Population protocols & Ehrenfest urns scheme

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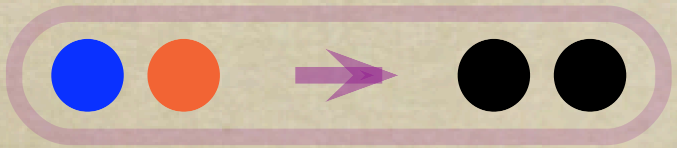

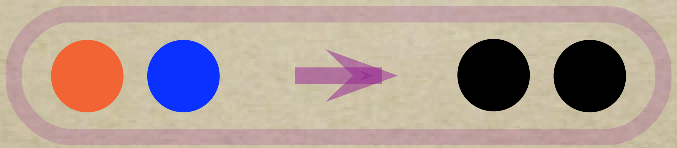

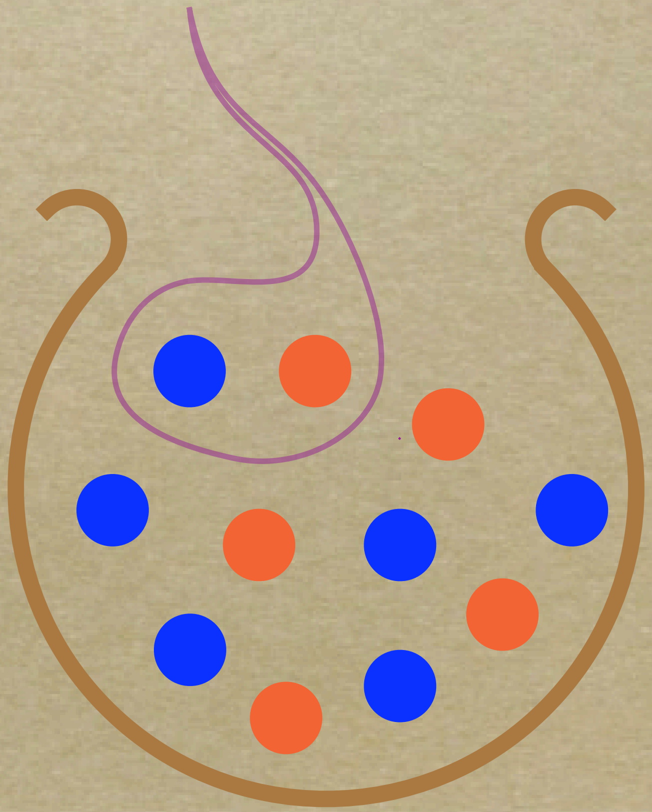

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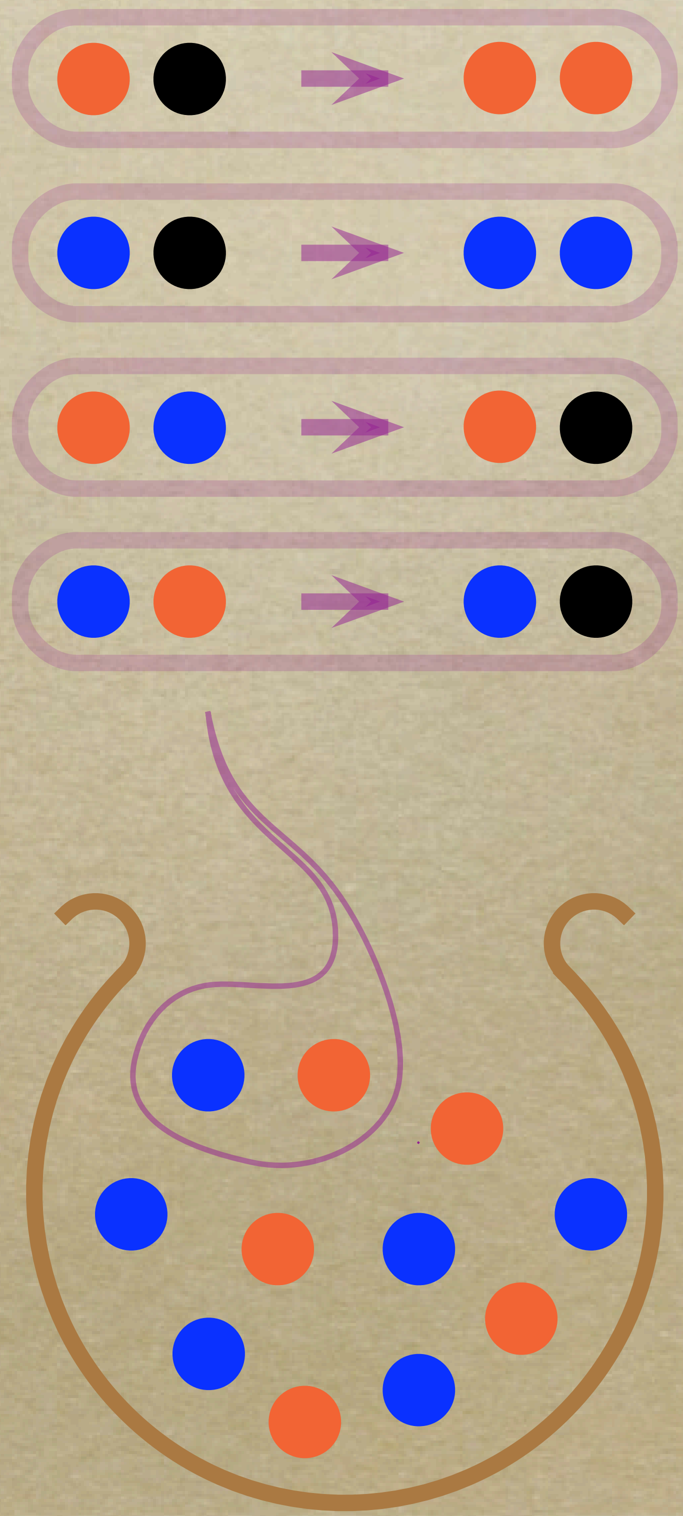

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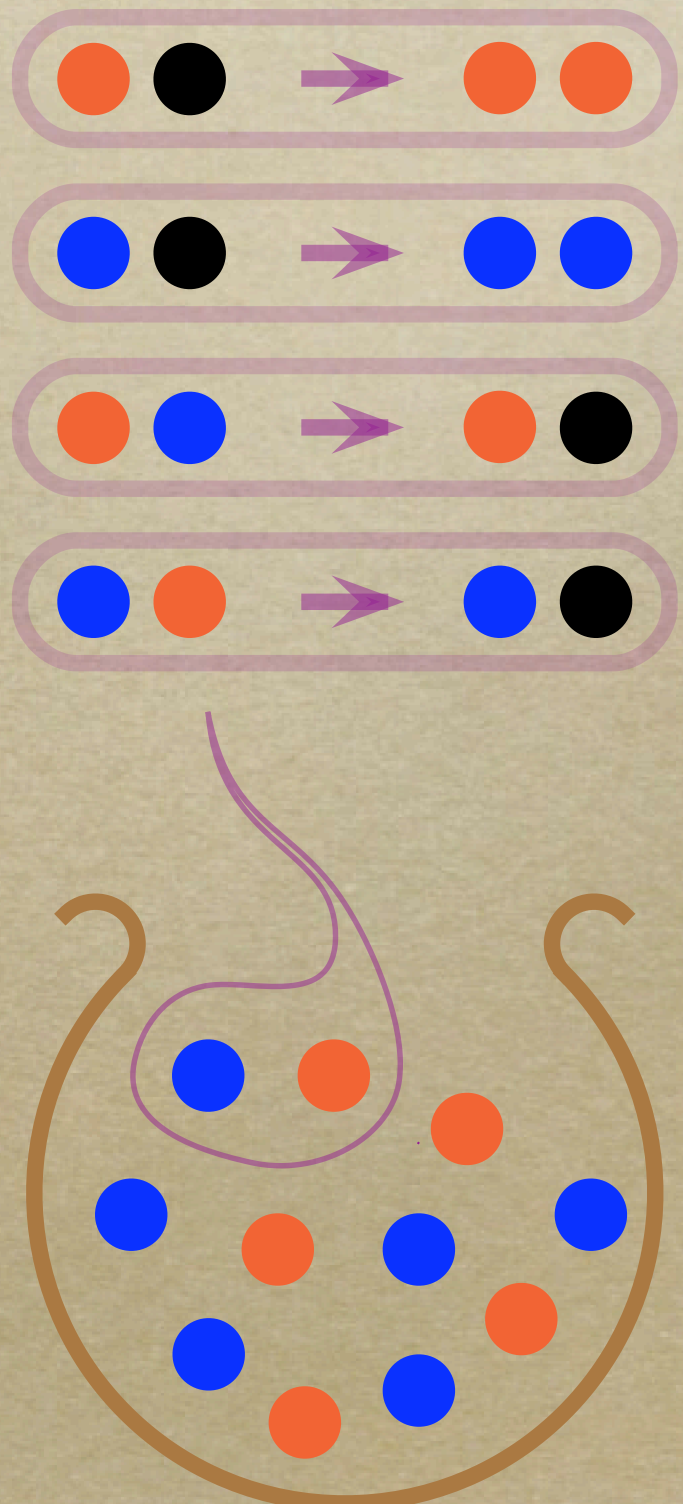

