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PÓLYA FESTOONS

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Abstract. This note[1] proposes a natural combinatorial setting for results stated by Pólya regarding the enumeration of 'diagonally convex lattice polygons' also known as parallelogram polyominoes, staircase polyominoes. A brief bibliographical update is also provided.

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In 1969, Pólya published a four page note [11] listing several results relative to the enumeration of *lattice polygons*, by now often referred to as *polyominoes*. Pólya's statements were in fact results entered into his diary in 1938 [11, footnote 2]. They concern both vertically convex and diagonally convex polygons.

(A lattice polygon is a simple closed polygonal line whose vertices are in $\mathbb{Z}^2$ and whose edges are parallel to the x, y axes. A polygon is convex with respect to the direction d if any line parallel to d intersects the domain enclosed by the polygonal line in one segment.)

For vertically convex polygons (polygons convex according to the 90° direction) counted according to area, Pólya gives the generating function

$$\frac{q(1-q)^3}{1-5q+7q^2-4q^3}, \tag{1}$$

a result that was independently derived by Klarner who published several proofs, see [9, p. 32] and Stanley's discussion in [12].

Diagonally convex polygons (i.e., polygons convex according to the $-45°$ direction) are also referred to as parallelograms or staircase polyominoes. They are related to general convex polygons, and by now a fairly extensive literature exists with roots in recreational mathematics, enumerative combinatorics, theoretical computer science or statistical physics.

Pólya gives a bivariate generating function for diagonally convex polygons counted according to both perimeter and area. I am not aware of any published proof by

[1]Research was done while the author was visiting the Department of Mathematics, The University of Melbourne, Parkville, Victoria 3052, Australia.

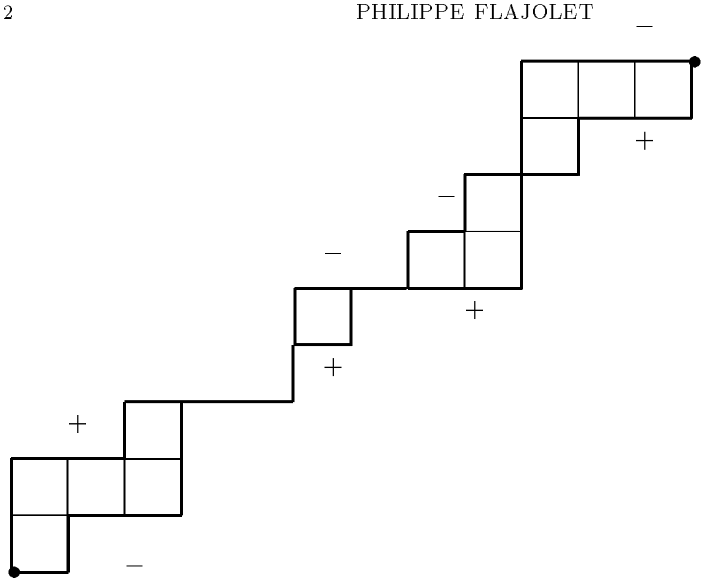

Figure 1. A festoon linking the origin to the point $(12, 9)$, with area -3.

Pólya's of his results[2]. The question of supplying a proof was posed to me by Tony Guttmann and Richard Brak in Melbourne, and I am also indebted to them for several stimulating discussions on the subject. I propose here an extremely simple and elegant argument which is almost certainly that employed by Pólya. In support of this claim, note that 'standard' arguments tend to give ordinary generating functions, while the argument described below leads naturally to a Laurent series, which is precisely the form of Pólya's result. Actually, the argument gives a little bit more than Pólya's original statement.

