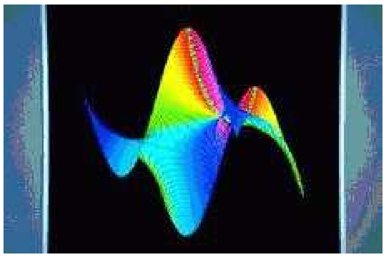
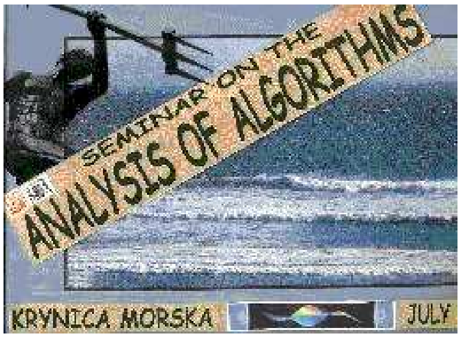




Some Airy Phenomena of Analytic Combinatorics

PHILIPPE FLAJOLET, INRIA, FRANCE

Cyril Banderier, Gilles Schaeffer, Michèle Soria

Krinica Morska — July 2000

Analytic combinatorics, like probabilistic combinatorics, strives to recognize **"universal phenomena"**. For analytic combinatorics, this is based on **generating functions** and **complex-analytic asymptotics.**

Saddle-point in relation to Cauchy coefficient formula applies to: large powers, large singularities. Implications are, e.g., tree enumeration [univariate] as well as Gaussian laws [multivariate] and LLT's.

Here: **Coalescing saddles** are of some generality and lead to **Airy functions**. This is illustrated *inter alia* by **random planar maps**.

Looking for UNIVERSALITY

Probabilistic combinatorics

$$\mathcal{M}_n \Longrightarrow \mathcal{M}_\infty \Longrightarrow \mathcal{M}_n|_{n\to\infty}$$

discrete continuous discrete

♣ $\sum$-RV's; Brownian motion, branching processes, Aldous' continuum random tree, &c.

Analytic combinatorics

$$\mathcal{M}_n \Longrightarrow M(z,\ldots) \Longrightarrow \mathcal{M}_n|_{n\to\infty}$$

discrete continuous discrete

♡ $\sqrt{\ }$-sing., saddle-point, Bender-Hwang's Quasi-powers, &c.

The simple saddle point (review)

$$\boxed{I(n) = \frac{1}{2i\pi} \int e^{h_n(z)}\, dz}$$

saddle point at $\zeta \equiv \zeta_n \;:\; h_n'(\zeta) = 0$

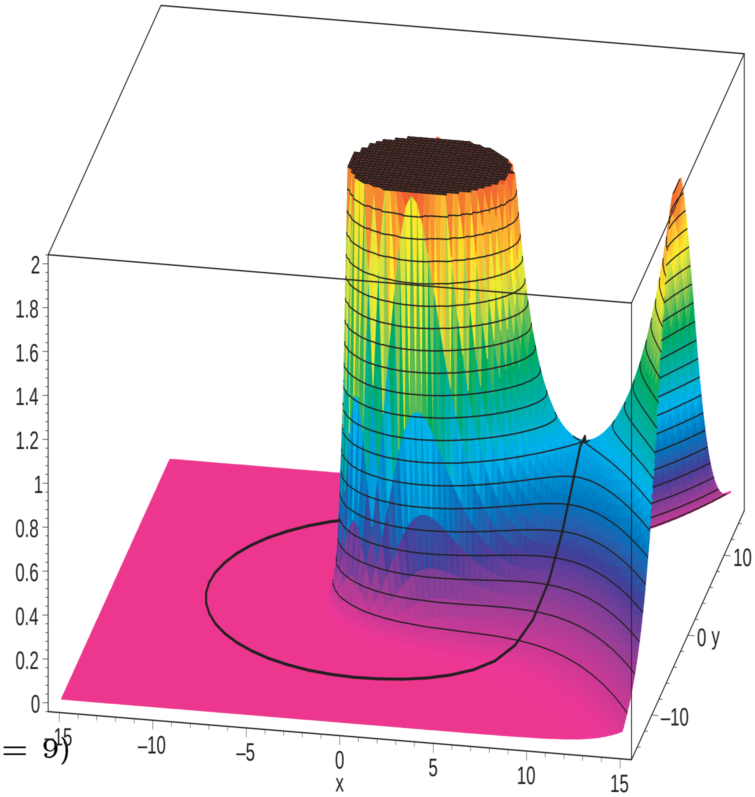
via: $e^{h_n(\zeta)} \int \exp(\frac{t^2}{2} h_n''(\zeta))\, dt$

$+$ concentration as $n \to \infty$

$$\boxed{I(n) \sim \frac{1}{2\pi} \left((2!)^{1/2}\Gamma(1/2)\right) \frac{1}{\sqrt{h_n''(\zeta)}} e^{h_n(\zeta)}.}$$

Normalization to:

Exp $\circ$ Quadratic; scales $\sqrt{n}$, ...



