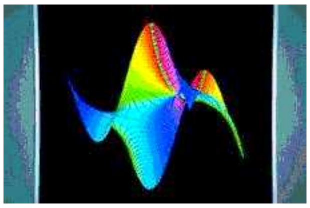

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Analytic Aspects of Digital Trees

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A largely historical perspective on rich algorithmic and analytic aspects:
—- initially a data structure for textual data,
— also an abstract splitting process,
$\rightsquigarrow$ with various applications to:
— data bases
— communication protocols
— computational geometry
— computer algebra, semi-numerical algorithms

The digital tree (trie) data structure

The Unix dictionary has:

23,000 words for total 194,000 characters $\approx$ 8 characters per word Information is redundant:

```
    bestial bestir bestowal bestseller
    bestselling
```

- Save space by grouping common prefixes,

```
BEST
---- I
---- - AL
---- - R
---- - OWAL
---- SELL
---- ---- ER
---- ---- ING
```

- Save search time compared to binary search

Get about 4^- letter per word.

The mathematical definition

— Split $E \subset$ Words into $E_A, \ldots, E_Z$;

— Recursively continue with next position

— until all words are separated

++ Use branching structure impl. by pointers/edges

The algorithmic construction

— Insert by splitting at external nodes

```
ABSE(ntia), ABSO(lve) + absolute
                   ...............
ABSE(-ntia), ABSOLU(-te), ABSOLV(-e)
```

The underlying sort (Radix exchange sort)

— Partition & Sort on leading digits, second leading, etc

Refs: Fredkin, Trie memory, 1960; Knuth, vol 3, 1973

1. First analyses

Binary alphabet, Unbiased memoryless model

$$\Pr(\texttt{char} = 0) = \Pr(\texttt{char} = 1) = \frac{1}{2}$$

Probability of splitting n into $\langle k, n-k \rangle$ is

$$\pi_{n,k} = \binom{n}{k} 2^{-n}$$

Probabilistic divide-and-conquer recurrence for expectations

$$f_n = t_n + \sum_k \pi_{n,k}(f_k + f_{n-k})$$

where toll t_n is 1 (size) or n (construction cost $\gg$ depth).

Introduce exponential generating functions (EGF)

$$f(z) := \sum_n f_n \frac{z^n}{n!}$$

and get functional equation

$$\boxed{f(z) = t(z) + 2e^{z/2} f(\frac{z}{2})}$$

solved by iteration

$$f(z) = t(z) + 2e^{z/2} t(\frac{z}{2}) + 4e^{z/2} e^{z/4} t(\frac{z}{2}) + \cdots$$

Asymptotic analysis

For construction cost, aka path length $\gg$ depth: $t_n = n$, $t(z) = z(e^z - 1)$:

$$E(D_n) = \frac{f_n}{n} = \sum_{k \geq 0} \left[1 - \left(1 - \frac{1}{2^k} \right)^{n-1} \right]$$

De Bruijn and Knuth, ca 1965: $E(D_n) = F(n) + o(n)$

$$F(x) := \sum \left(1 - e^{-x/2^k} \right)$$

The way $F(x)$ behaves at $+\infty$ is attained via the Mellin transform

Mellin analysis of harmonic sums. [F. Gourdon, Dumas, TCS'95]

Nowadays: Asymptotics of harmonic sums is often simple

Some 50 analyses!!

$$f(x) \quad \mapsto \quad f^*(s) := \int_0^\infty f(x) x^{s-1}\, dx$$

$$F(x) = \sum \lambda_k \phi(\mu_k x)$$

$$\downarrow$$

$$\boxed{\text{Mellin transform}}$$

$$\downarrow$$

$$F^*(s) = \left(\sum \lambda_k \mu_k^{-s} \right) \cdot \phi^*(s)$$

$$\text{Asympt}(F) = \text{Asympt}(\phi) \times \text{Sing}(\textstyle\sum \lambda_k \mu_k^{-s})$$

