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# Analytic Urns

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General: **BALLS** and one or more **URNS**.

## Two kinds of models



- “Balls-and-bins”: Throw balls at random into a number of urns.

= **Random allocations**. Basic in the analysis of hashing algorithms; also SAT problem, cf V. Puyhaubert.

= Techniques: Exponential generating functions and saddle point. Poissonization &c.

Kolchin et al., *Random Allocations*, 1978.



- “Urn models”: One urn contains balls whose nature may randomly change according to ball drawn and finite set of rules.

## Here: URNS with BALLS of TWO COLOURS

Type I	Type II
Black	White

**RULES** are given by a  $2 \times 2$  Matrix

The composition of the urn at time 0 is fixed. At time  $n$ , a ball in the urn is randomly chosen and its colour is inspected (thus the ball is drawn, looked at and then placed back in the urn): if it is black, then  $\alpha$  black and  $\beta$  white balls are subsequently inserted; if it is white, then,  $\gamma$  black balls and  $\delta$  white balls are inserted.

drawn	added	
$\downarrow$	$B$	$W$
$B$	$\alpha$	$\beta$
$W$	$\gamma$	$\delta$ .

- Drawing with replacement =  $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ .

- Drawing without replacement =  $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ .

- Laplace's "melancholic" model (1811): if a ball is drawn, it is repainted black no matter what its colour is.

$$\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$$

- Ehrenfest & Ehrenfest = *Über zwei bekannte Einwände gegen das Boltzmannsche H-Theorem*, 1923. Irreversibility contradicts Ergodicity.

Exchanges of basic balls ("atoms of heat") between two urns, one cold and one hot  $\rightsquigarrow$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

Bernoulli (1768), Laplace (1812).

- **Pólya Eggenberger model.** A ball is drawn at random and then replaced, together with  $s$  balls of the same colour.

$$\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}$$

A model of positive influence. Closed form.

- **“Adverse influence” model**

$$\begin{pmatrix} 0 & s \\ s & 0 \end{pmatrix}$$

Used in epidemiology, etc.

- **The special search tree model**

$$\begin{pmatrix} -2 & 3 \\ 4 & -3 \end{pmatrix}$$

Yao (1978); Bagchi and Pal (1985); Aldous et al (1988);  
Prodinger & Panholzer (1998)

Johnson & Kotz, *Urn Models and their Application*,  
Wiley 1973

**Here case of a  $2 \times 2$ -matrix**

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

**with constant row-sum** ♡♡

$$s := \alpha + \beta = \gamma + \delta.$$

At time  $n$  size  $t_n$  satisfies  $t_n = t_0 + sn$ .

**Constant increment  $s$**

A problem with **three parameters** + two initial conditions.

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♠ Kotz, Mahmoud, Robert (2000) show  
“pathologies” in some of the other cases.

Huge literature: *Math. Reviews*

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Lead to amazingly **wide variety of behaviours**, special functions, and limit distributions.

## **Methods**

- Difference equations and explicit solutions.
- Same but with probability generating functions.
- Connection with branching processes.
- Stochastic differential equations (KMR)
- Martingales (Gouet)

**Here:** A frontal attack:

- PDE of snapshots at time  $n$
- Usual solution for quasilinear PDE
- Bivariate GF and singularity perturbation

**Conformal mapping argument**, Abelian integrals over Fermat curves  $z^h + y^h = 1$

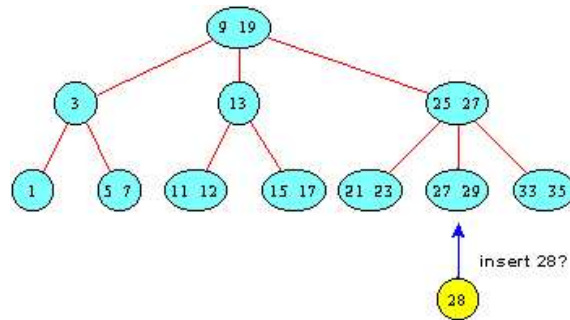
+ Special solutions with elliptic functions

## Part I

# The $\mathcal{T}_{2,3}$ model—basic equations

$$\begin{pmatrix} -2 & 3 \\ 4 & -3 \end{pmatrix}$$

- Insertions in a 2–3 tree: 2–node  $\mapsto$  3–node;  
3–node  $\mapsto$  (2–node + 2–node).



- Fringe-balanced 2–3 tree analogous to median-of-three quicksort.