$\int e^z z^{-n-1}\, dz\ (n = 9)$

Applications of the simple saddle point

♡ Large singularities.

$e^{P(z)}$ & perms with cycles $< m$

$e^{z/(1-z)}$ & increasing subseq. in perms, int. partitions.

e^{e^z-1} & set partitions

Differential eq. and irregular singularity, ...

♡ Large powers

Trees via Lagrange [after Darboux]: $T = z\phi(T)$ ($\not\approx$ S.A.)

♡ Quasi-powers (Gardy, Hwang, etc)

$[z^n]F(z,u) \approx \rho(u)^{-n}$. S.P. gives local limit thm.

$[z^n]F(z,u) \approx e^{h_n(\zeta_n(u))}$. SP gives local limit thm.

The double saddle point ♡ Easy!

$$\boxed{I(n) = \frac{1}{2i\pi} \int e^{h_n(z)}\, dz}$$

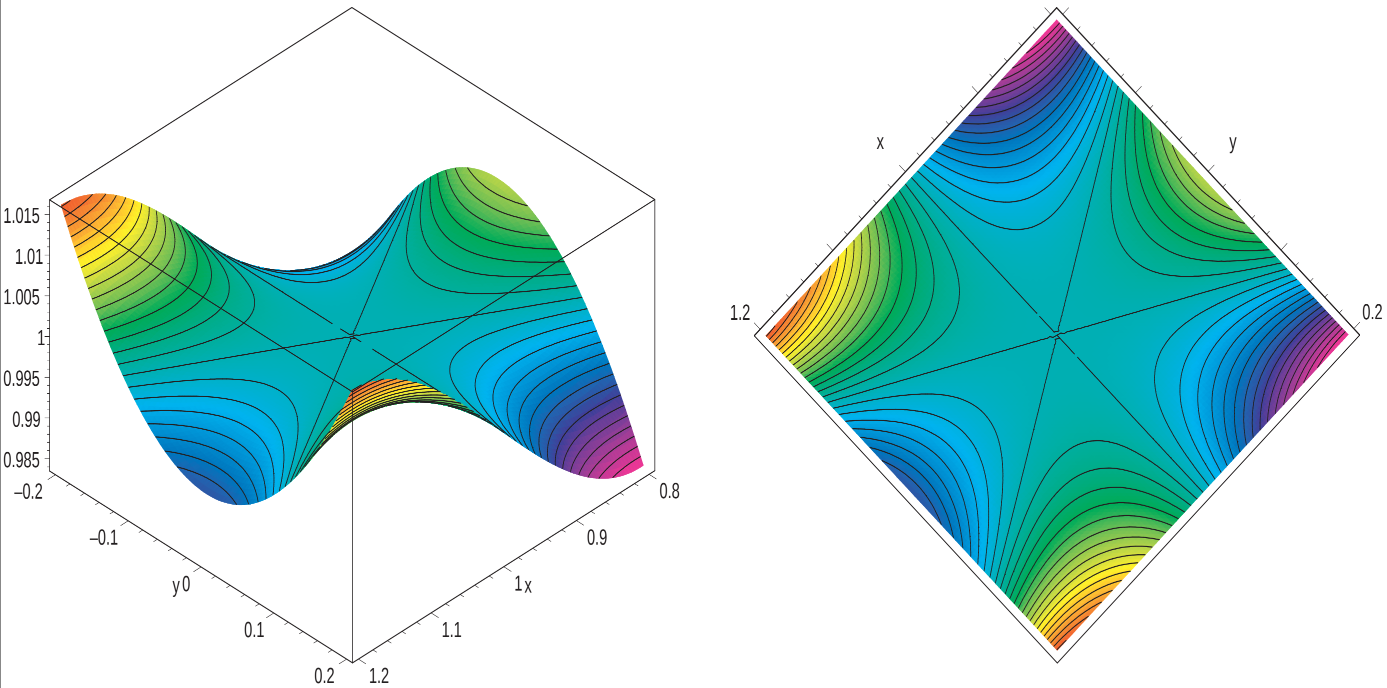
double saddle point at $\zeta \equiv \zeta_n \;:\; h_n'(\zeta) = h_n''(\zeta) = 0$