Derives from scaling property of M.T.'s, inversion theorem, and residues

Here:

$E(D_n) = F(n) + o(1), \qquad F(x) := \sum \left[1 - \exp(-x/2^k)\right]$

$e^{-x} \;\mapsto\; \Gamma(s)$ with poles at $0, -1, -2, \ldots$

$\sum (2^{-k})^{-s} = \dfrac{1}{1-2^s}$ has singularities at

- $s = 0$: residue is $\dfrac{1}{\log 2}$
- $s = 2ik\pi/\log 2$: complex poles $\Longrightarrow$ fluctuations

— Completes the basic analysis of basic trie

— Adapts to biased memoryless models

- S_n = Size; D_n = Depth

Theorem 1 [DBK 1965$^{\pm}$; K 1973]. For unbiased memoryless source, size S_n and D_n

$$\begin{cases} E(S_n) & = & \frac{2}{\log 2}(n + P(\log_2 n)) + o(n) \\ E(D_n) & = & \log_2 n + Q(\log_2 n) + o(n) \\ P(w), Q(w) & \mapsto & \sum_{k \in Z} \Gamma(\frac{2ik\pi}{\log 2}) e^{2ik\pi w} \end{cases}$$

---

m-ary trie under general Bernoulli model

Theorem 2 [K 1973, ...]. Memoryless sources

$$E(S_n) \approx \frac{n}{h} \quad E(D_n) \sim \frac{1}{h} \log n,$$

Involves **entropy**: $h = \sum p_i \log \frac{1}{p_i}$.

[Fayolle, Flajolet, Hofri 1986]: Fluctuations in size are small and depend on arithmetic prop. $\frac{\log p}{\log q} \in, \notin \mathbb{Q}$.

Eg. Fluctuation for $p = \frac{2}{1+\sqrt{5}}$ but not for $p = \frac{1}{\pi}$ (!!).

2. Database applications: bucketing and height

Around 1978, several people invent

adaptive hashing schemes

= split overflow buckets instead of chaining

- Larson's dynamic hashing

— Build trie on hashed values;

— Stop splittings when sets have cardinality $\leq b$.

For large data bases, get an index tree stored in main memory; 1 external page access.

Theorem 3 [K 1973, Larson]. Filling ratio of bucket tries depends on entropy h

$$E(S_n^{(b)}) \approx \frac{1}{b \cdot h} n.$$

For "good" hash function, buckets are $\log 2 = 69\%$ full.

Extendible hashing

For very large data bases, the index itself must be paginated.

- Fagin, Nievergelt, Pippenger, Strong's algorithm

— Embed the index tree into a perfect tree;

— Reorganize occasionally when height increases.

0	1	2*3	4	5	6	7	
α	α	α	α	β	γ	δ	δ

Grants 2 disk accesses: 1 (index) + 1 (data). Storage cost is $I_n = 2^{H_n}$ Analysis by Yao, Régnier, Flajolet 1978–1983.

Theorem 4 [F. 1983] Height H_n

$$E(H_n) \sim \left(1 + \frac{1}{b}\right) \log n, \quad E(2^{H_n}) \sim R(\log n) n^{1+1/b}.$$

Safe if $b \gg 20$ (otherwise go hybrid!) and with typical losses of a factor of 4 on index sizes. Well-configured extendible hashing provide direct access to very large files.

1. The Poisson(z) model is easier: Independence of subintervals entails

$$\Pr_z\{H \le k\} = \left(e^{-z/2^k} e_b\left(\frac{z}{2^k}\right)\right)^{2^k} \qquad e_b(z) := 1 + \cdots + \frac{z^b}{b!}.$$

2. Use principle of algebraic depoissonization

Poisson expect. $\times\ e^z \equiv$ Bernoulli[a] GenFun

3. Invert asymptotically by saddle point method applied to Cauchy integral coefficient formula.

See JaSz98, TCS 1998 as analytic depoissonization.

Nonprobabilistic situation of Poisson with complex rate $\mapsto$ Bernoulli model!

