arctic phenomena in diamonds, groves and fortresses, or generating functions at work

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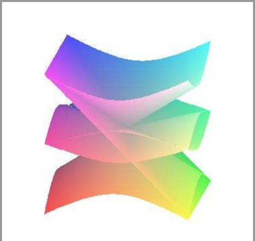

motivation

This work is a part of a larger program on developing machinery for reading asymptotics of the coefficients of multivariate generating functions similar to that perfected by Philippe in univariate case.

His work was and remains a source of motivation and inspiration for us.

All faults are still ours.

motivation

In large random spatial objects one can observe sometimes a peculiar phenomenon, separation into regions with different local statistics.

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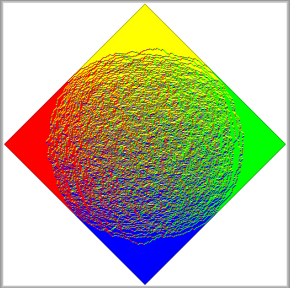
In large random spatial objects one can observe sometimes a peculiar phenomenon, separation into regions with different local statistics.
The simplest way to explain what we mean is to look at some examples:



This is a colorful representation of a random matching in a special planar graph, *Aztec diamond*.

large Aztec diamonds

Remarkably, a *large random* tiling exhibits emergence of order:

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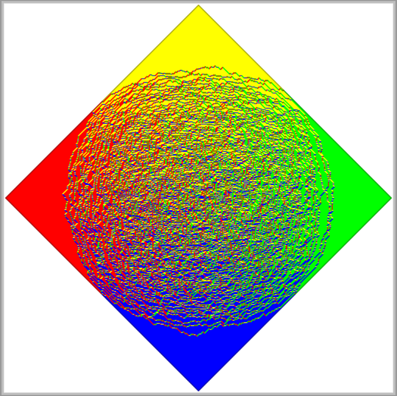
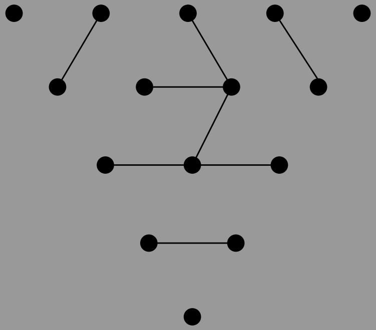


The Arctic Circle Theorem states that outside a $(1+\epsilon)$ enlargement of the inscribed circle the orientations of the dominos are converging in probability to a deterministic brick wall pattern (yielding monochrome regions), while inside a $(1-\epsilon)$ reduction of the inscribed circle the measure has positive entropy. (Propp, Kuperberg, Shor, Jokusch,...) They proved in several different ways the Arctic Circle phenomenon and found the densities of each color within the circle, *the temperate zone*.

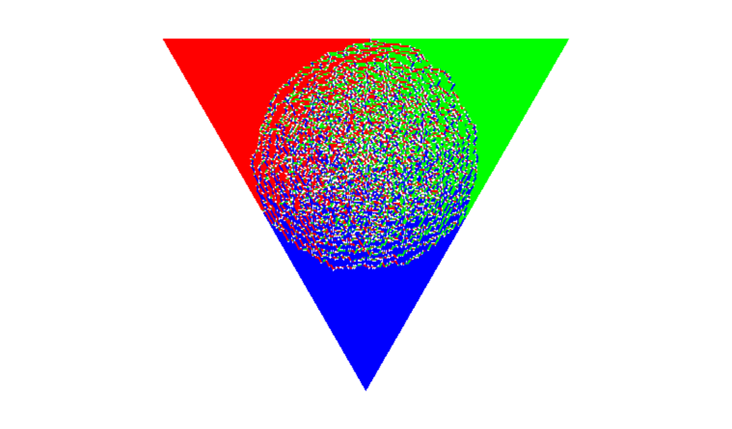
Random Groves

Consider another random object, the (cube) groves introduced by Speyer, Carroll, Petersen. Cube groves are random subgraphs of triangular lattices.
Here is an example:



Arctic Circles in random groves

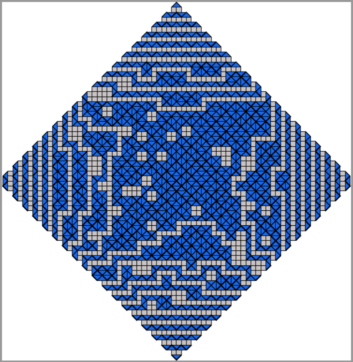
Coloring differently the triangles with edges in each direction, we can visualize large random groves. They, like Aztec Diamond tilings, exhibit different regions!