via: $e^{h_n(\zeta)} \int \exp(-\frac{t^2}{2}|h_n''(\zeta)|\, dt$

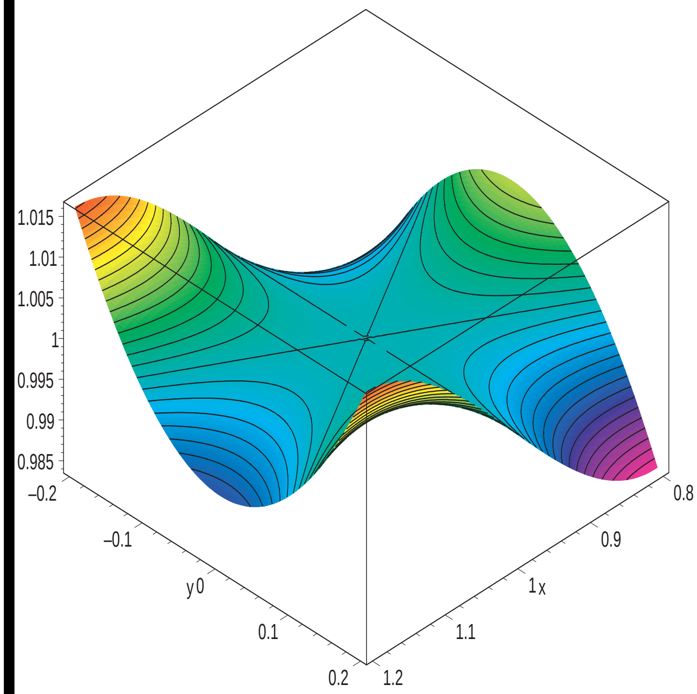
$$\boxed{I(n) \sim \frac{1}{2\pi} \left((3!)^{1/3}\Gamma(1/3)\right) \frac{1}{\sqrt[3]{h_n'''(\zeta)}} e^{h_n(\zeta)}.}$$

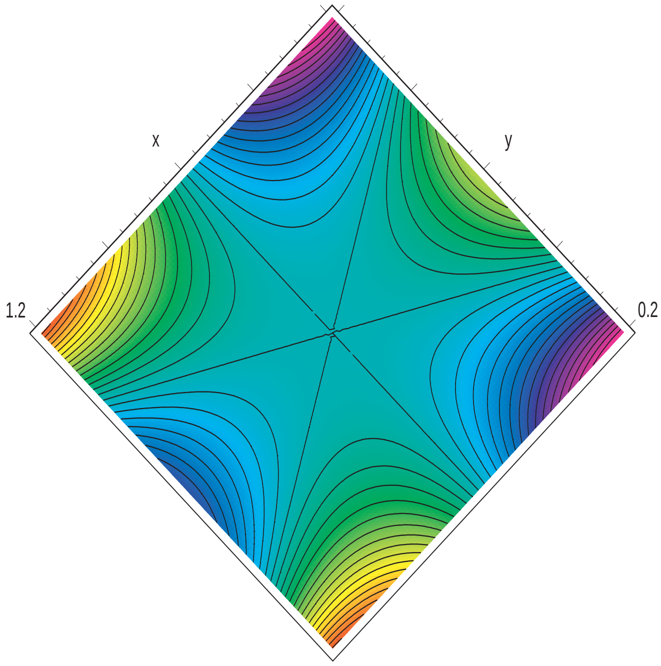
<u>Normalization to:</u>

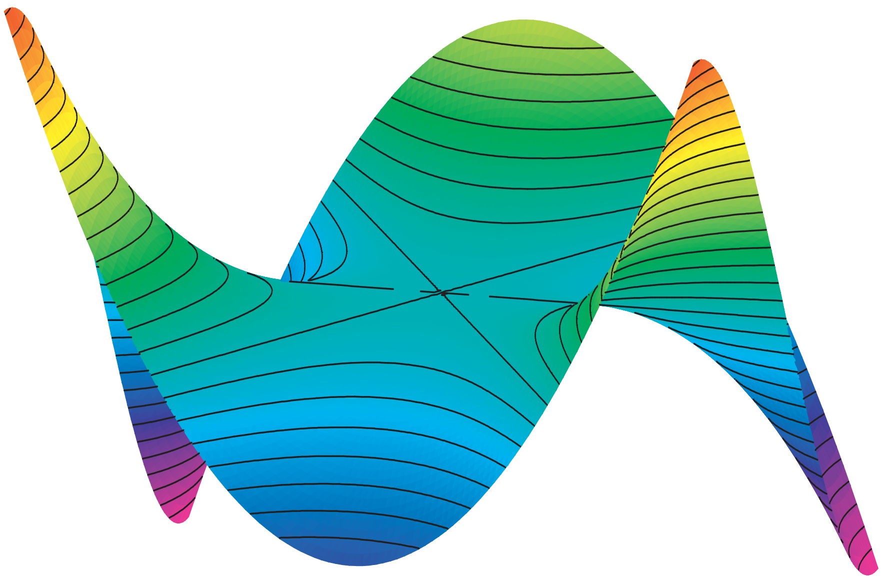
Exp ∘ Cubic; scales $\sqrt[3]{n}$, ...