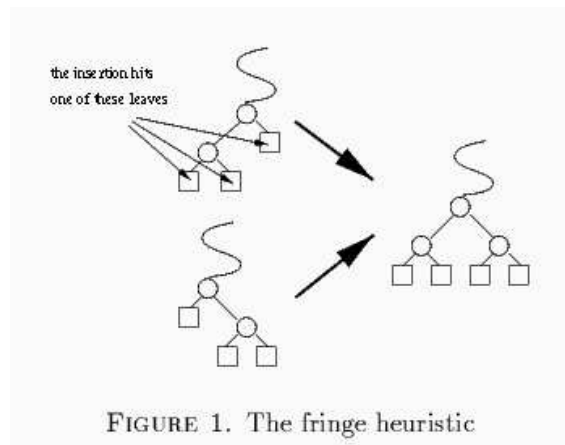


FIGURE 1. The fringe heuristic

Mahmoud (1998); Panholzer–Prodinger (1998)

Evolution is:

$$X_1 = 2; \quad X_n - X_{n-1} = \begin{cases} -2 & \text{with probability } \frac{X_{n-1}}{n} \\ +4 & \text{with probability } 1 - \frac{X_{n-1}}{n}. \end{cases}$$

Let  $p_{n,k} = \mathbb{P}(X_n = k)$ ,  $p_n(u) := \sum_k p_{n,k} u^k$ , and

$$F(z, u) := \sum_{n \geq 1} p_n(u) u^n = \sum_{n,k} p_{n,k} u^k z^n,$$

whose elicitation is our main target.

**Lemma:** PDE satisfied by BGF of probabilities is

$$(u^5 z - u) \frac{\partial F}{\partial z} + (1 - u^6) \frac{\partial F}{\partial u} + u^5 F + u^3 = 0.$$

PROOF. Each  $p_n$  is determined from previous one by  $\partial_u$  = a differential recurrence. Gives PDE for bivariate generating function  $F$ .

Take  $p_0(u)$  that satisfies PDE and write

$G := p_0(u) + F(z, u)$ . then, we get a *homogeneous PDE*.

$$(u^5 z - u) \frac{\partial G}{\partial z} + (1 - u^6) \frac{\partial G}{\partial u} + u^5 G = 0.$$

with

$$p_0(u) = (1 - u^6)^{1/6} \int_0^u t^3 (1 - t^6)^{-7/6} dt.$$

## Quasilinear first-order PDE's are reducible to ODEs.

$$A(z, u, G) \frac{\partial G(z, u)}{\partial z} + B(z, u, G) \frac{\partial G(z, u)}{\partial u} + C(z, u, G) = 0$$

**1.** Look for a solution in implicit form  $X(z, u, G) = 0$ .

$$A(z, u, w) \frac{\partial X}{\partial z} + B(z, u, w) \frac{\partial X}{\partial u} - C(z, u, w) \frac{\partial X}{\partial w} = 0.$$

**2.** Consider the ordinary differential system

$$\frac{dz}{A} = \frac{du}{B} = -\frac{dw}{C}.$$

The solution of two "independent" ordinary differential equations, e.g.,

$$\frac{du}{B} = -\frac{dw}{C} \quad \text{and} \quad \frac{dz}{A} = \frac{du}{B},$$

leads to two families of integral curves,

$$U(u, z, w) = C_1 \quad \text{and} \quad V(u, z, w) = C_2.$$

**3.** The generic solution of the PDE is provided by

$$X(z, u, w) = \Phi(U(u, z, w), V(u, z, w)),$$

for arbitrary bivariate  $\Phi$ . Solving for  $w$  in  $X(z, u, w) = 0$  provides a relation  $w = R_\Phi(z, u)$ . General solution is

$$G(z, u) := R_\Phi(z, u).$$

$$(u^5 z - u) \frac{\partial G}{\partial z} + (1 - u^6) \frac{\partial G}{\partial u} + u^5 G = 0.$$

Consider

$$\frac{du}{1 - u^6} = \frac{dz}{u^5 z - u} = -\frac{dw}{u^5 w}.$$

- $du \leftrightarrow dw$  first integral by separation:

$$w(1 - u^6)^{-1/6} = C_1.$$

- $du \leftrightarrow dz$  variation of constant:

$$z(1 - u^6)^{1/6} + \int_0^u \frac{t}{(1 - t^6)^{5/6}} dt = C_2.$$

Bind the two integrals by arbitrary  $\Phi$  &  $w \equiv G$

$$\Phi \left( \frac{G}{(1 - u^6)^{1/6}}, z(1 - u^6)^{1/6} + \int_0^u \frac{t}{(1 - t^6)^{5/6}} dt \right) = 0,$$

Solve for  $G$ , introducing arbitrary  $\psi$ :

$$G(z, u) = \delta(u) \psi(\delta(u) z + I(u)), \quad I(u) := \int_0^u \frac{t}{(1 - t^6)^{5/6}} dt,$$

with  $\delta(u) := (1 - u^6)^{1/6}$ .

Initial conditions identify  $\psi$ .

**Theorem 1.** *Define the quantities*

$$\begin{aligned}\delta(u) &= (1 - u^6)^{1/6}, \\ I(u) &= \int_0^u \frac{t}{(1 - t^6)^{5/6}}, \quad J(u) = \int_0^u \frac{t^3}{(1 - t^6)^{7/6}} dt.\end{aligned}\tag{1}$$

*Then, the bivariate generating function of the probabilities is*

$$G(z, u) = \delta(u)\psi(z\delta(u) + I(u)),\tag{2}$$

*where  $\psi$  is the function defined parametrically for  $|u| < 1$  by*

$$\psi(I(u)) = J(u).\tag{3}$$

## Dominant singularities of $\psi$ ?

The diagram that summarizes  $\psi$  is

$$\begin{array}{ccc} & u & \\ \swarrow & & \searrow \\ z = I(u) & \xrightarrow{\psi} & \psi(z) = J(u). \end{array}$$

The radius of analyticity of  $\psi$  is

$$\rho = I(1), \quad I(u) := \int_0^u \frac{t}{(1-t^6)^{5/6}} dt,$$

Proof: There is local (analytic) invertibility of  $I(u)$  along  $(0, \rho)$ . Thus  $\psi$  is analytic along  $(0, \rho)$ .