- ***Theorem*** *(Angluin et al., 2007). A predicate is computable in the basic population protocol model if and only if it is semilinear.*

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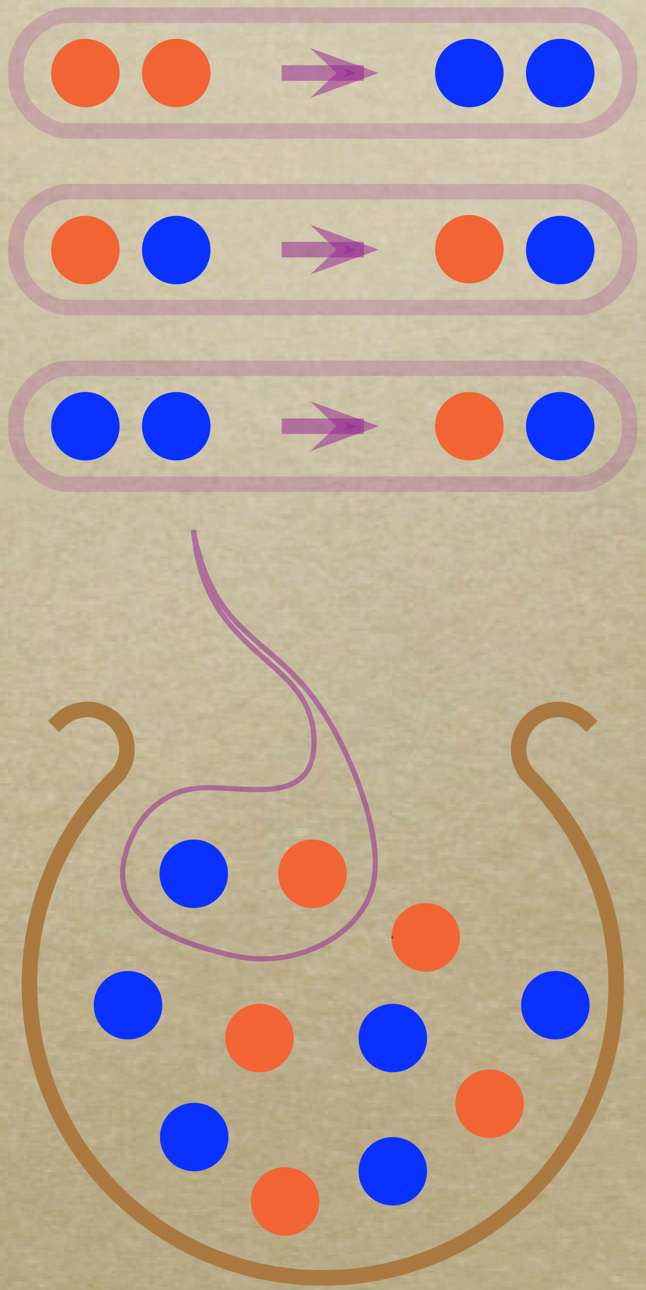

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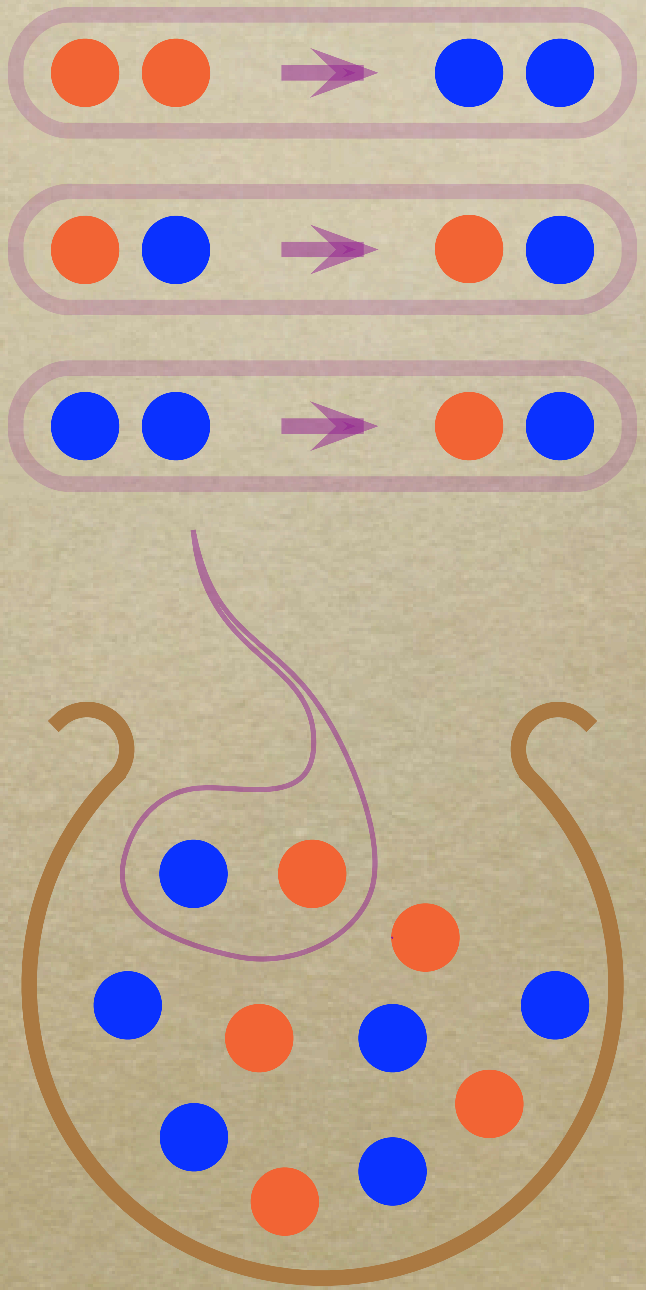