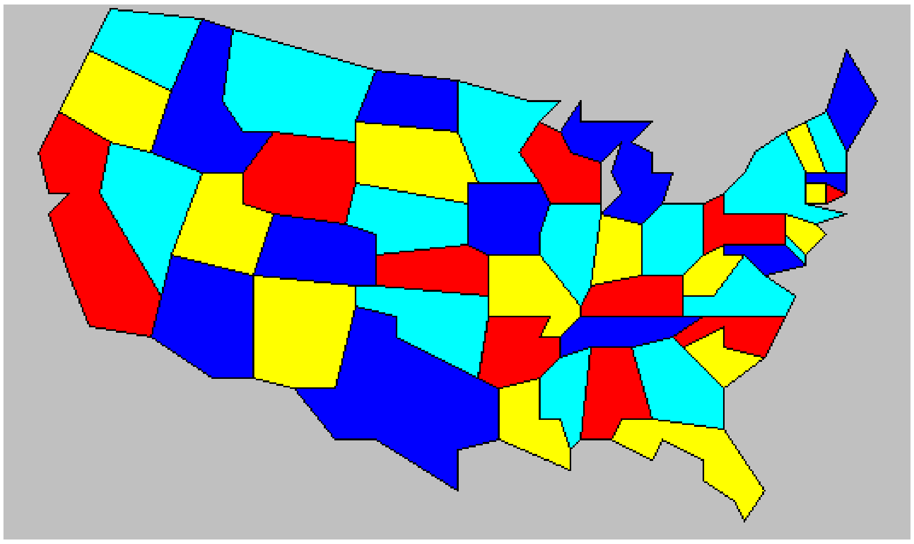
Monkey Saddles







Maps and a double saddle point



Maps are planar graphs with an embedding in the plane (or sphere) plus a *rooted* edge.

Consider:

- Maps $\mathcal{M}$ of sorts;
- Core maps $\mathcal{C} \subset \mathcal{M}$ with higher connectivity;
- The core of map $\in \mathcal{M}$ is the maximal $\mathcal{C}$-component of the root.

Problem is: Analyse Core-size of random map.

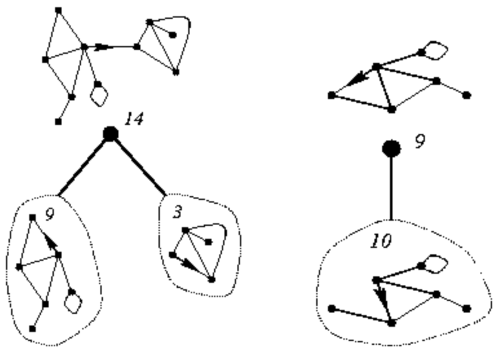
Examples:

— General 1-connected Maps $\mathcal{M}$; 2-connected Cores $\mathcal{C}$;

— 2-connected Maps $\mathcal{M}$; 3-connected Cores $\mathcal{C}$; &c.

Tutte's analyses I

I. Root-face decomposition: delete root edge and get some pieces.



$$F(z,u) = 1 + xu^2 F(x,u)^2 + xu \frac{F(x,1) - uF(x,u)}{1-u}.$$

Complete trinomial $A^2 = B$; Bind x, u by $A = 0$; express that $B = B'' = 0$; get algebraic functions $\mapsto$ Banderier.

$\Longrightarrow u(z)$, $F(z,1)$, $F(z,u)$ are algebraic:

$$z = \frac{(1-u)(3-u)}{u^2}, \qquad F(z,1) = u\frac{4-3u}{(3-2u)^2}.$$

Here: $\quad M_n = 2 \cdot 3^n \frac{(2n)!}{n!(n+2)!} \sim \frac{2}{\sqrt{\pi}} \frac{12^n}{n^{5/2}}.$

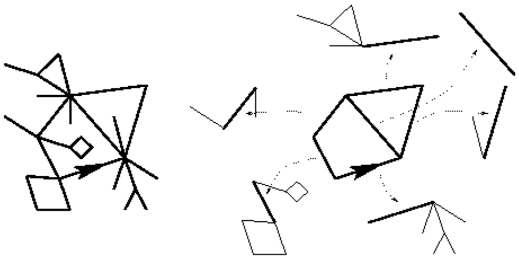
Use Lagrange (+SP) or singularity analysis with singular exponent $\frac{3}{2}$. Works in general:

$$U = z\phi(U), \qquad M = \psi(U); \qquad M_n \sim \mu \frac{A^n}{n^{5/2}}.$$

Tutte's analyses II

II. Grafting (Composition) of maps:

$\mathcal{C}$: class of 2-connected loopless maps.