We have  $\psi(z) = G(z, 0)$  which has nonnegative coeffs and is Pringsheim.

We have  $I(1) < \infty$  while  $J(1) = \infty$ . Thus  $\rho$  is a singularity.

By Eulerian Beta integrals:

$$\rho \equiv I(1) = \frac{1}{6} B\left(\frac{1}{6}, \frac{1}{3}\right) = \frac{1}{6} \frac{\Gamma(1/3)\Gamma(1/6)}{\Gamma(1/2)} \doteq 1.40218\ 21053\ 25454.$$

$$\left(\frac{\psi_{47}}{\psi_{50}}\right)^{1/3} \doteq 1.40218\ 21053\ 2545\underline{6},$$

Local expansions near  $u = 1$  plus symmetries of the problem are compatible with:

**Proposition 1.** *There are no singularities of  $\psi(z)$  on  $|z| = \rho$  other than  $\rho, \rho\omega, \rho\omega^2$  that are simple poles. Precisely, let*

$$S(z) = \frac{1}{\rho - z} + \frac{1}{\rho\omega - z} + \frac{1}{\rho\omega^2 - z} = \frac{3z^2}{\rho^3 - z^3}.$$

*The function*

$$\psi(z) - S(z)$$

*is analytic in a disc  $|z| < R$  for some  $R$  satisfying  $R > \rho$ . (One can take  $R = 2\rho$ .)*

Why singularities of  $\psi$ , BTW?

$$G(z, u) = \delta(u)\psi(z\delta(u) + I(u)), \quad \psi(I(u)) = J(u).$$

Set  $u = 0$  and estimate  $[z^n]\psi(z)$ : Get **extremely large deviations**, all balls of one colour.

Know approximately  $[z^n]G(z, u) = \text{PGF of distribution} \rightsquigarrow$  **LIMIT LAW**.

Set  $u$  to value  $\neq 1$  and get **LARGE DEVIATIONS**.

**And a good deal more...**

## Part II

# The $\mathcal{T}_{2,3}$ model—elliptic structure

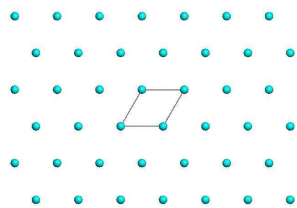
Recall: an **elliptic function** is a **doubly periodic meromorphic** function in  $\mathbb{C}$ .

**Historically:** Integration over a conic  $\int Q(z, y)$  where  $y = \sqrt{P(x)}$  and  $\deg P = 1, 2$ , yields functions like  $\arctan$ ,  $\arcsin$  and hyperbolic counterparts. Such functions satisfy  $\arctan(z) \cong \arctan(z) + k\pi$  so that inverses are **simply periodic**. This is a way to (re)build trigonometry from integrals over conics.



Integration over a cubic or a quartic  $y = \sqrt{P(x)}$  with  $\deg P = 3, 4$ , which are topologically “doughnuts” leads to **double periodicity**. Such things occur when rectifying the ellipse hence the name **elliptic integrals** and **elliptic functions** for inverses.

$$\sum 1/(z - \omega)^3$$



For a **parameterized curve**,  $\psi(I(u)) = J(u)$ , examine all possible **paths** in the  $u$ -plane, and the corresponding determinations of  $I(u)$ . Reflect on

$$\psi \left( \int_1^u \frac{dt}{t} \right) = u,$$

which defines  $\psi(\log u) = u$ , that is,  $\psi(z) = e^z$ .