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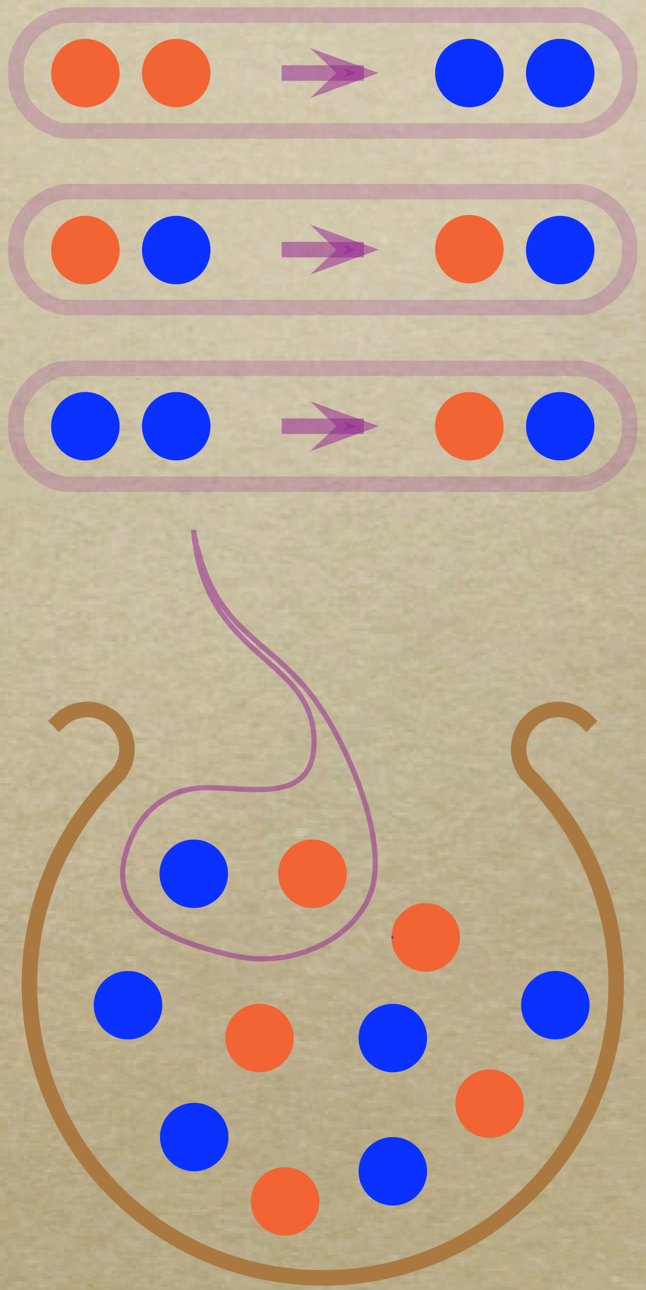
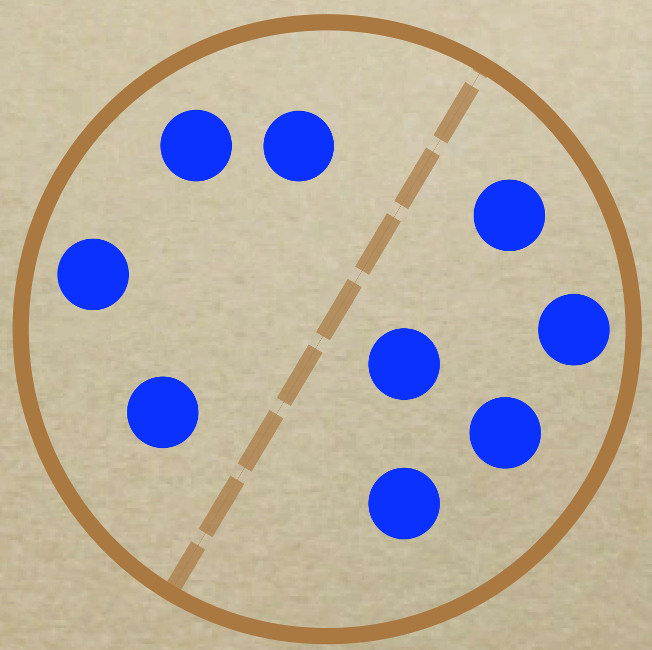


- ***Aim:*** *to compute a number p, i.e. to converge to a given proportion p of blue balls.*
- ***Example.*** *The protocol pictured on the right computes $p=2-\sqrt{2}=0.58\ldots$. More precisely,*
- ***Theorem*** *(Ph. Ch., Bournez et al.)*. *The stationary distribution of the proportion of blue balls converges to p, when the number N of balls goes to ∞. The error term is $O(N^{-1/2})$.*

Ehrenfest: the two-chambers urn

- *An urn contains N (here N=9) balls in its 2 chambers.*
- *Pick a ball at random and move it from its chamber to the other chamber, and repeat.*



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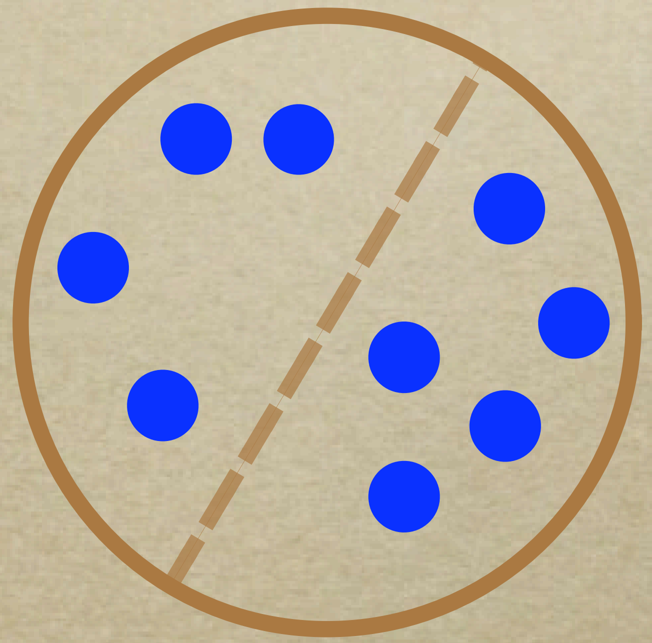

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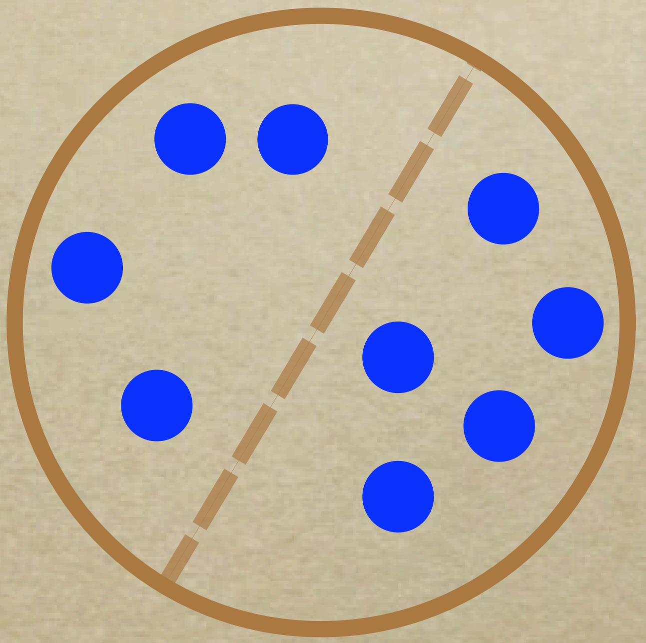

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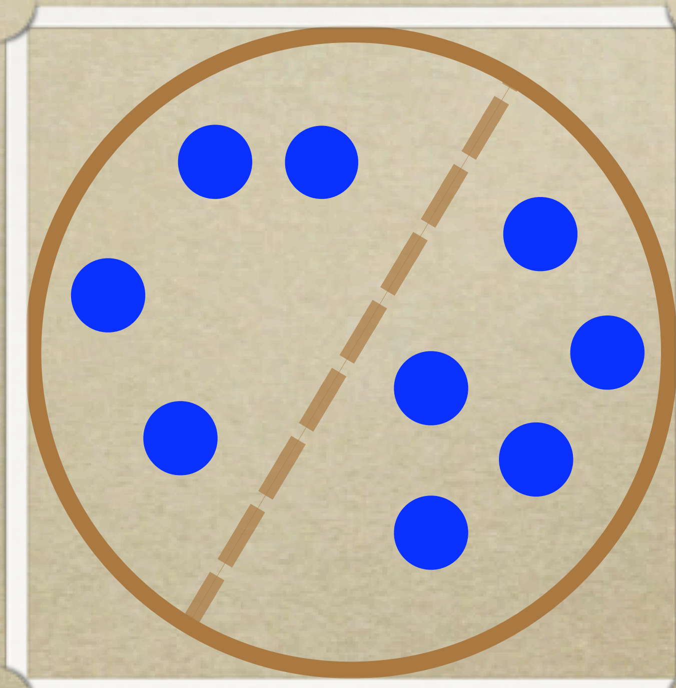
Ehrenfest: the hypercube



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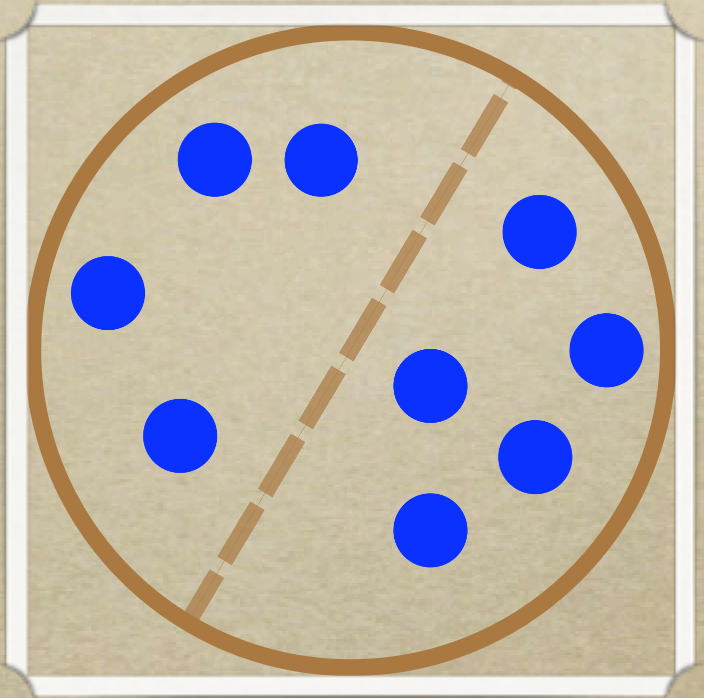
Each configuration can be encoded by a vector $x=(x_1,x_2, \ldots ,x_N) \in \{0,1\}^N$, *in which*