$$M(z) = C(zM(z)^2)$$

$$C_k \sim \mu' \frac{B^k}{k^{5/2}}.$$

By same principle, size of core = max. 2-connected component of root is

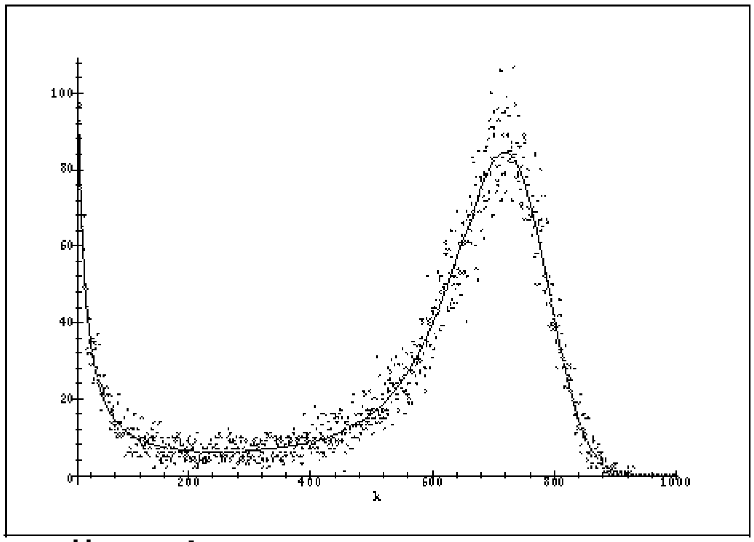
$$M(z,u) = C(uzM(z)^2)$$

The probability distribution of the core is given by

$$\Pr(X_n = k) = C_k \frac{[z^n]M(z)^k}{[z^n]M(z)}.$$

In the sequel [BFSS 2000] $\mathcal{M}$aps are 2-connected and $\mathcal{C}$ores are 3-connected.

A. The distribution of core size $C(uM(z))$.



$\mapsto$ G. Schaeffer

— Small region

— Left tail

— Central region & center of central region

— Right tail

Theorem 0. Small region ($k = O(1)$) is given by singularity analysis: $C_k \cdot [z^n]M(z)^k \propto C_k k \lambda^{k-1} \approx k^{-5/2}$.

Theorem 1. [Gao, Wormald] Away from $k = \frac{n}{3}$, usual saddle point analysis gives with $\lambda(n) \to \infty$

$$\Pr(X_n = k) \sim \frac{32}{243\sqrt{\pi}} \cdot \frac{n^{5/2}}{k^{3/2}(n-3k)^{5/2}}, \quad \lambda(n) < k < \frac{n}{3} - n^{2/3}\lambda(n)$$

$$\Pr(X_n = k) = O\left(\exp(-n(k/n - 1/3)^3)\right), \quad k > \frac{n}{3} + n^{2/3}\lambda(n).$$

Two saddle points and only closer one to 0 counts:

$$z_+(n,k) = \frac{1}{2}, \qquad z_-(n,k) = \frac{n-k}{n+k}.$$

B. The "centre" itself: a double saddle point

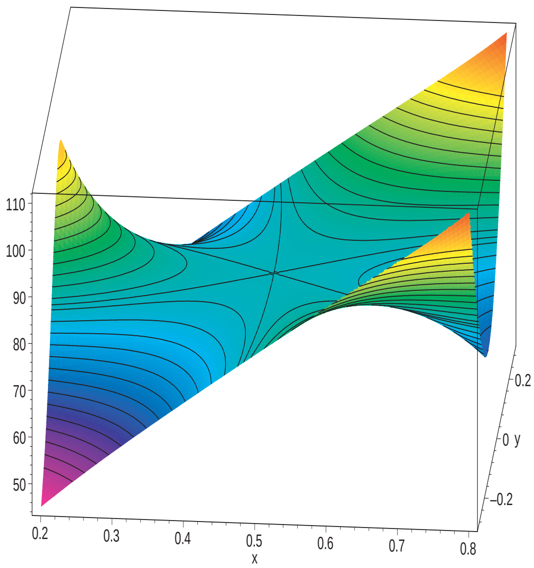
Theorem 2. "Centre" of central region at $n = 3k$ is

$$\Pr(X_{3k} = k) = \frac{4}{27}\frac{\Gamma(2/3)}{3^{1/6}\pi}k^{-2/3}\left(1 + O((\ln(k))^4 k^{-1/3})\right)$$

Proof. Double saddle point:

$$[z^{3k}]M^k(z) = \frac{1}{6i\pi}\int G(z)P(z)^k\,dz.$$

Here: $P(z) = (1-z)(1+z)^9/z^2$ has a double saddle point at $z_+ = z_- = \frac{1}{2}$



Note: Gao & Wormald use integration on circle, i.e., a level line, resort to Van der Corput's method and do not get any error term.