[a] Bernoulli $\equiv$ "fixed-size"!

3. Distributed algorithms: Leader election

Simple algorithm: Elect a leader in $\approx \log_2 n$ steps:
— Transmit desire to be leader
— If collision, persist with probability $\frac{1}{2}$.

This is equivalent to length of leftmost branch in trie. See Sedgewick-F. 1996.

Improved algorithm [Prodinger]: If unsuccessful, recursively restart with last nonempty group.

This is equivalent to length of leftmost border in trie and relates to max of geometric random variables [Prodinger].

Proposition. Simplified leader election succeeds with probability $1/(2\log 2) \pm 10^{-5}$ in $\log_2 n + O(1)$ rounds.

[Prodinger 1993] Improved leader elecetion succeeds with probability 1 in $\log_2 n + P(\log_2 n) + o(1)$ rounds.

Distributed algorithms: the tree (stack) protocol

Shared channel access, e.g., Ethernet.
— Everybody speaks when ready; all listen
— Resolve collisions probabilistically
(Ethernet uses "procrastination".)

The tree protocol based on dichotomy (Capetanakis'77)
— If collision, then split into 2 subgroups G_0, G_1;
— Transmit G_0 first, G_1 afterwards.

Implement by means of a stack (Tsybakov et al.)
— Transmit if `stack` $=$ 0;
— Flip coin of `stack` $=$ 0 and Collision;
— Do `stack` $=$ `stack` ± 1 based on channel

```
A  B  -  D
B  D       B
C             C  A
D             A     C
-------------------------
x  x  - I  I  x  I  I
```

Massey, Gallagher, Berger, . . . ca 1980.
(Connection with tries not immediately recognized!)

Blocked access: Mean time is $(1 + 2S_n)/n$ to serve a request. Hence, under Poisson arrivals,

Stable: $\lambda < \frac{\log 2}{2} - 10^{-5}$; unstable: $\lambda > \frac{\log 2}{2} + 10^{-5}$.

Free access: Fayolle, F, Hofri, Jacquet 1985/6

Theorem 5 [FFH 1986] Binary tree protocol with free access is stable until

$$\lambda^{(2)}_{\max} = 0.36017 > \frac{\log 2}{2} = 0.34657$$

Theorem 5' [Mathys F. 1985] Ternary tree protocol with free access is stable until

$$\lambda^{(3)}_{\max} = 0.401599 > \lambda^{(2)}_{\max} = 0.36017.$$

Gain 10% in throughput by changing `stack=stack+1` into stack=stack+2 upon collision.

$$F(z) = t(z) + F(\lambda + pz) + F(\lambda + qz)$$

1. Iteration semigroup generated by two linear transformations,

$$\text{fbox}\sigma_0(z) = \lambda + pz, \quad \sigma_1(z) = \lambda + qz.$$

2. Mellin analysis
3. Dynamics: use globally invariant measure (uniform) under $\{\sigma_0, \sigma_1\}^\star$ to recover singularities of Dirichlet series.

Bidirectional Cable TV networks[a] using hybrid fiber coaxial systems ... The IEEE 802.14 draft specifications [1] include Tbound Access for FTR, and a cluster implementation for the tree-based collision resolution algorithm.

[1] IEEE 802.14 Working Group, Media Access Control, IEEE Draft Std. 802.14, Draft 3 R3, October 1998.

[a] `http://www.comsoc.org/pubs/surveys/1q99issue/golmie.html` by Golmie, Saintillan, Su; NIST 1999.

Probabilistic estimation algorithms for data bases

Database query optimization, needs low cost **estimate**:
Cardinality of a multiset = number of <u>distinct</u> elements.

Question 1. How many different words is there approximately (within 5%, say) in Shakespeare's writings? Allow a single scan and a single sheat of paper.

Question 2. If you and your friend are 1,000 miles aprt with a phone line, can you do it? (Data are distributed.)