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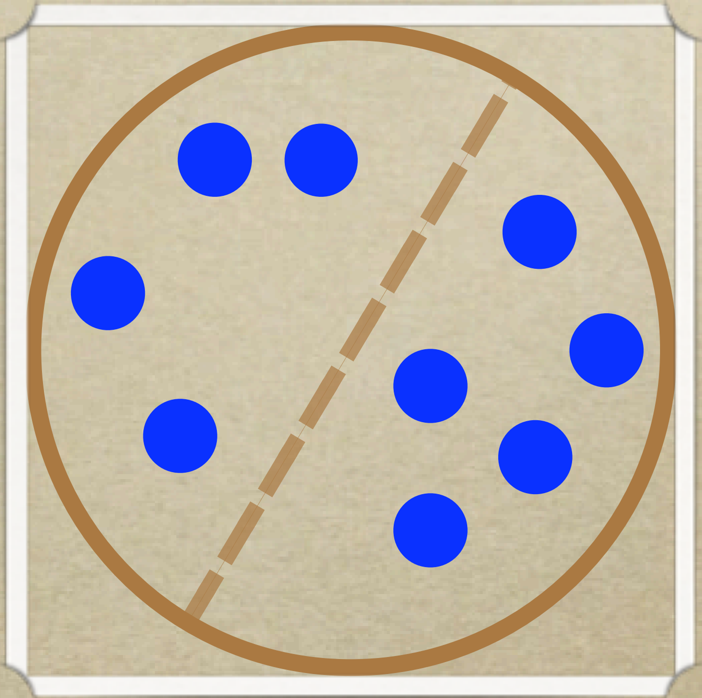

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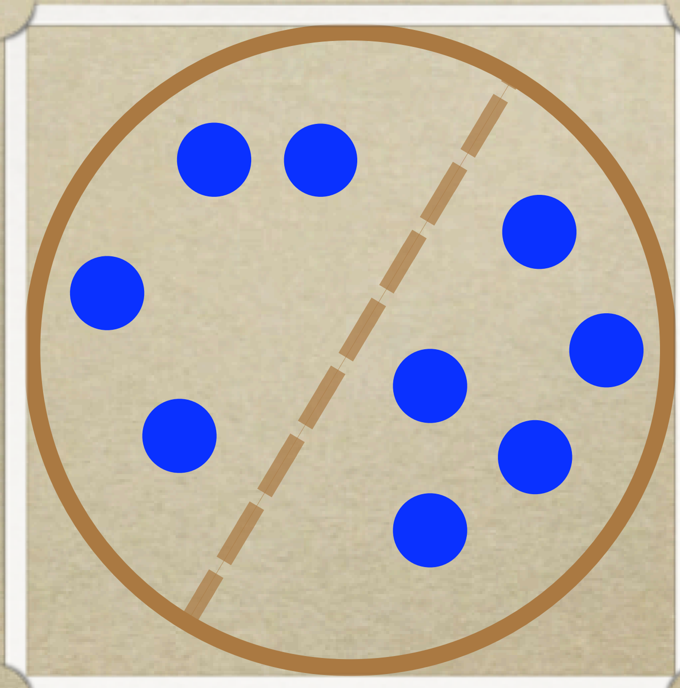

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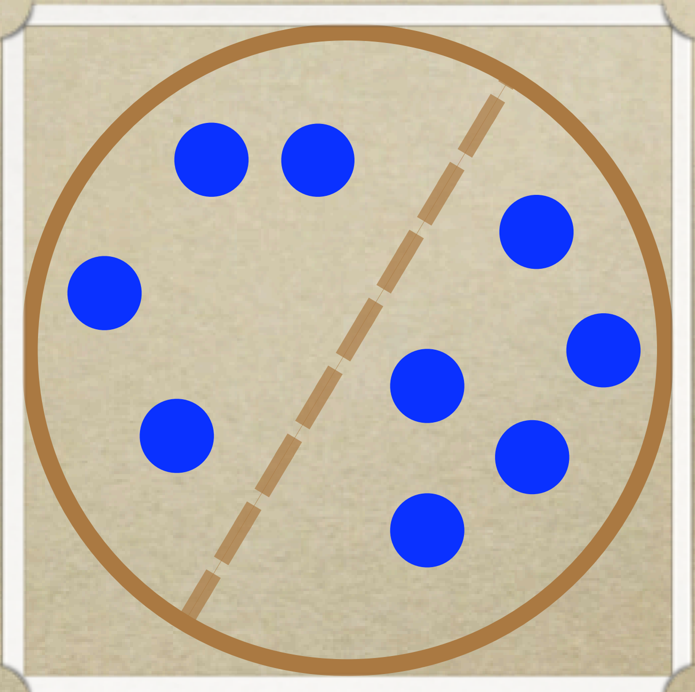

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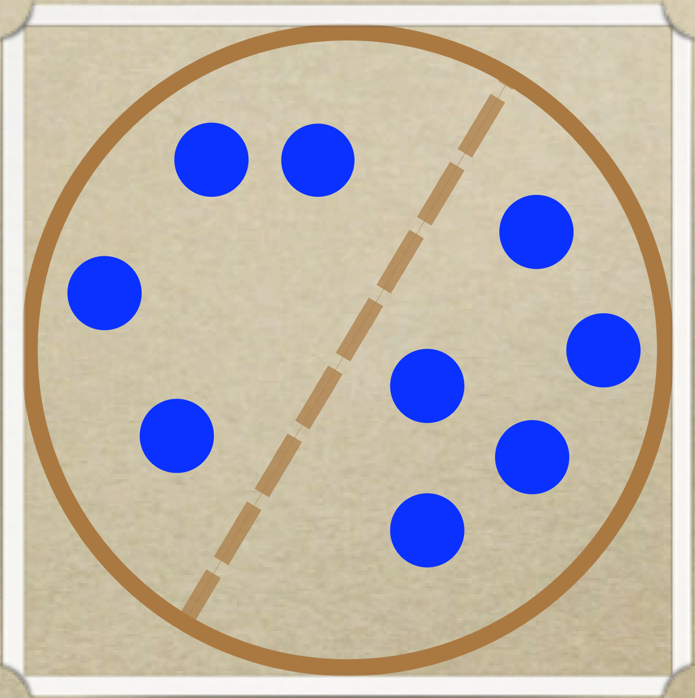

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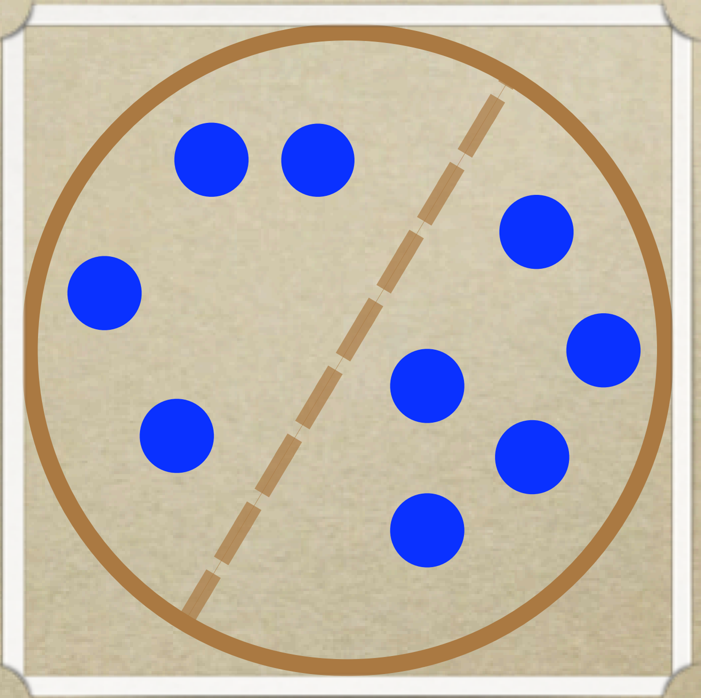
Thus the binomial distribution $(N,0.5)$ *is stationary for the Markov chain* X.