C. The central region and nearby saddle points

Problem is

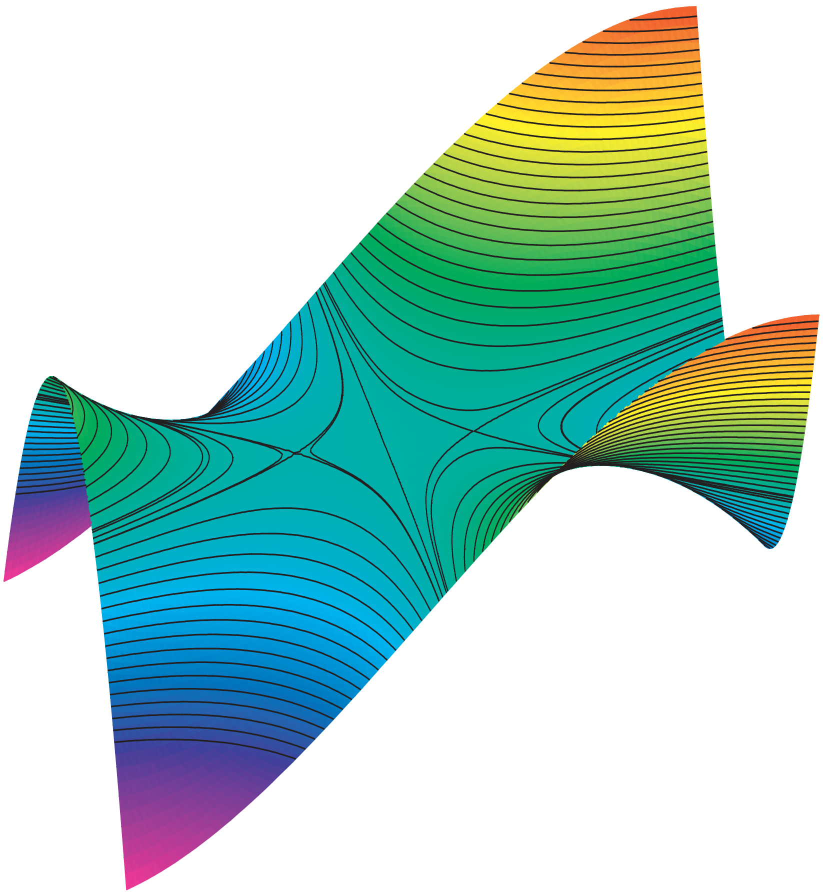
$$[z^n]A(z)^k B(z)^n,$$

when $k \approx k_0 = n/3$ with $A(z)^{k_0} B(z)^n$ having a double saddle point.

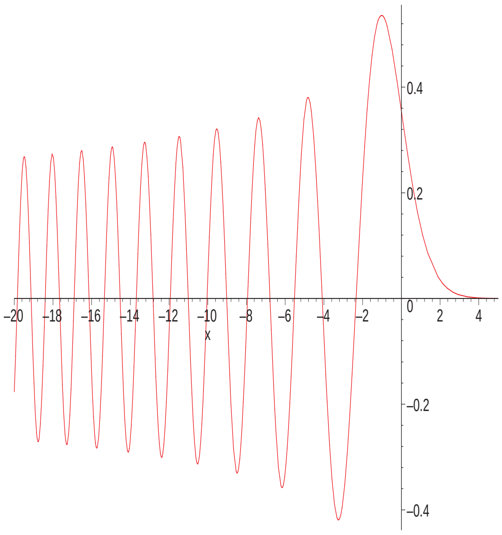
There are two saddle points at

$$z_+(n,k) = \frac{1}{2}, \qquad z_-(n,k) = \frac{n-k}{n+k}.$$

Idea: use a contour emanating from the midpoint and with an angle $= \pm\frac{2\pi}{3}$ like in double saddle-point case.

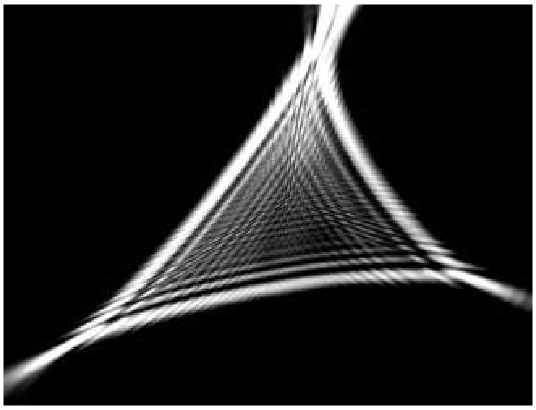


The Airy function



A solution of $y'' - zy = 0$

$$\begin{aligned}\mathrm{Ai}(z) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i(zt+t^3/3)}\,dt \\ &= \frac{1}{\pi 3^{2/3}}\sum_{n=0}^{\infty}\frac{\Gamma((n+1)/3)}{n!}\sin\left(\frac{2(n+1)\pi}{3}\right)\left(3^{1/3}z\right)^n.\end{aligned}$$