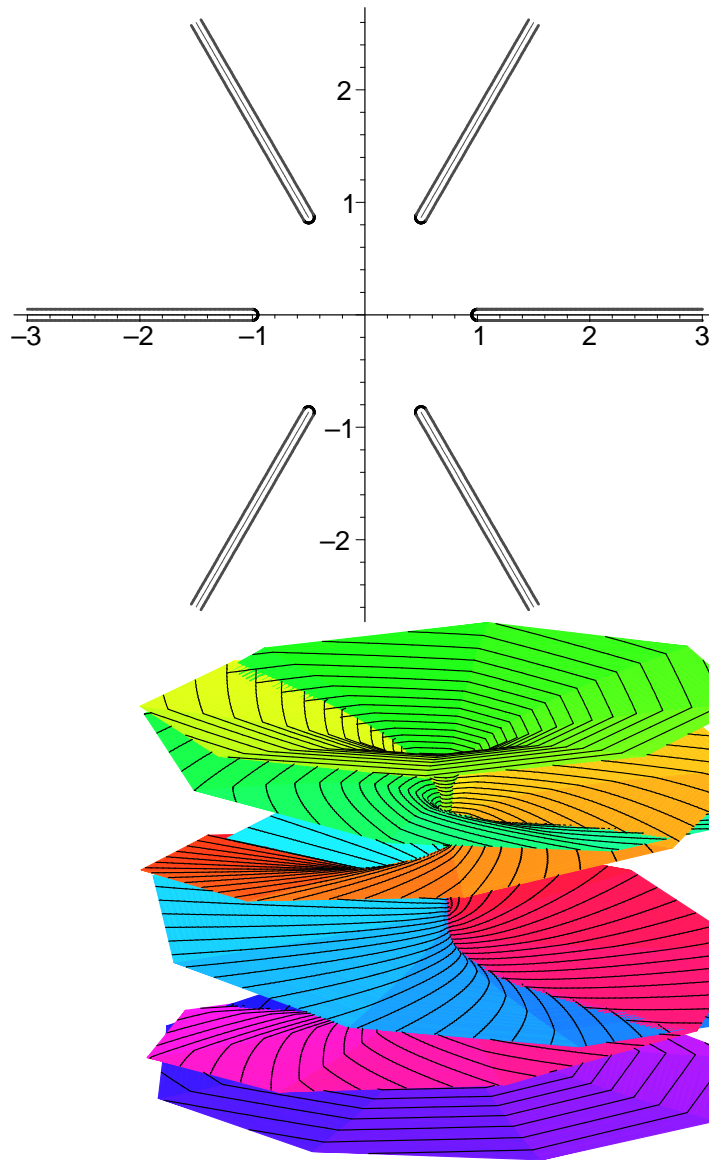
Here:

$$\psi(I(u)) = J(u)$$
$$I(u) = \int_0^u \frac{t dt}{(1-t^6)^{5/6}}, \quad I(u) = \int_0^u \frac{t^3 dt}{(1-t^6)^{7/6}}.$$

The curve is  $t^6 + y^6 = 1$  and has **genus 10**.

Go step by step.

- **The elementary triangle**
- **The fundamental triangle**



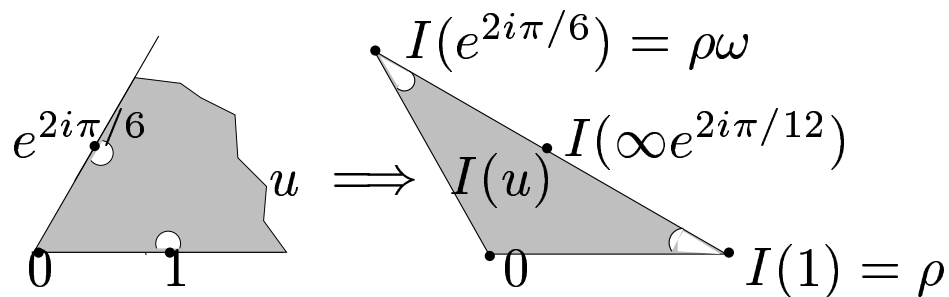
The region  $R_0$  (left) and a rendering of the six-sheeted Riemann surface  $\mathfrak{R}$  of  $\delta(u) \equiv (1 - u^6)^{1/6}$  for  $u$  near 1 (right).

Because of double parameterization, taking  $u$  in a half-plane suffices.

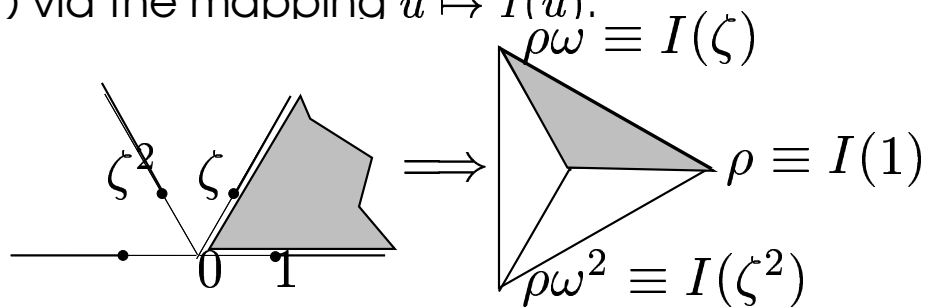
**Lemma 1.** The function  $\psi$  maps the interior of  $(R_0 \cap H)$  in a one-to-one manner on the interior of the equilateral triangle  $T$  with vertices  $\rho, \rho\omega, \rho\omega^2$ , where  $\omega := e^{2i\pi/3}$ .

**Proof.** Folds angles in an appropriate way...

Start with Elementary triangle.