Balanced urn models

Urn evolution rule. *A ball is chosen from the urn, with all balls present in the urn at that time having equal chances of being chosen, and its colour is inspected. If the ball chosen is orange, then* α *new orange balls and* β *new blue balls are placed into the urn; if the ball chosen is blue, then* γ *new orange balls and* δ *new blue balls are placed into the urn.*

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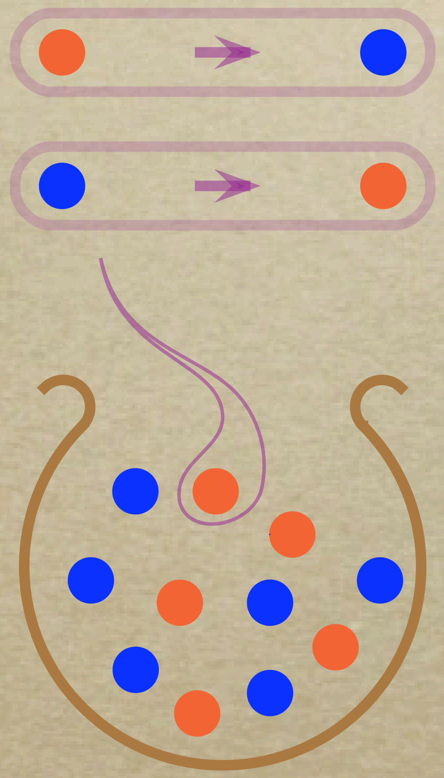

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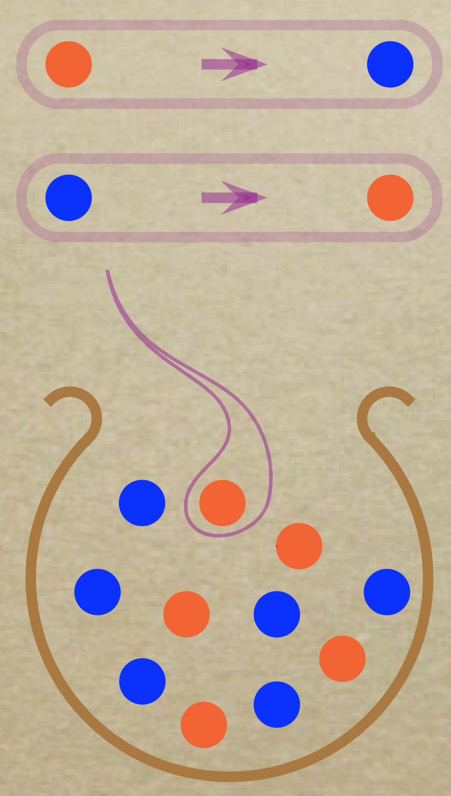

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

The associated differential system

Theorem (Flajolet et al., 2006). *For a balanced urn model, the generating function of histories starting from state (a_0,b_0) and ending at (a,b) after n steps,*

$$H\left(x,y,z \,\middle|\, \begin{matrix} a_0 \\ b_0 \end{matrix}\right) = \sum_{n,a,b} H_n \begin{pmatrix} a_0 & a \\ b_0 & b \end{pmatrix} x^a y^b \frac{z^n}{n!},$$

is given by

$$H(x_0,y_0,z) = X\left(z \,\middle|\, \begin{matrix} x_0 \\ y_0 \end{matrix}\right)^{a_0} Y\left(z \,\middle|\, \begin{matrix} x_0 \\ y_0 \end{matrix}\right)^{b_0},$$

in which X and Y are solutions of

$$\Sigma : \begin{cases} \dot{x} = x^{\alpha+1} & y^{\beta} \\ \dot{y} = x^{\gamma} & y^{\delta+1} \end{cases}$$

with initial conditions (x_0,y_0).

The associated differential system : the case of the Ehrenfest model

For the Ehrenfest model, the associated differential system is quite simple:

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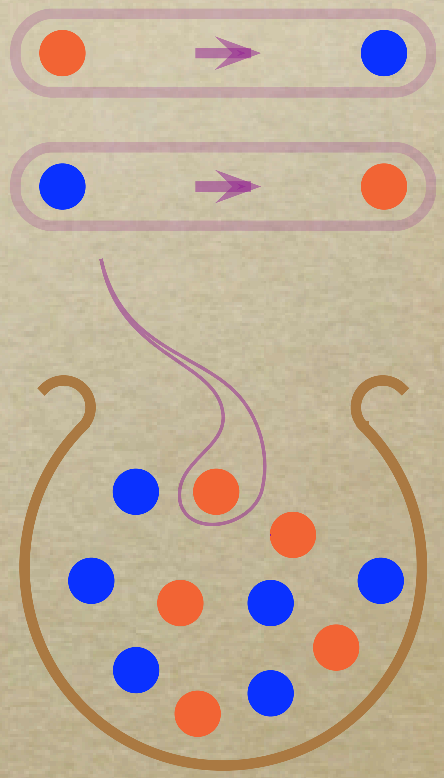

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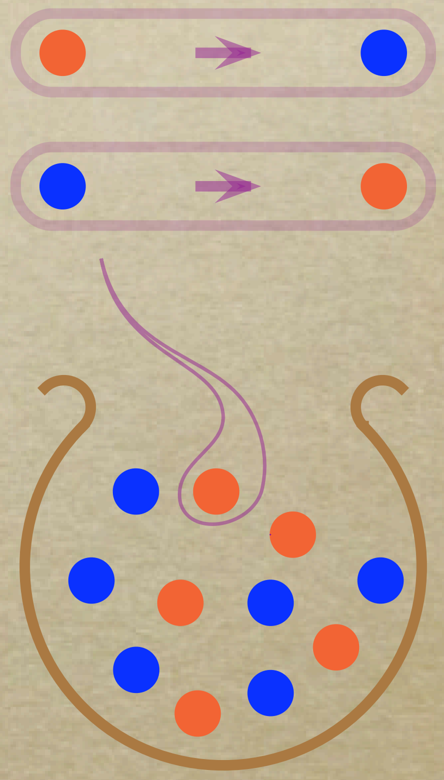

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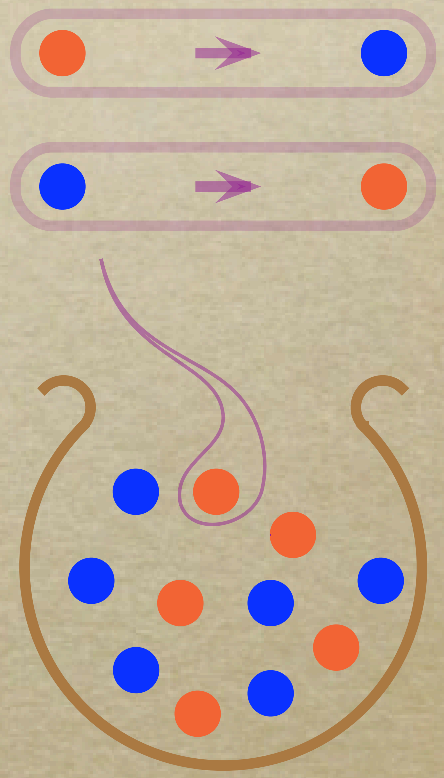

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$$H(x,y,z) = \left(\frac{x+y}{2}e^{z} + \frac{x-y}{2}e^{-z}\right)^{a_0} \left(\frac{x+y}{2}e^{z} - \frac{x-y}{2}e^{-z}\right)^{b_0}.$$

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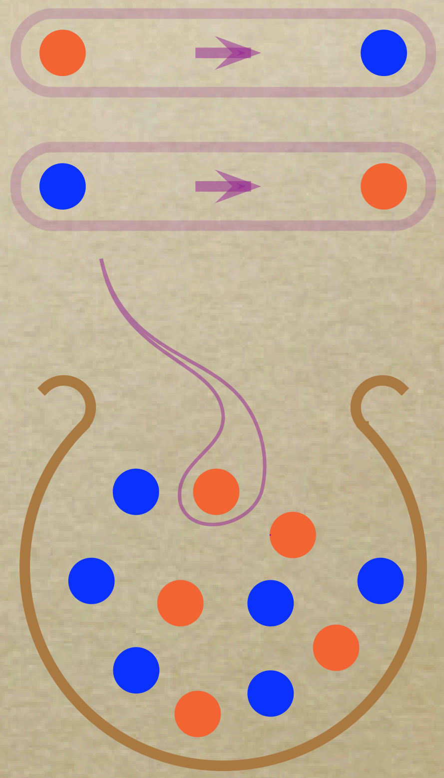