Wegman's Adaptive Sampling based on hashing:
— Scan file, keep a buffer of maximal size b (e.g., $b = 100$), and start with $\delta = 0$;
— If buffer overflows, then increase depth δ by 1, and filter out elements that don't have a 0 in position δ
Return as estimate: $2^{\delta-1}X$ with X='current buffer size'.

Theorem 6 [F. 1990] Adaptive sampling is unbiased with std. dev. of estimate: $\frac{\sigma_n}{n} \approx \sqrt{\frac{1}{(b-1)\log 2}} \approx \frac{1.20}{\sqrt{b}}$.

$\mapsto$ Analyse leftmost branch of a bucket trie!

Probabilistic counting [F. Martin 1995]

Keep a BITMAP vector $b = (b_1 b_2 \cdots)$ where $b_j = 1$ if $0^j 1$ has been observed;
Return $2^{MaxRank}/\phi$ where $MaxRank$=max rank of a 1.

Theorem 7 [F., Martin 1985] Probabilistic counting is asymptotically unbiased with

$$\phi = \frac{e^\gamma}{\sqrt{2}}(\frac{2}{3})(\frac{4}{5})^{-1}(\frac{6}{7})^{-1}(\frac{8}{9})\cdots \approx 0.77351,$$

Related to left branch of trie.

++ With "stochastic averaging", get accuracy of $\approx \frac{0.78}{\sqrt{b}}$.

4. Advanced data models for tries

Devroye 1986 considers binary digits on skewed density distribution over $[0,1]$.

Theorem 8 [Devroye 1986] (i) For initial density $f(x)$,

$$E(H_n) \sim 2\log_2 n \quad \text{if} \quad \int_0^1 f^2(t)\,dt < +\infty,$$

and $E(H_n) = +\infty$ otherwise.
(ii) Accordingly, $E(S_n) = O(n)$ or $E(S_n) = +\infty$.

Jacquet, Szpankowski et al. considered Markov sources

Theorem 9 [Sz-Ja 1991] (i) For a Markov source,

$$E(H_n) \sim 2\frac{\log n}{\log 1/\lambda(2)}$$

where $\lambda(2)$ is the coincidence probability:

$$\lambda(2) := \text{max. eigenval} P^{\odot 2}.$$

(ii) Size/depth involve entropy
$h = \sum \pi_k \left(\sum p_{kj} \log \frac{1}{p_{kj}}\right)$.

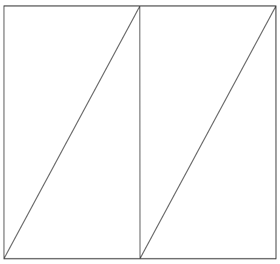
The model of "dynamical sources"

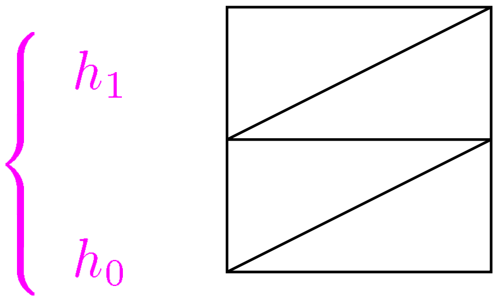
[Vallée 1999; Clément-F.-Vallée 1999]

A general framework of analysis based on fundamental intervals and transfer operators.

Binary system: $Tx = \{2x\}$ has 2 sections T_0, T_1;
$T^{-1}x = \{\frac{x}{2}, \frac{x+1}{2}\}$ has 2 branches h_0, h_1.