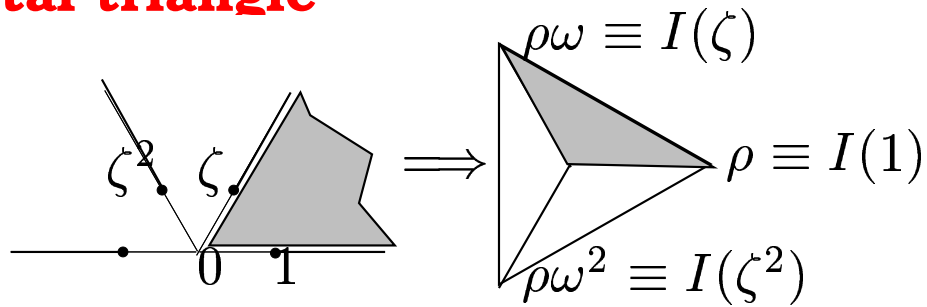


The “elementary triangle”  $T_0$  (right) is the image of the basic sector  $S_0$  (left) via the mapping  $u \mapsto I(u)$ .

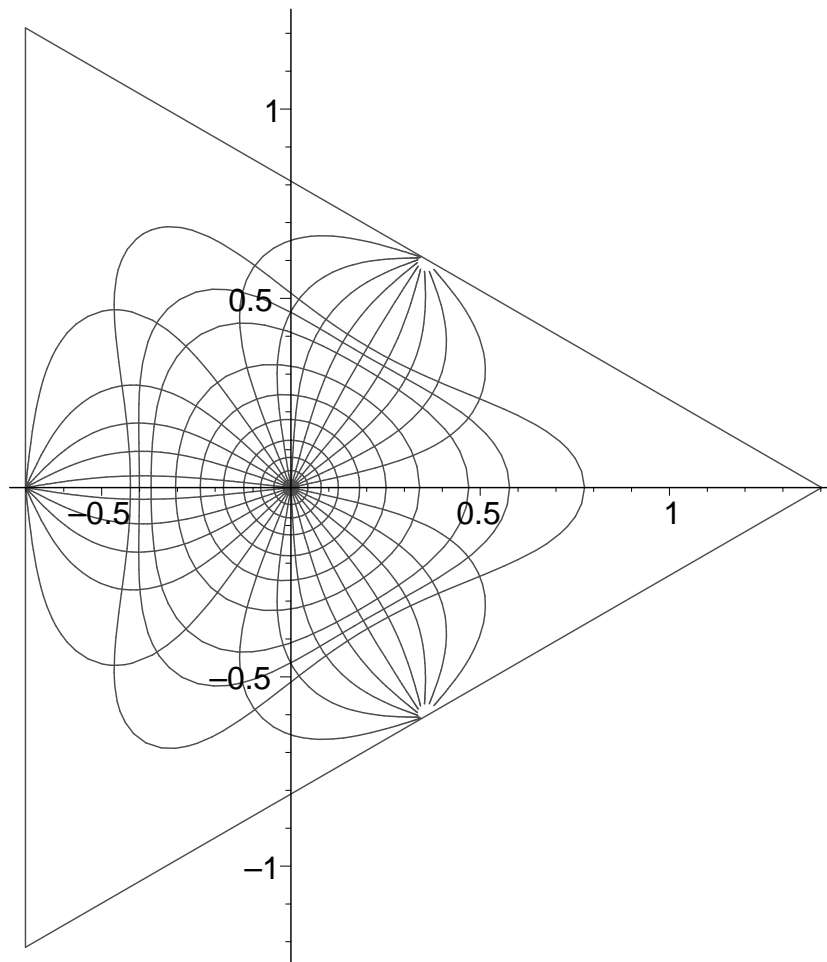


The “fundamental triangle”  $T$  (right) is the image of the slit upperhalf plane  $(R_0 \cap H)$  (left) via the mapping  $u \mapsto I(u)$ .