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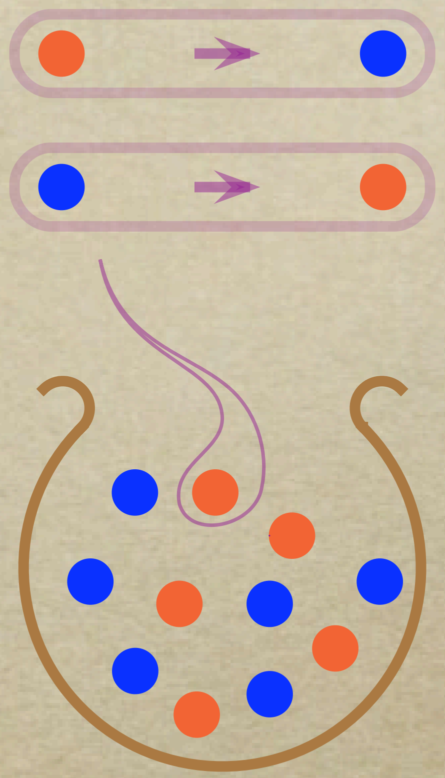
$$p_{N,0}^{(n)} = \frac{n!}{N^n}[z^n]\cosh^N z = \frac{1}{N^n 2^N}\sum_{k=0}^{N}\binom{N}{k}(N-2k)^n.$$

The rescaled Markov chain

Set:

$$Y_t = \frac{2X_{\lfloor Nt \rfloor} - N}{2\sqrt{N}}, \quad \Delta Y_t = Y_{t+\frac{1}{N}} - Y_t.$$



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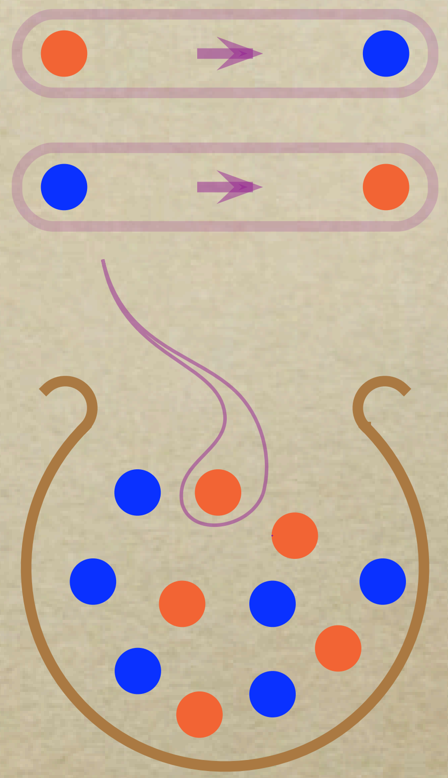

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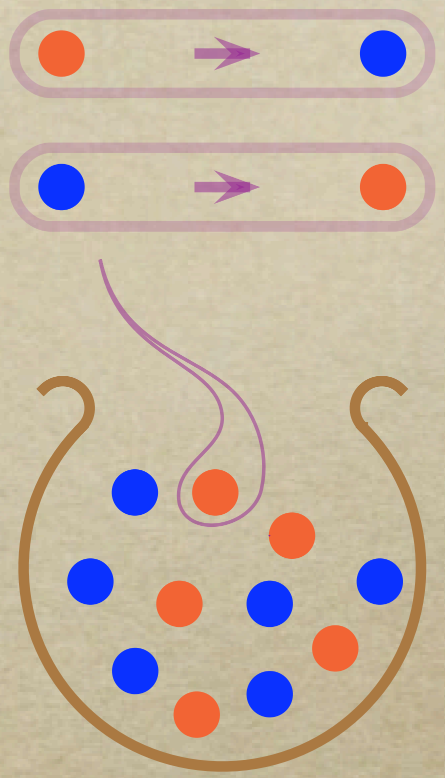

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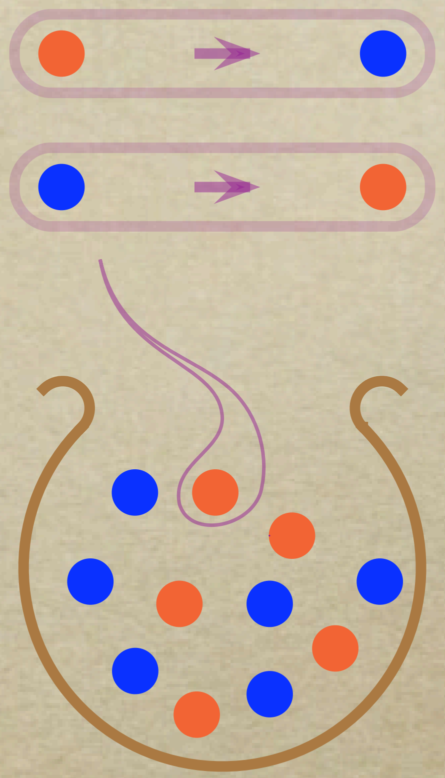

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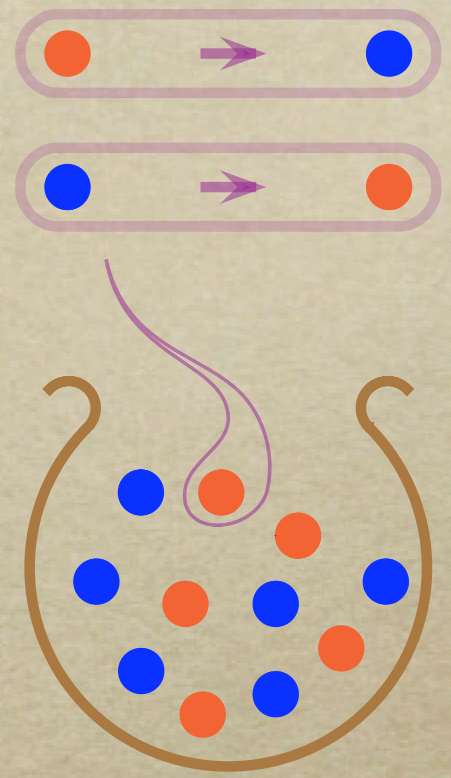

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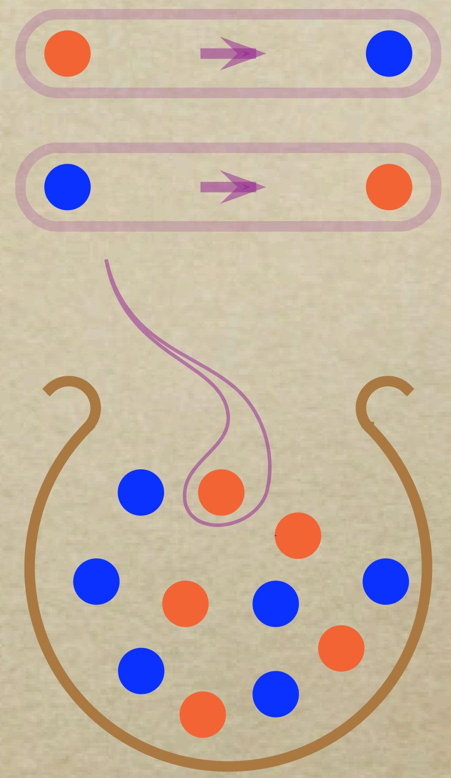

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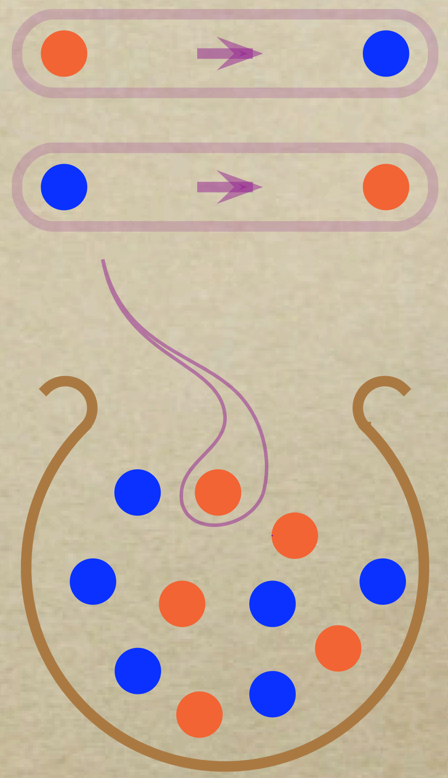