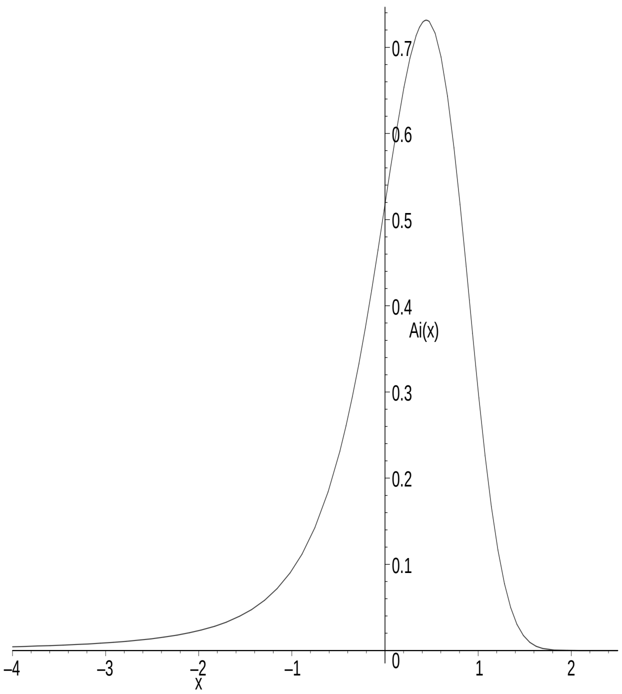
The Airy distribution for maps

Definition. The Airy density

$$\mathrm{Ai}(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i(zt+t^3/3)}\,dt$$

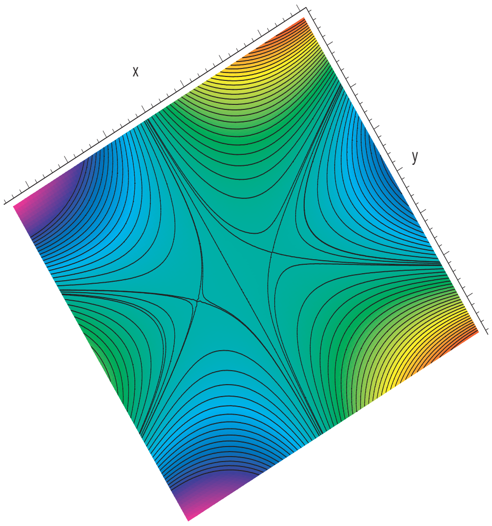
$$\mathcal{A}(x) = 2\exp\left(-\frac{2}{3}x^3\right)\left(x\mathrm{Ai}(x^2) - \mathrm{Ai}'(x^2)\right).$$

Airy distribution:



$$\mathcal{A}(x) \underset{-\infty}{=} O\left(|x|^{-5/2}\right), \quad \mathcal{A}(x) \underset{+\infty}{=} O\left(x^{1/2}\exp\left(-\frac{4}{3}x^3\right)\right).$$

The Airy distribution for maps



With midpoint $\zeta = (z_+ + z_-)/2$, scaling is

$$k = \frac{n}{3} + xn^{2/3}, \quad |z_- - z_-| \approx n^{-1/3}, \quad z = \zeta + vn^{-1/3}.$$

Normalize to something like

$$\int e^{-\cdot v^3 - \cdot vx^2} (\ldots)\, dv.$$

Complete the square and get Airy distribution.

Theorem 3. Core-size ($\Pr(X_n = k)$) admits a local limit law of the map–Airy type:

$$\Pr(X_n = k) \sim Kn^{-2/3}\mathcal{A}\left(\frac{3^{4/3}}{4}x\right), \qquad k = \frac{n}{3} = xn^{2/3}.$$

uniformly in compact subintervals.

C″. Coalescing saddles.

Full method of Friedman, Chester, and Ursell gives more.

(i) Match integrand with a cubic;
(ii) Check global validity of cubic change of variables (possibly restrict range)
(iii) Perform change of variables, expand, and unify with Airy integrals.

Here: get a global result.

Theorem 4. For "whole" range $[k \to \infty,\ k = o(n)]$, let $k = n(1/3+\beta)$; then $\Pr(X_n = n/3 + \beta n)$ is uniformly

$$\sim \frac{16}{81(1+3\beta)^{3/2} n^{2/3}} \left[\frac{a_1}{2} \mathcal{A}(\chi) + \frac{a_4}{n^{2/3}} \exp\left(-\frac{2}{3}\chi^3\right) \mathrm{Ai}(\chi^2) \right]$$

$$\mathcal{L}(x) = x \log x$$

$$\chi = n^{1/3}\gamma$$

$$\gamma = \left(2\mathcal{L}(1+3\beta/4) - \tfrac{1}{2}\mathcal{L}(1-3\beta/2) - \tfrac{1}{4}\mathcal{L}(1+3\beta) - \tfrac{9}{4}\beta \ln 2\right)^{1/3}$$

$$\frac{a_1}{2} = \frac{9}{8}\left(\frac{\beta/\gamma}{(1+3\beta/4)(1-3\beta/2)(1+3\beta)}\right)^{1/2} \quad a_4 = \frac{4}{9\beta^2}\sqrt{\frac{\gamma}{\beta}} - \frac{a_1}{4\gamma^2}$$

Maps of various types.