Speyer and Peterson proved that the orientations are frozen outside the inscribed circle. But what are the densities *inside*? That was unknown that far.

arctic regions in diabolo fortress tilings

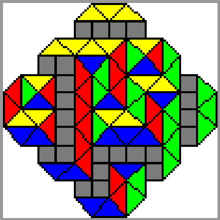
One more example of the frozen/moderate regions:

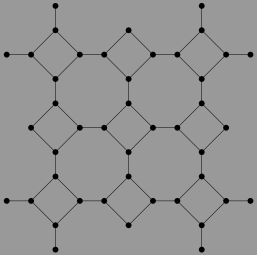


Discovered and analyzed by Jim Propp and collaborators. The shape of the frozen region was conjectured by Cohn and Pemantle, proved by Kenyon and Okounkov using variational principle.

fortresses

Let us describe what is depicted here.





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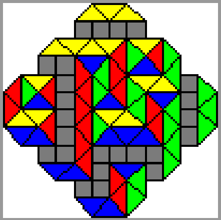
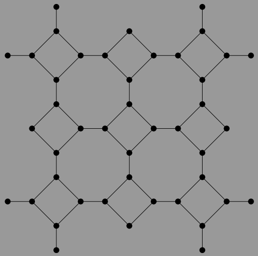

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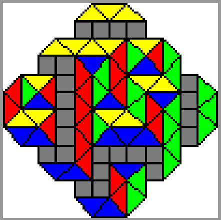

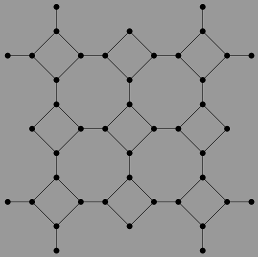

This is a fancy way to represent a *random matching*, or *dimer configuration* on the right graph (*fortress*).
Some combinatorial magics (invented first by Jim Propp) yields *exact* generating functions for the *probabilities of particularly oriented edges* (not for faint-hearted, though fits on 2 pages).

generating functions

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Let p_{kln} be the probability to have red domino (red triangle) at position (k,l) in n-th random tiling of Aztec diamond or n-th random grove or n-th fortress, and set

$$F(x,y,z) = \sum_{k,l,n} p_{kln} x^k y^l z^n.$$

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$$F_a(x,y,z) = \frac{z/2}{(1-yz)(1-\frac{z}{2}(x+x^{-1}+y+y^{-1})+z^2)}$$

for Aztec diamonds

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for Aztec diamonds and

$$F_g(x,y,z) = \frac{2z^2}{3(1-z)(1+xyz-\frac{1}{3}(x+y+z+xy+xz+yz))}$$

for groves.

The fact that the coefficients p_{kln} are probabilities imply that the asymptotic information about the behavior of p along the ray

$$n \to \infty, k/n \to u, l/n \to v$$

is encoded in the local geometry of the *pole divisor* near $(1, 1, 1)$: pole divisor is the variety

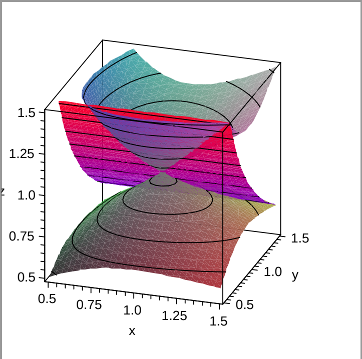
$$\mathcal{A} = \{Q = 0\},$$

where Q is the denominator of the rational function

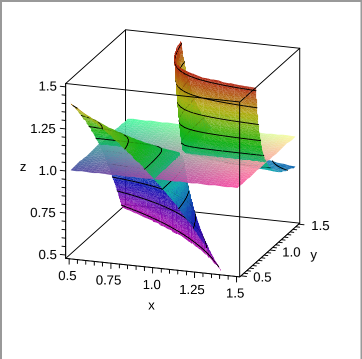
$$F = \frac{P(x, y, z)}{Q(x, y, z)}.$$

Let us look at $\mathcal{A}$ near the point $(1,1,1)$. Here is the pole varieties:

for the Aztec diamond



and for the random groves



In both cases there are two components near $(1,1,1)$: a smooth one ($\{yz = 1\}$ in the Aztec diamond case; $\{z = 1\}$ for groves), and a quadratic singularity, *intersecting the smooth component in the real domain*.

geometry near $(1,1,1)$

In fact, the asymptotics of p_{kln} is already reflected in the geometry of the pole divisor near $(1,1,1)$. Consider the principal homogeneous part of the singular component of $\{Q_g = 0\}$ near $(1,1,1)$: in coordinates

$$u := x-1, v := y-1, w := z-1$$

it is given by

$$q_g = w(uv + uw + vw).$$

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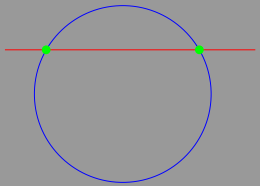
Consider the projectivization of variety $\{q_h = 0\}$:



Blue is the projectivization of the quadric $\{uv+uw+vw=0\}$, red line is for $\{w=0\}$.