Three **elementary triangles** assemble to form a **fundamental triangle**

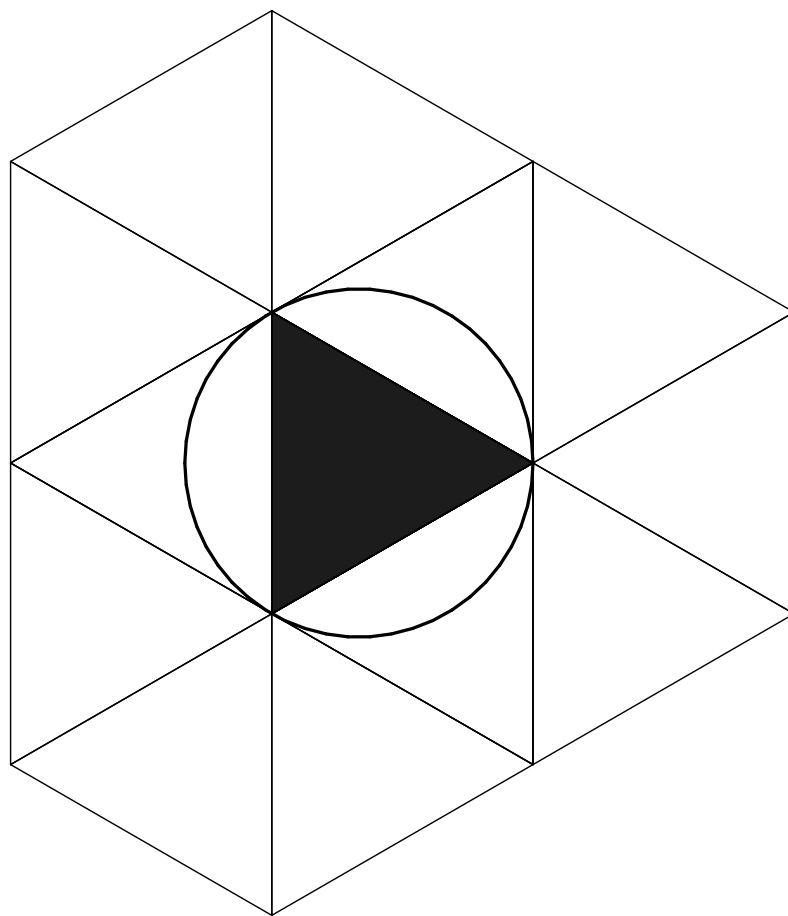


Based on  $I(\zeta u) = \omega I(u)$ , where  $\zeta^6 = 1, \omega^3 = 1$ .



Another view of the image of  $(R_0 \cap H)$  by  $I(u)$  giving the fundamental triangle  $T$ : a representation of the images of rays emanating from 0 and of circles centred at 0

**Lemma 2.** *The function  $\psi$  is analytic (holomorphic) in the disk  $|z| < 2\rho$  stripped of the points  $\rho, \rho\omega, \rho\omega^2$ . (The function admits these three points as simple poles, as asserted in Prop. 1.)*



Rotated copies of the fundamental triangle around  $\rho, \rho\omega, \rho\omega^2$  shown against the circle of convergence of  $\psi(z)$ .

**Proof.** Laces around  $u = 1$  and changes of variables:  $I(1) - I(u) \sim 6^{1/6}(1 - u)^{1/6}$ .

## The full story and the elliptic connection

A lattice  $\Lambda$  with generators  $\xi, \eta \in \mathbb{C}$ :

$$\Lambda(\xi, \eta) = \{n_1\xi + n_2\eta \mid n_1, n_2 \in \mathbb{Z}\}.$$

The Weierstraß *zeta function* relative to  $\Lambda$  is classically defined as

$$\zeta(z; \Lambda) := \frac{1}{z} + \sum_{w \in \Lambda \setminus \{0\}} \left( \frac{1}{z-w} + \frac{1}{w} + \frac{z}{w^2} \right).$$

**Theorem 2.** *The  $\psi$ -function of the  $\mathcal{T}_{2,3}$  model initialized with 2 balls of the first type ( $a_0 = t_0 = 2$ ) is exactly*

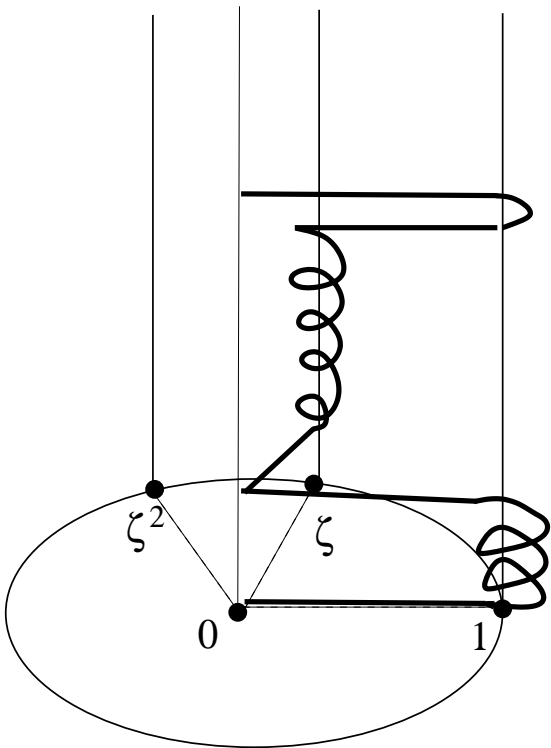
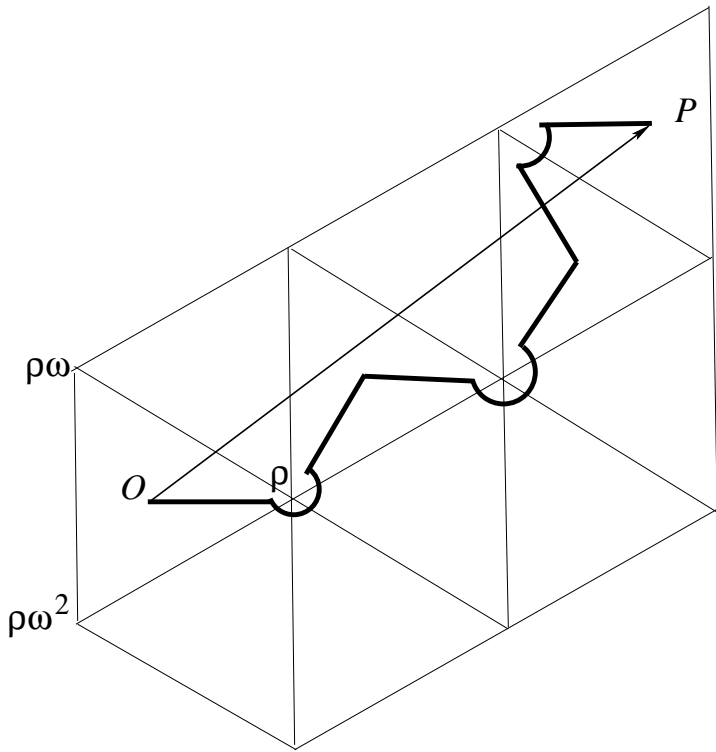
$$\psi(z) = \frac{1}{\rho\sqrt{3}} \left( -\zeta \left( \frac{z-\rho}{\rho\sqrt{3}} \right) + \zeta \left( -\frac{1}{\sqrt{3}} \right) \right), \quad \rho := \frac{1}{6} \frac{\Gamma(\frac{1}{3})\Gamma(\frac{1}{6})}{\Gamma(\frac{1}{2})}, \quad (4)$$