Let $C_{m,n,k}$ denote the number of diagonally convex lattice polygons having area k, comprising $2m$ steps parallel to the x-axis and $2n$ steps parallel to the y-axis. The corresponding trivariate generating function is

$$C(x, y; q) = \sum_{m,n,k} C_{m,n,k} x^m y^n q^k.$$

Theorem 1. *The trivariate generating function of diagonally convex polygons counted according to height, width, and area satisfies*

(2)

$$C(x, y; q) + C(x, y; q^{-1}) + x + y = 1 - \left(\sum_{m,n \geq 0} x^m y^n \binom{m+n}{m}_q \binom{m+n}{m}_{q^{-1}} \right)^{-1}.$$

where $\binom{m+n}{m}_q$ *denotes the* q*-Gaussian binomial coefficient,*

$$\binom{m+n}{m}_q = \frac{(1-q)(1-q^2)\cdots(1-q^{m+n})}{((1-q)(1-q^2)\cdots(1-q^m)) \cdot ((1-q)(1-q^2)\cdots(1-q^n))}.$$

[2]This is confirmed by inspection of the *Math. Reviews*.

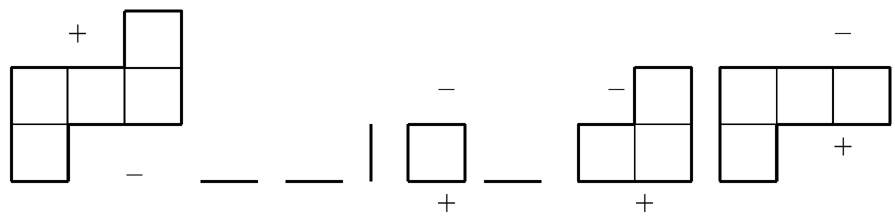

FIGURE 2. The eight components that arise from the decomposition of the example festoon of Figure 1

PROOF. (i). *Festoons and polygons.* The argument rests on the fact that if C is a generating function (GF) for combinatorial objects of some sort, then

$$F = \frac{1}{1-C} \tag{3}$$

is the GF for arbitrary *sequences* of objects of the type counted by C, see for instance [7]. In particular, if F can be determined directly, then the GF for the components, C, is given by

$$C = 1 - \frac{1}{F}. \tag{4}$$

Define a *lattice path* in the plane $\mathbb{Z} \times \mathbb{Z}$ as a path that starts at $(0,0)$ and is formed with either horizontal $(0,+1)$ steps or vertical $(+1,0)$ steps. A *festoon* is an ordered pair of paths which have a common end point. The first path will be referred to as the +path, the second path as the −path.

The essence of the argument is that festoons (F) and polygons (C) have GF's that are linked by the relations (3,4).

(ii). *The GF of festoons.* Define the area below a path as the number of unit squares between the path and the x-axis in the usual way. As is well known [7], the GF of paths ending at a point (m,n) counted according to area is the q-binomial coefficient,

$$\binom{m+n}{n}_q.$$

Define the area of a festoon as the difference of the areas of its +path and its −path. The GF of festoons ending at point (m,n) is

$$\binom{m+n}{n}_q \binom{m+n}{n}_{q^{-1}}.$$

Thus, the trivariate GF of festoons with variables x and y marking the end point coordinates and q marking area is

$$F(x,y;q) = \sum_{m,n} x^m y^n \binom{m+n}{n}_q \binom{m+n}{n}_{q^{-1}}.$$

(iii). *The GF of polygons.* Create two twin copies of the set of polygons, and call them +polygons and −polygons: in a +polygon, the upper side and the lower side are marked with a '+' and a '−' respectively; in the −polygons, this is reversed.

Define the area of an oriented polygon as a signed quantity that is positive or negative depending on whether one has a +polygon or a −polygon.

Clearly, every festoon is decomposable (see Figure 2) as a sequence of elementary components that are of one of the following four types: a horizontal unit step; a vertical unit step; a +polygon; a −polygon. There results the relation

$$F(x,y;q) = \left(1 - x - y - C(x,y;q) - C(x,y;q^{-1})\right)^{-1}. \tag{5}$$

This completes the proof of the theorem. □

The number of festoons comprising a total number of $2p$ steps is

$$\sum_{k=0}^{p} \binom{p}{k}^2 = \binom{2p}{p},$$

by Vandermonde convolution. The corresponding GF is

$$F(z,z;1) = \sum_{p=0}^{\infty} \binom{2p}{p} z^p = \frac{1}{\sqrt{1-4z}}.$$

Thus, the GF of polygons counted according to perimeter is by a specialization of Theorem 1,

$$\begin{aligned} C(z,z;1) &= \tfrac{1}{2}(1 - 2z - \sqrt{1-4z}) \\ &= z^2 + 2\,z^3 + 5\,z^4 + 14\,z^5 + 42\,z^6 + 132\,z^7 + O(z^8), \end{aligned}$$

which is a generating function for the Catalan numbers.