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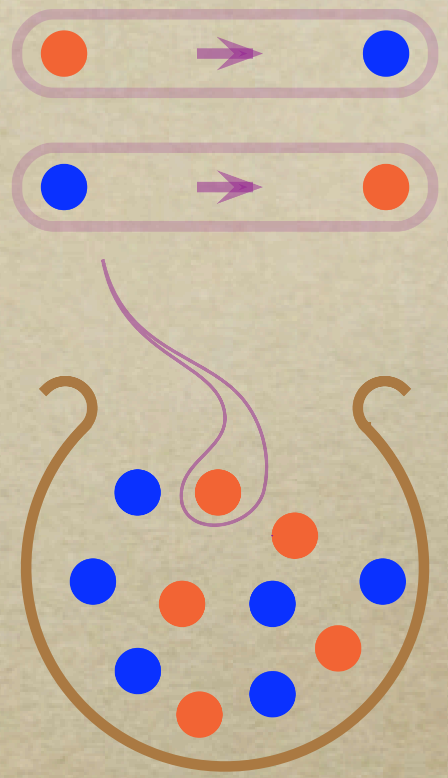
The Ornstein-Uhlenbeck process

Any solution Y_t of the SDE:



$$dY_t = -\theta(Y_t - \mu)\,dt + \sigma\,dB_t.$$

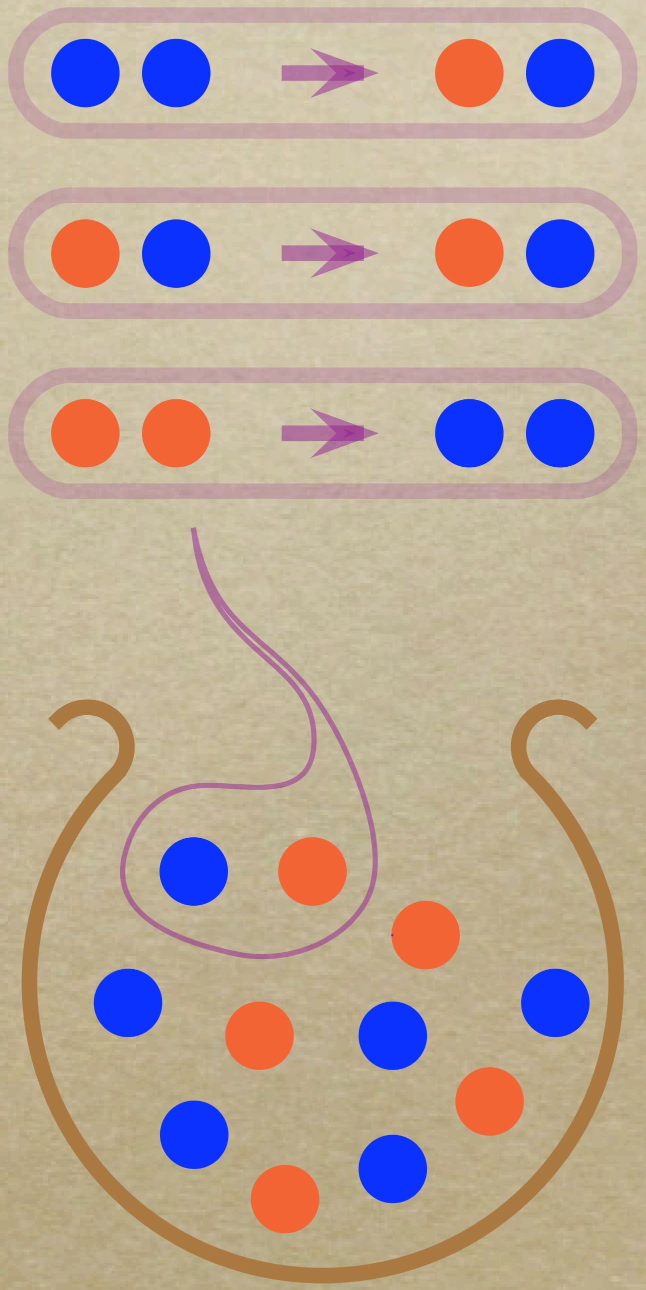
is an Ornstein–Uhlenbeck process. It admits a Gaussian stationary distribution. The stationary variance is given by:

$$\mathrm{Var}(Y_\infty) = \frac{\sigma^2}{2\theta}.$$

An Ornstein–Uhlenbeck process Y_t has closed-form representation in terms of the Brownian motion $B(t)$:

$$Y_t = Y_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \frac{\sigma}{\sqrt{2\theta}} B\left(e^{2\theta t} - 1\right) e^{-\theta t}.$$

Population protocols : the general case



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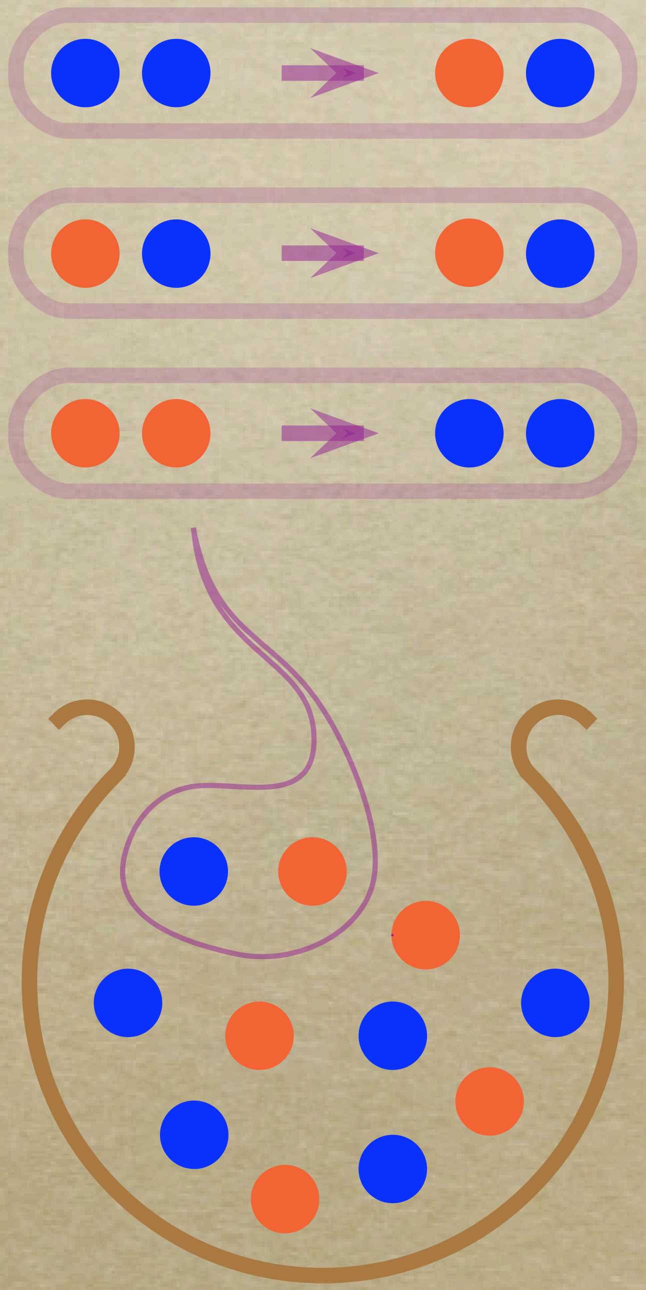