11 classes of maps/cores $\mapsto$ G. Schaeffer

maps ($\mathcal{M}$)	$\mathcal{M}_n$	cores ($\mathcal{C}$)	α_0	c
general	n e.	nonseparable	1/3	$3/4^{2/3}$
general	n e.	bridgeless	4/5	$(5/3)^{2/3}/4$
general	n e.	loopless	2/3	$3/2$
loopless	n e.	simple	2/3	$3^{4/3}/4$
bipartite	n e.	bip. simples	5/9	$3^{8/3}/20$
bipartite	n e.	bip. nonsep.	5/13	$\frac{3}{10}(13/6)^{5/3}$
bipartite	n e.	bip. bridgeless	3/5	$(15/2)^{5/3}/18$
nonseparable	n e.	simple nonsep.	4/5	$15^{5/3}/36$
nonseparable	$n+1$ e.	3-connected	1/3	$3^{4/3}/4$
cubic nonsep.	$n+2$ f.	cubic 3-conn.	1/2	$(3/2)^{1/3}$
cubic 3-conn.	$n+2$ f.	cubic 4-conn.	1/2	$6^{2/3}/3$

Consequences

Theorem 5. For 11 cases, core size (for root edge) is asymptotically Airy distributed in the sense of a *local limit law*.

Theorem 6. ($\mapsto$ Schaeffer) Same for largest core-component.

Understand bimodal law of core size: one large component of size $O(n)+$ many small ($O(n)$) components. Root may fall on either.

Theorem 7. ($\mapsto$ Schaeffer), STOC'98. Fast random generation in $O(n^{5/3}$ (exact size); plus optimization by $\approx$ 30% ("centre" is not peak); plus approximate random generation in $O(n)$.

♡ Also suggests right contours for large powers of GF's with algebraic singularities when singular exponent is $\neq \frac{1}{2}$.

Theorem 8. Singularity composition schemas. $\mapsto$ Michèle Soria

Related situations I

Look out for confluences of saddle points in analytic combinatorics. Expect Airy functions to appear!

♡ Analyse forests of unrooted trees (cf Kolchin for probabilistic approach).

$$\text{rooted: } T = ze^T, \qquad \text{unrooted: } U = T - \frac{1}{2}T^2$$

Unrooting divides counts by n and corresponds to $(\cdot)^{3/2}$ singularity.

Take $T = t$ as independent variable and get situation of coalescing saddles in critical region. Alternatively: modified singularity analysis.

♡ The Random Graph with n vertices, m edges has phases

Gas $(m = o(n))$; Liquid $(m \sim \frac{n}{2})$;
Solid $(m \sim \frac{1}{2} n \log n)$; Hypersolid $(m \gg n \log n)$

At critical stage $m \sim \frac{n}{2}$, RG is "almost" a forest of unrooted trees and unicyclic graphs.

$$W_{-1} = T - T^2/2, \quad W_0 = \frac{1}{2}\log\frac{1}{1-T} - \frac{T}{2} - \frac{T^2}{4}.$$

Airy functions have been observed in Fl-Knuth-Pittel and the "Giant paper" by Janson-Knuth-Łuczak-Pittel!

Related situations II. ♡ For probabilists, the Airy law is a stable law.

Sums of independent RV's. Assume PGF of X is S.A..

$$\Pr(X = k) \propto k^{-5/2}, \qquad S_n = (X_1 + X_2 + \cdots + X_n).$$

$$\frac{S_n - n\mu}{n^2/3} \xrightarrow{\text{law}} \mathcal{A}\text{iry}$$

$E(e^{iX}) = 1 + \mu i\theta + \cdot\theta^{3/2}$. $\hat{S}_n = \dfrac{S_n - n\mu}{n^{2/3}}$.

$$\begin{aligned} E(e^{i\theta S_n}) &= e^{-n\mu i\theta/n^{2/3}} \left(1 + \frac{\mu i\theta}{n^{2/3}} + \cdot\frac{\theta^{3/2}}{n}\right)^n \\ &\sim e^{\cdot\theta^{3/2}} \end{aligned}$$

This is characteristic function/Fourier transform. Go back to density by Fourier inversion. Set $\theta = w^2$:

$$\text{density} \approx \int_{\cdot} e^{\cdot w^3 + \cdot x w^2}\, dw.$$

Complete square to get:

$$\exp(x^3)\left[\mathrm{Ai}(x^2) + x\mathrm{Ai}'(x^2)\right].$$

Fact. If tail is $\propto k^{-5/2}$, then limit law is Airy (classical). Furthermore, if PGF is S.A., then local limit law.

Question: Is there a natural **microscopic model** of core-size in maps???

Related situations III

♡ There is another class of Airy-related distributions:

Theorem. [Louchard; Takács; Fl-Poblete-Viola] Area below excursions, path length of trees, and displacement in linear probing hashing all converge to a law determined by its moments as (roughly)

$$MGF \approx \frac{\mathrm{Ai}'(z)}{\mathrm{Ai}(z)}$$

Work by Prellberg indicates that a bivariate GF's of area has an Airy behaviour. This is based on integral representations and coalescing saddle points. One might (should?) be able to deduce Airy laws (of the area type), including speed of convergence, local limit theorems, tails, etc.

Cf Aldous, Chassaing-Marckert-Yor for probabilistic alternatives.