$T(x)$:



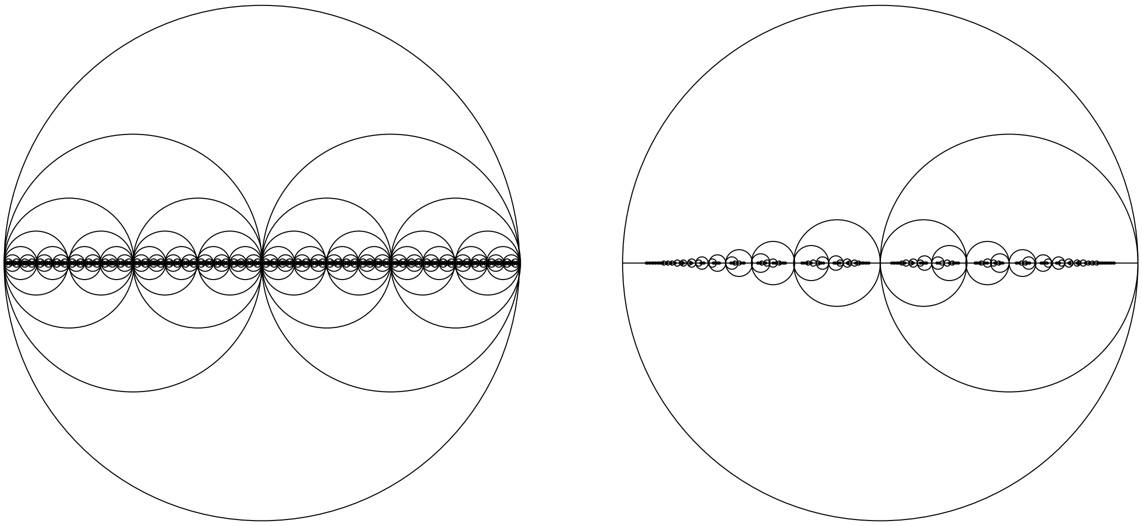


— Assume analytic initial density $f(x)$;

— Assume symbolic coding given by analytic expanding map T, with $|T'| > 1$, i.e., $|h'| < 1$ for $h \in T^{-1}$.

The Mincing Process of dynamical sources

[Vallée 1999]



Property A. "Moments" $M_k^{(s)} := \sum_k |u|^s$ exist,

$\left(M_k^{(s)}\right)^{1/k} \to \lambda(s)$

Property B. Entropy exists and does derives from moments $\frac{1}{k} H_k \to h \equiv -\lambda'(1)$

Based on spectral properties of transfer operators (dominant eigenvalue $\lambda(s)$ + analytic perturbation)

$$\mathcal{G}_s[f](x) = \sum_{h \in T^{-1}} (h'(x))^s f(h(x)).$$

and $(I - \mathcal{G}_s)^{-1}$ "generates" fundamental intervals.

++ new brand of **secant operators**.

Theorem 10 [Clément-F.-Vallée 1999] Under general dynamical sources, size, path length and height satisfy

$$\begin{cases} E(S_n) \approx \dfrac{n}{h}, \qquad E(D_n) \sim \dfrac{1}{h}\log n, \\ E(H_n) \sim 2\dfrac{\log n}{\log 1/\lambda(2)} \end{cases}$$

♡ Unifies memoryless and Markov analyses.

♡ Applies to other sources, e.g., continued fraction digits

♡ Applies to fast data structures [Bentley-Sedgewick'97]

ternary search tree (TST)
= skeleton trie + binary search trees on chars

Theorem 10' [Clément-F.-Vallée 1999] Under general dynamical sources,

$$E(D_n) \sim \frac{K}{h}\log n.$$

= Beats hashing, useful on OCR, unicode characters, &c.

Distributions & compression: some analytic jewels!

Variance $\mapsto$ modular form identities of Ramanujan!

Theorem 11. [Jacquet-Régnier] Size and Path length are asymptotically Gaussian.

[Jacquet-Szpank'95] Same for **digital search trees**, where root keeps one elem. and rest splits,

$$\boxed{\frac{\partial F(z,u)}{\partial z} = F(pz,u)F(qz,u) + \mathtt{Poly}(z,u)}$$

♠ **difference-differential equations**.

♣ An analytic theory based on Mellin tr. and Rice method + analytic depoissonization.

$\rightsquigarrow$ Related *by inversion* to Lempel-Ziv compression.

