpha,R}.$$

$\boxed{W \text{ is infinitely divisible}}$

More on these distributions:

Define more general continuous-time $\mathbb{R}^2$-valued Markov processes.

Initial condition: $(x_0, y_0) \in \mathbb{R}_+ \times \mathbb{R}_+$

Activities α and jumps governed by matrix $R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

State vector $U_{x_0,y_0,\alpha,R}(t)$,

renormalized → martingale

$\xrightarrow[t\to\infty]{} W_{x_0,y_0,\alpha,R}$ with the property:

for every $n \in \mathbb{N}$,

$$W_{x_0,y_0,\alpha,R} = [n]W_{\frac{x_0}{n},\frac{y_0}{n},\alpha,R}.$$

$\boxed{W \text{ is infinitely divisible}}$

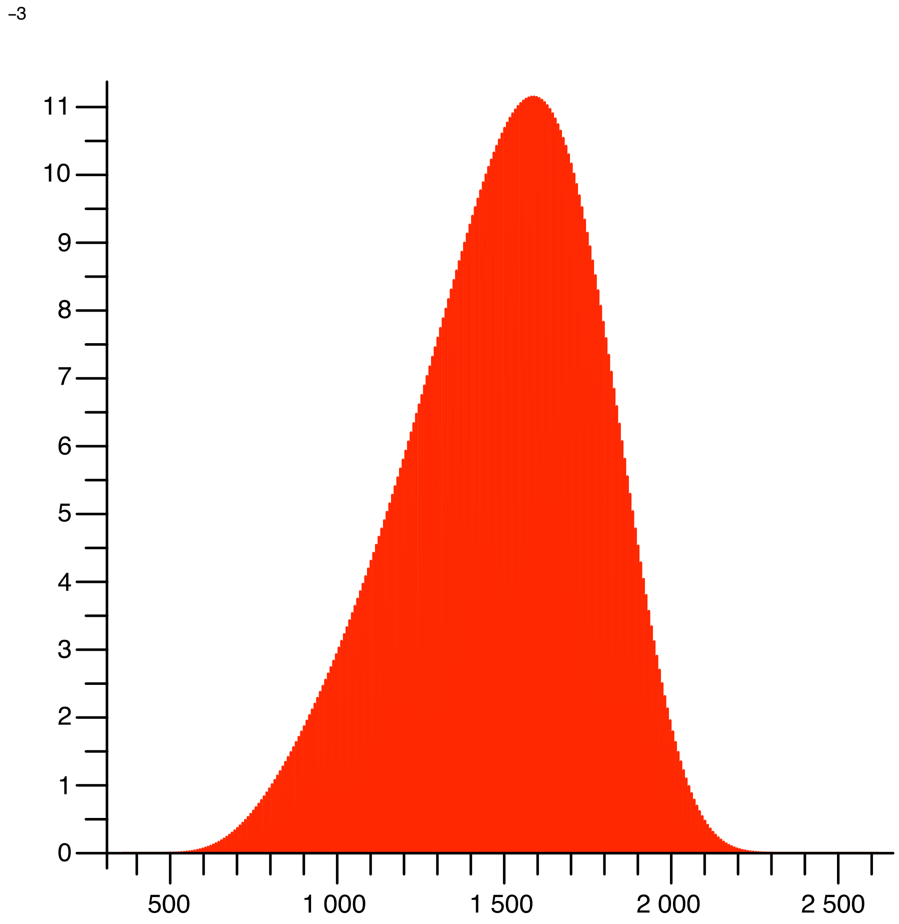
Remember:

$$\boxed{W^{CT} = \xi^{\sigma} W^{DT}}$$

$$\underset{\text{unimodal}}{\uparrow} \qquad \underset{\text{non unimodal}}{\uparrow}$$

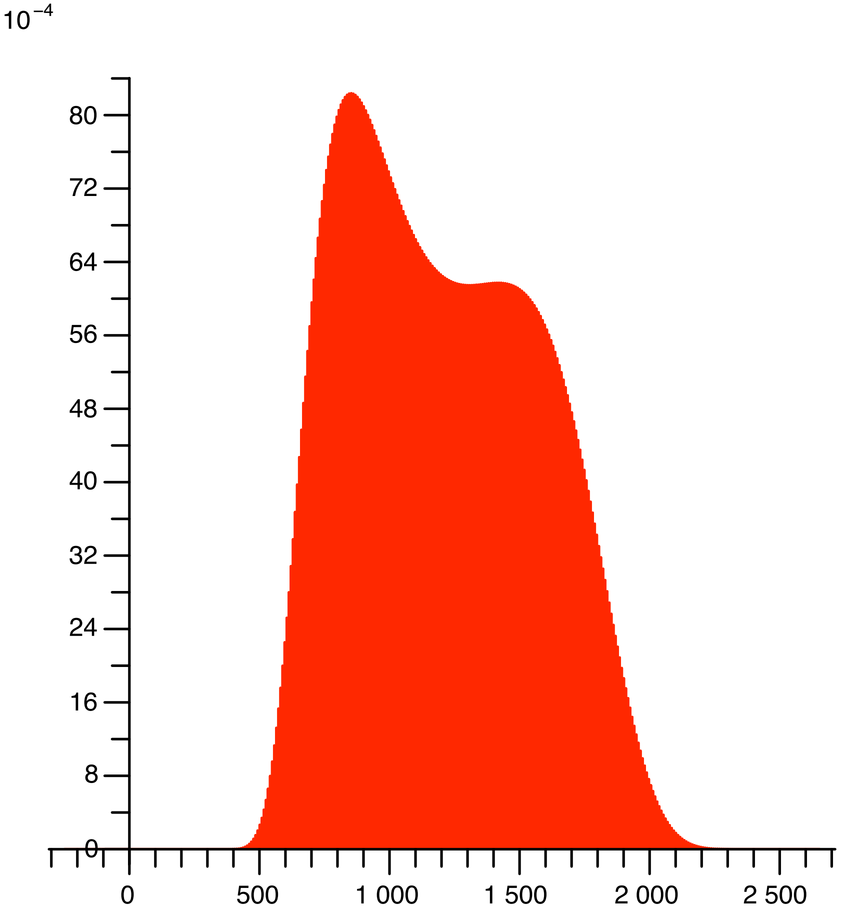
Replacement matrix $R = \begin{pmatrix} 9 & 2 \\ 1 & 10 \end{pmatrix}$

Starting from 1 red ball and 0 black ball; Figure $= W^{DT}$



Replacement matrix $R = \begin{pmatrix} 9 & 2 \\ 1 & 10 \end{pmatrix}$

Starting from 1 red ball and 1 black ball; Figure $= W^{DT}$



Replacement matrix $R = \begin{pmatrix} 6 & 1 \\ 2 & 5 \end{pmatrix}$

Starting from 9 red balls and 9 black balls; Figure $= W^{DT}$