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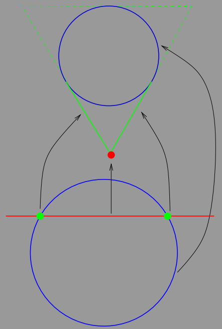
Construct its projective dual:



Under duality, lines go to points; points to lines; quadrics to quadrics.

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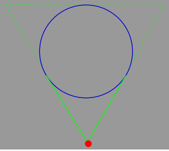
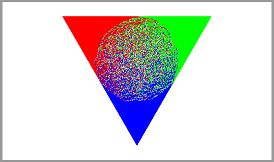
The dual variety reflects the shape of asymptotic support of p:





Support is the convex hull of the quadric and the point corresponding to $\{w = 0\}$.

precise asymptotics

In fact, a much more precise result can be derived.

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Assume that the essential singularity governing the asymptotics of p_{kln} is locally a quadratic cone Q and a smooth stratum with the tangent plane H at the critical point, intersecting the quadratic cone transversally in the real domain (as it happens in the case of Aztec diamond tilings and cube groves)

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Then the asymptotics in the direction

$$k/n \to u, l/n \to v, n \to \infty$$

such that the plane

$$X_{u,v} = \{ux + vy = z\}$$

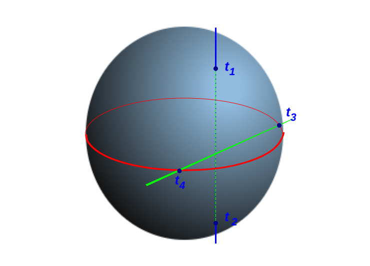
does *not* intersect the quadratic cone in the real domain, is given by

Precise asymptotics

is given by

$$\frac{1}{2\pi i}\log\frac{(t_1-t_3)(t_2-t_4)}{(t_1-t_4)(t_2-t_3)},$$

where t_1, t_2 are the point of intersection of the line $\mathbb{P}X$ with the quadric $\mathbb{P}Q$ in $\mathbb{C}P^1$, and t_3, t_4 are the (real) points of intersections of $\mathbb{P}H$ with $\mathbb{P}Q$:



(Depicted are: the quadric $\mathbb{P}Q$ i.e. a Riemann sphere, with its real part represented as the equator; (projective) line $\mathbb{P}X$, real as well and imaginary $\mathbb{P}H$.)

precise asymptotics

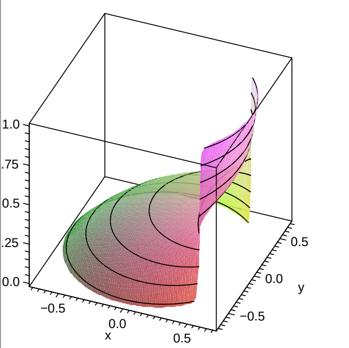
Translating this all back to our respective problems, we obtain the final results: as $k/n \to u, l/n \to v, n \to \infty$, p_{kln} tends to

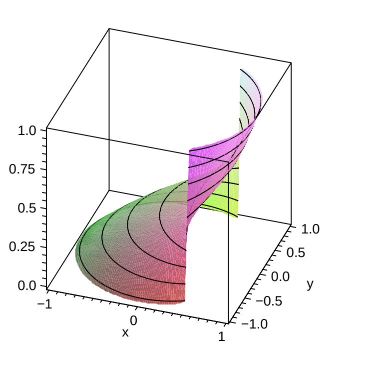
for Aztec diamonds

$$1/2 + 1/\pi \arctan \frac{u - 1/2}{\sqrt{1 - 2u^2 - 2v^2}}$$

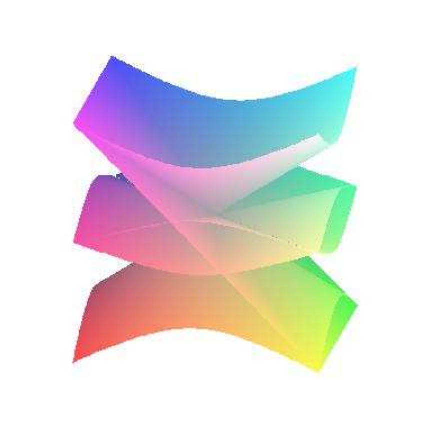
for cube groves:

$$1/2 + 1/\pi \arctan \frac{v - 1/2}{\sqrt{1 - u^2 - v^2}}$$





The pole variety near the point $(1,1,1)$ looks like this:



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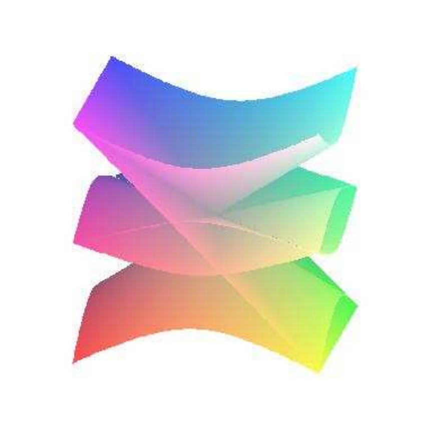
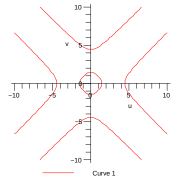


Its principal homogeneous part is a cone over a (reducible) projective curve,

$$(-400 + 200u^2w^2 + 200v^2w^2 + \\ 18u^2v^2 - 9u^4 - 9v^4)(1 + v),$$



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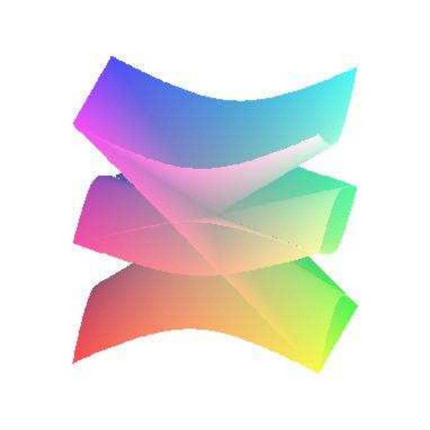

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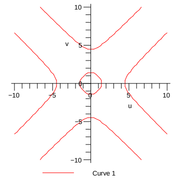
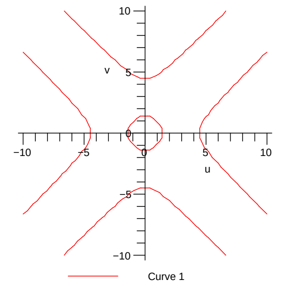
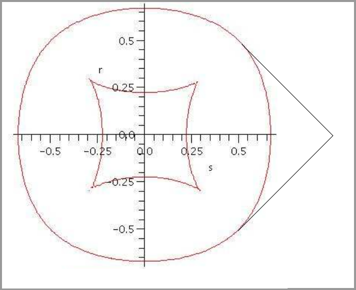


Its component are a projective line, and a singular curve (with 2 double points) which becomes elliptic upon normalization.

some corresponding points/lines

and the dual to which is given by the octic curve





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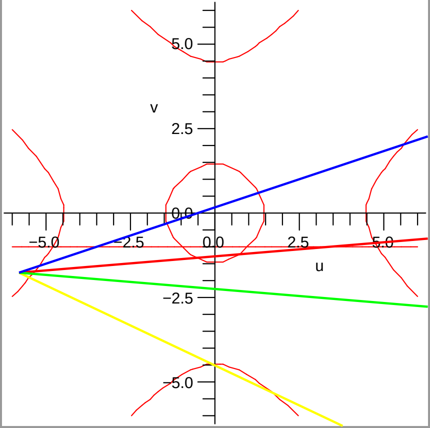

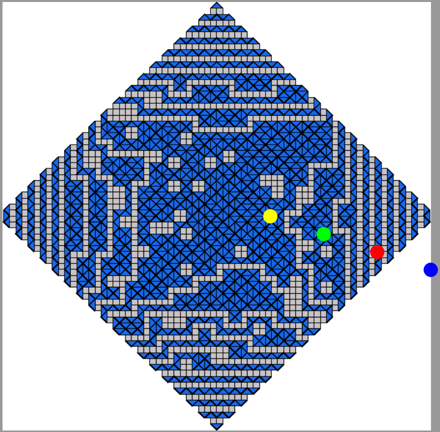

explanations

The theory behind these correspondences involves

- Some multidimensional complex analysis (*contour deformation*)
- A bit of hard-core calculus (*Fourier transforms for distributions*)
- Theory of hyperbolic polynomials (*all polynomials we saw are hyperbolic for a reason*)
- Topology (*resulting contours of integration are somewhat unusual cycles in somewhat unusual homology groups*)
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TBC!