Corollary. For LZ78: Gaussian fluctuations in number of "segments"; redundancy for memoryless is

$$E(r_n) \sim \frac{A + \delta(n)}{\log n}$$

Cf Louchard, Szpankowski, Savari.

Unexpected applications

Cantor-Zassenhauss factorization. [F-Go-Panario'98]

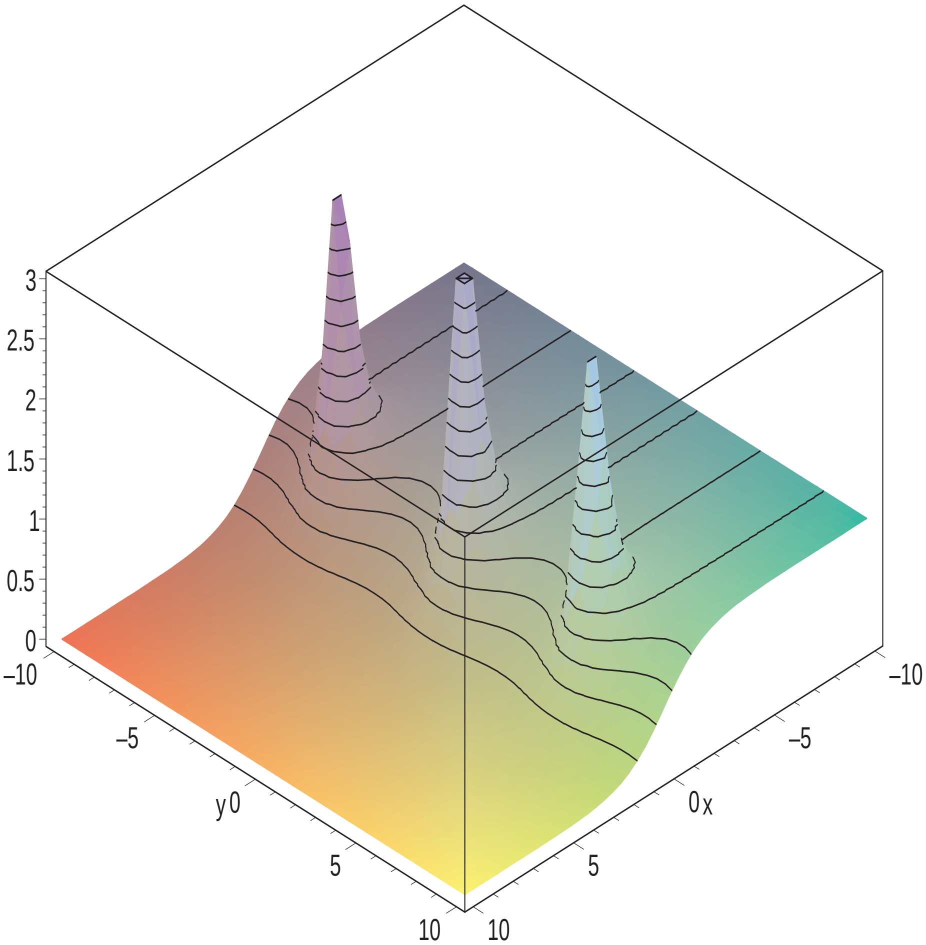
```
                 5      2             3    2
factor(1 + z + z ) = (z  + z + 1) (z  - z  + 1)
```

HAKEM ca 1970: *To compare two rationals, lazily expand them into continued fractions.*

Theorem 12. Number of digits necessary to distinguish n real numbers based on continued fraction rep's has mean

$$S_n^{CF} \sim \frac{6 \log 2}{\pi^2} n + \Phi(n) + O(1).$$

Order of $\Phi(n) \in [n^{1/4}, n^{1/2}] \leftrightarrow$ Riemann hypothesis.



3
2.5
2
1.5
1
0.5
0
–10
–5
y 0
5
10
10
5
0 x
–5
–10