Corollary 1 (Pólya). *The number of parallelogram polygons of perimeter $2n$ is given by the Catalan number*

$$\frac{1}{n}\binom{2n-2}{n-1}.$$

Corollary 2 (Pólya). *The bivariate generating function of parallelogram polygons by perimeter and area satisfies*

$$\begin{aligned} C(z,z;q) + C(z,z;q^{-1}) + 2z &= 1 - \frac{1}{1 + P_1(q)z + P_2(q)z^2 + P_3(q)z^3 + \cdots} \\ &\qquad \textit{where } P_n(q) = \textstyle\sum_{r=0}^{n} \binom{n}{r}_q^2 q^{r(n-r)}. \end{aligned}$$

Despite its unusual form as a Laurent series, Pólya's symmetrical GF *is* an answer to the counting problem for parallelogram polygons. For instance it implies a counting algorithm to determine the number of polygons of area q that has polynomial time complexity; in this way, we determine easily the first few values,

$$\begin{aligned} C(1,1;q) = q &+ 2\,q^2 + 4\,q^3 + 9\,q^4 + 20\,q^5 + 46\,q^6 + 105\,q^7 + 242\,q^8 + 557\,q^9 \\ &+ 1285\,q^{10} + 2964\,q^{11} + 6842\,q^{12} + 15793\,q^{13} + 36463\,q^{14} + O(q^{15}). \end{aligned}$$

Subsequent researchers have concentrated on standard generating functions. Klarner and Rivest [9] found a GF for polygons counted according to area that involves q–Bessel functions,

(6)
$$C(1,1;q) = \frac{\dfrac{q}{1-q} - \dfrac{q^3}{(1-q)^2(1-q^2)} + \dfrac{q^6}{(1-q)^2(1-q^2)^2(1-q^3)} - \cdots}{1 - \dfrac{q}{(1-q)^2} - \dfrac{q^3}{(1-q)^2(1-q^2)^2} + \dfrac{q^6}{(1-q)^2(1-q^2)^2(1-q^3)^2} + \cdots}.$$

Admittedly, such forms are better suited to asymptotic analysis than Laurent series, and Bender [1] proved

$$[q^n]\, C(1,1;q) \approx 0.29745 \cdot 2.30913859330^n,$$

by considering singularities of the GF (6).

The counting technique employed by [1, 9] is that of 'adding a new slice', and it is shown there to provide a GF which takes into account area, height, and sizes of the left and right borders.

Delest and Viennot [4] introduced a $-45°$ scan that transforms a parallelogram into a well parenthesized expression ('Dyck word'), and they derive in this way a GF which takes into account both width and height. This approach has been extended by Fédou [5], and further by Bousquet–Mélou [2] in order to include area. In relation to the continued fraction approach of [6], one derives the representation,

$$C(x,y;q) = \cfrac{xyq}{1-(x+y)q - \cfrac{xyq^3}{1-(x+y)q^2 - \cfrac{xyq^5}{1-(x+y)q^3 - \cfrac{xyq^7}{\cdots}}}}.$$

Independently motivated by statistical physics problems, Brak and Guttmann [3] as well as Lin and Tzeng (see [10]) have obtained GF's according to area and perimeter. The general approach is a recurrence based on the length of the leftmost side, combined with an 'Ansatz' for solving q–linear recurrences with coefficients linear in q^n.

All the generating functions based on area and obtained by these various approaches are refinements of the Klarner and Rivest generating function (6), and they involve one form or another of q–Bessel function. Perimeter generating functions, for reasons well accounted for by Delest and Viennot, are plainly algebraic.

Finally, there are by now many papers dealing with the enumeration of a variety of convex polygons, and the reader is advised to consult a recent survey by Guttmann [8] for an extensive bibliography.

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