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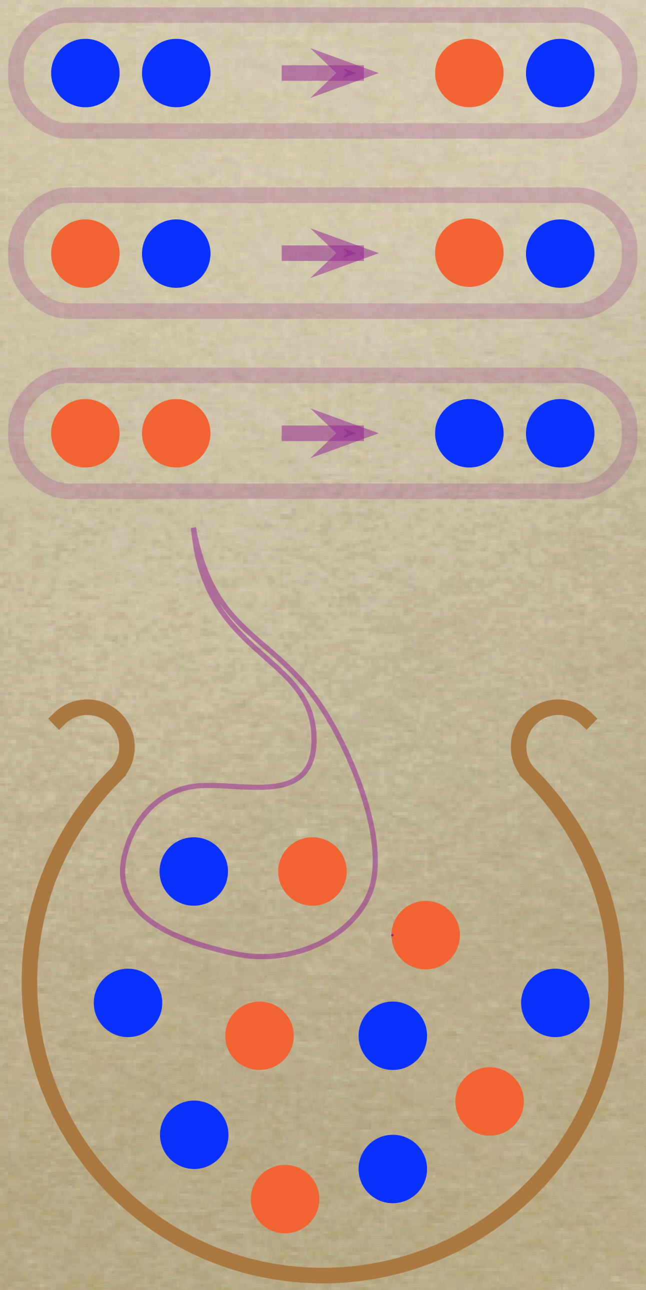
- *Let* p *denote the unique root in (0,1), of the polynomial* P *below.*



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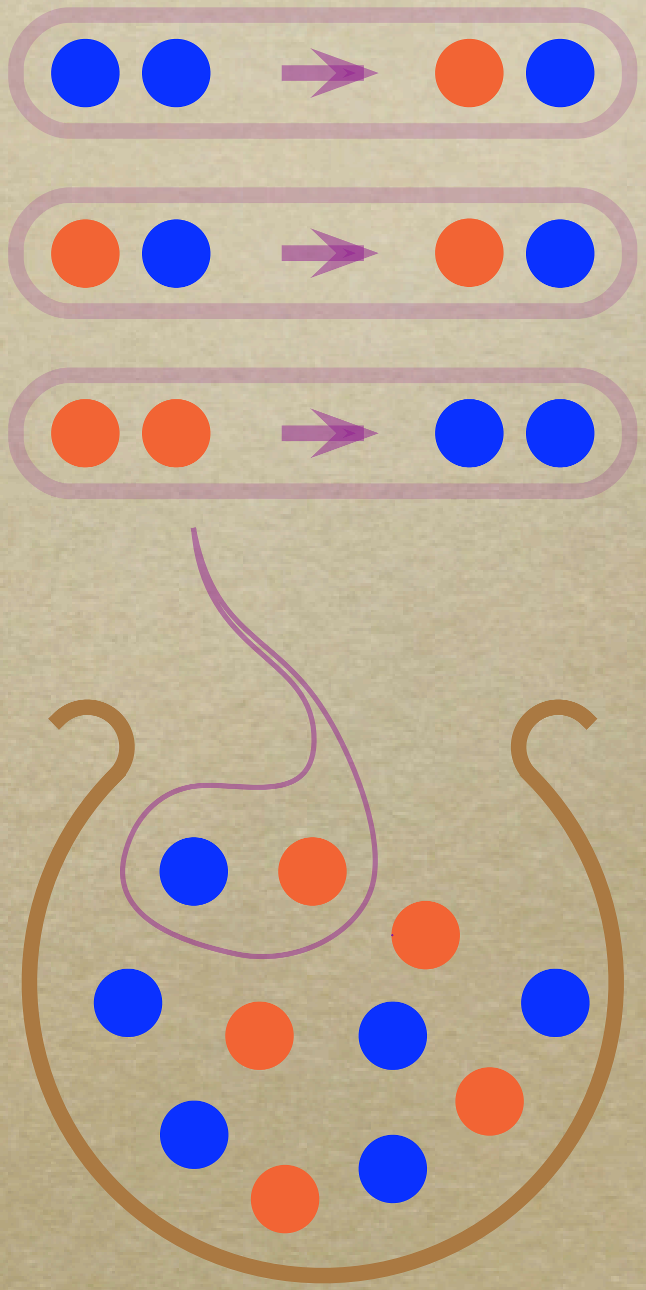
- *There exist 27 rules, described by triplets (a,b,c) in $\{0,1,2\}^3$. In the example, (a,b,c)=(1,1,2).*
- *Let p denote the unique root in (0,1), of the polynomial P below.*

$$P(X) = (a-2b+c)X^2 + (2b-2c-2)X + c.$$



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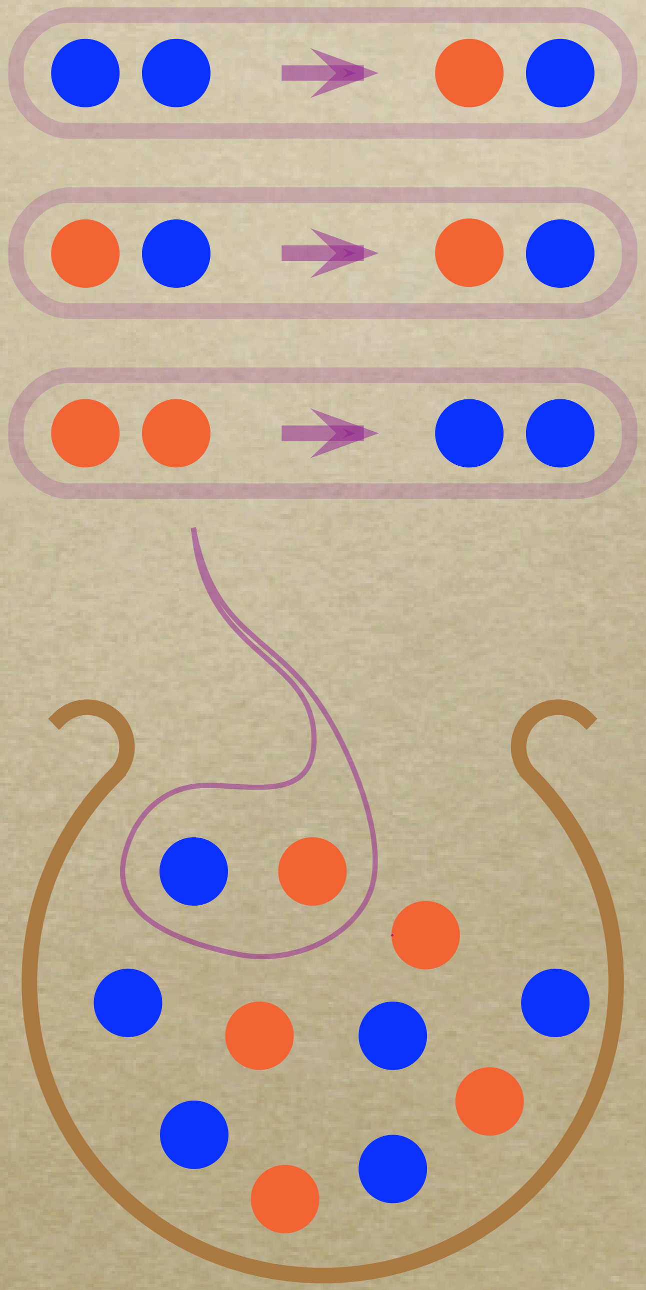
- ***Theorem** (Ph. Ch., Bournez et al.)**.** Among the 26 possible rules, one is trivial, 10 converge to 0 or 1. For the 16 other rules, the stationary distribution of the proportion of blue balls is concentrated around p, when the number N of balls goes to ∞. The error term is $O(N^{-1/2})$.*



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