where  $\zeta(z) := \zeta(z; \Lambda_{\text{hex}})$  is the Weierstraß zeta function of the hexagonal lattice:

$$\Lambda_{\text{hex}} := \left\{ n_1 e^{i\pi/6} + n_2 e^{-i\pi/6} \mid n_1, n_2 \in \mathbb{Z} \right\}.$$

### **Proof.**

- Follow all paths and examine  $I(\gamma(u))$ : any point  $z \in \mathbb{C}$  is reachable.
- There is a pole of  $\psi$  at lattice points and residue is  $-1$  since determinations of  $\delta(u)$  in  $I, J$  are the same.
- By Liouville,  $\psi(z)$  and  $\zeta$  coincide (up to normalization).



A path in the  $z$ -plane from  $0$  to  $P$  and the contour  $\gamma$  above the  $u$ -plane that realizes it via  $u \mapsto z = I(u)$ .

### Part III

## Probabilistic consequences

Extract coeffs in simple fractions:

**Corollary 1.** *For the  $\mathcal{T}_{2,3}$  model, the probability generating function  $p_n(u) = \mathbb{E}(u^{X_n})$  admits an exact formula valid for all  $n \geq 2$ ,*

$$p_n(u) = \sum_{n_1, n_2 = -\infty}^{+\infty} \left( K(u) + \frac{\rho\sqrt{3}}{\delta(u)} (n_1 e^{i\pi/6} + n_2 e^{-i\pi/6}) \right)^{-n-1},$$

where

$$K(u) := \frac{1}{\delta(u)} \int_u^1 \frac{t}{\delta(t)^5} dt, \quad \delta(u) = (1 - u^6)^{1/6}.$$

**Note:** when  $u \approx 1$ , this is like  $K(u)^{-n-1}$ .

## The Quasi-powers framework.

Classics are:

(Laplace) Given a random variable  $X$ , define its characteristic function aka **Fourier transform** as

$$\phi_X(t) := \mathbb{E}(e^{itY}) = \sum_k \mathbb{P}(Y = k)e^{itk} = p(e^{it}).$$

If  $S_n = X_1 + \dots + X_n$  with i.i.d.  $X_j$ , then:

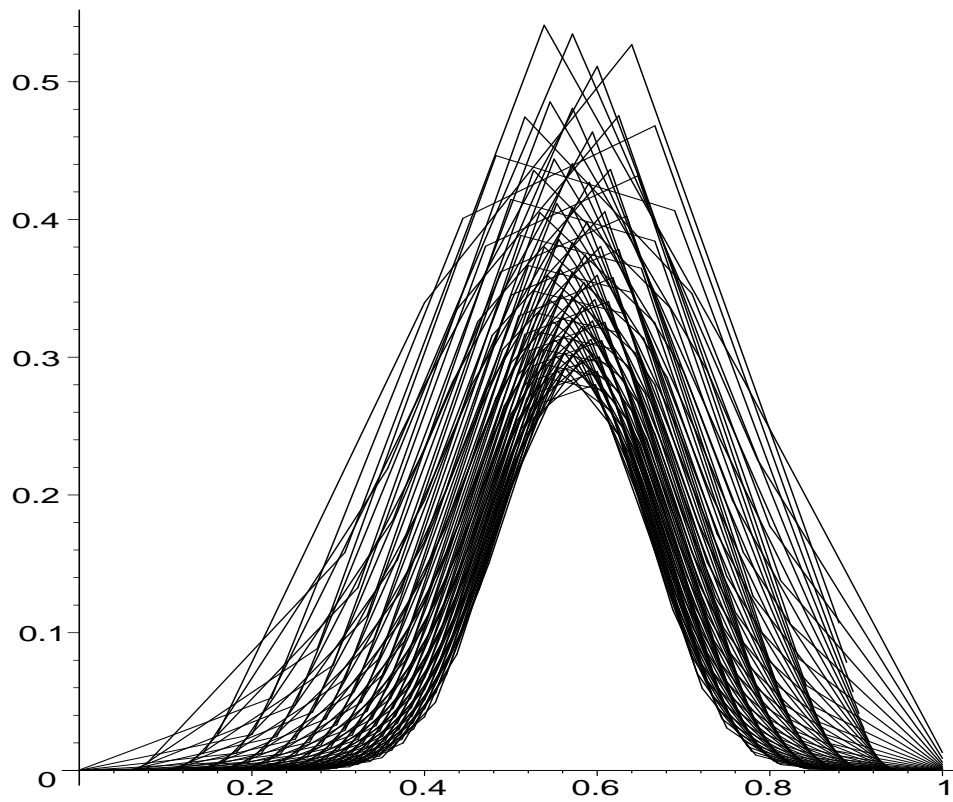
$$\phi_{S_n}(t) = (\phi_X(t))^n.$$

(Lévy et al.) Fourier inversion is continuous:  
convergence of F.T.'s

$$\lim_{n \rightarrow \infty} \phi_{Y_n}(t) = \phi_Z(t) \quad \text{pointwise}$$

implies  $Y_n \Rightarrow Z$  in distribution.

(Berry-Esseen) Uniform distance on F.T. furthermore gives bounds on uniform distance on distribution functions.



A Sedgewick plot of  $\{\mathbb{P}(x_n = k)\}_{k=0}^{n-1}$  for  $n = 24 \dots 96$  (the horizontal axis is normalized to  $n + 1$ ).

## Gaussian laws in analytic combinatorics

Classically  $S_n = X_1 + \cdots + X_n$ , where  $X_j$  have mean and variance. Calculation shows that

$$\log \mathbb{E} \left[ \exp \left( it \frac{S_n - n\mu}{\sigma \sqrt{n}} \right) \right] \xrightarrow[n \rightarrow \infty]{} -\frac{t^2}{2}.$$

Hence **Central Limit Theorem**.

A “good” uniform approximation  $p_n(u) \sim a(u) \cdot B(u)^n$  for  $u \approx 1$  (complex neighbourhood) is called **QuasiPowers approximation**.

From Bender, F.-Soria, Hwang (1995), one has:

- Moments result from differentiation (complex an.)
- Convergence to Gaussian distribution (erf)
- Speed of convergence is  $\frac{1}{\sqrt{n}}$ .
- Some large deviation estimates: probability of being far from mean at  $cn$  for  $c \neq \mu$  is exponentially small.

**Corollary 2 (Gaussian limit).** *For the  $\mathcal{T}_{2,3}$  model, the random variable  $X_n$  representing the number of balls of the first type at time  $n$  is asymptotically Gaussian with speed of convergence to the limit  $O(n^{-1/2})$ ,*

$$\mathbb{P}\left(\frac{X_n - \mathbb{E}(X_n)}{\sqrt{\mathbb{V}X_n}} \leq x\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-y^2/2} dy + O\left(\frac{1}{\sqrt{n}}\right).$$

**Proof.** From lattice sum, in complex neighbourhood  $u \approx 1$ :

$$p_n(u) = K(u)^{-n-1}(1 + O(2^{-n})).$$

Note that  $K(u)^{-1}$  plays the rôle of a probability characteristic function but it isn't!

$$K(u)^{-1} \doteq 0.713 + 0.254u^2 + 0.090u^4 - 0.086u^6 + 0.022u^8 + \dots$$

## The shape of moments.

In the literature, only a few moments are computed via (unpleasant?) recurrence manipulations from probabilities and original rec.

Here: everything is almost as though

$$p_n(u) = K(u)^{-n-1}.$$

$$P_1(\nu) = \frac{4\nu}{7}, \quad P_2(\nu) = \frac{4\nu}{637} (52\nu + 17), \\ P_3(\nu) = \frac{8\nu}{84721} (1976\nu^2 + 1938\nu - 11063).$$

**Corollary 3 (Moments).** *For the  $\mathcal{T}_{2,3}$  model, exact polynomial forms for moments of any order are available: the factorial moments satisfy*

$$\mathbb{E}((X_n)^r) = P_r(n+1), \quad n \geq 6r,$$

where the  $P_r$  are polynomials generated by

$$e^{vL(h)} = \sum_{r=0}^{\infty} \frac{h^r}{r!} P_r(v) \quad \text{and} \quad L(h) = -\log K(1+h).$$

Rota: polynomials of “binomial type” satisfying various convolution relations.

## Large deviations.

From dominant poles of  $\psi$ , corresponding to  $u = 0$ :

**Corollary 4 (Extreme large deviations).** *The probability that, in the  $\mathcal{T}_{2,3}$  model, all balls are of the first colour satisfies*

$$[z^{3n+2}]\psi(z) \sim 3\rho^{-3n-3} (1 + O(A^{-n})),$$

for any  $A < 8$ .

Moreover:

**Corollary 5 (Large deviations).** *Let  $\alpha$  be a number of the open interval  $(0, \frac{4}{7})$ . One has*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(X_n \leq \alpha \cdot n) = -\rho(\alpha), \quad (5)$$

where the rate function  $\rho$  is determined by

$$\rho(\alpha) = \log(\lambda_0^\alpha K(\lambda_0)), \quad (6)$$

and  $\lambda_0$  depending on  $\alpha$  is the implicitly defined root  $u \in (0, 1)$  of

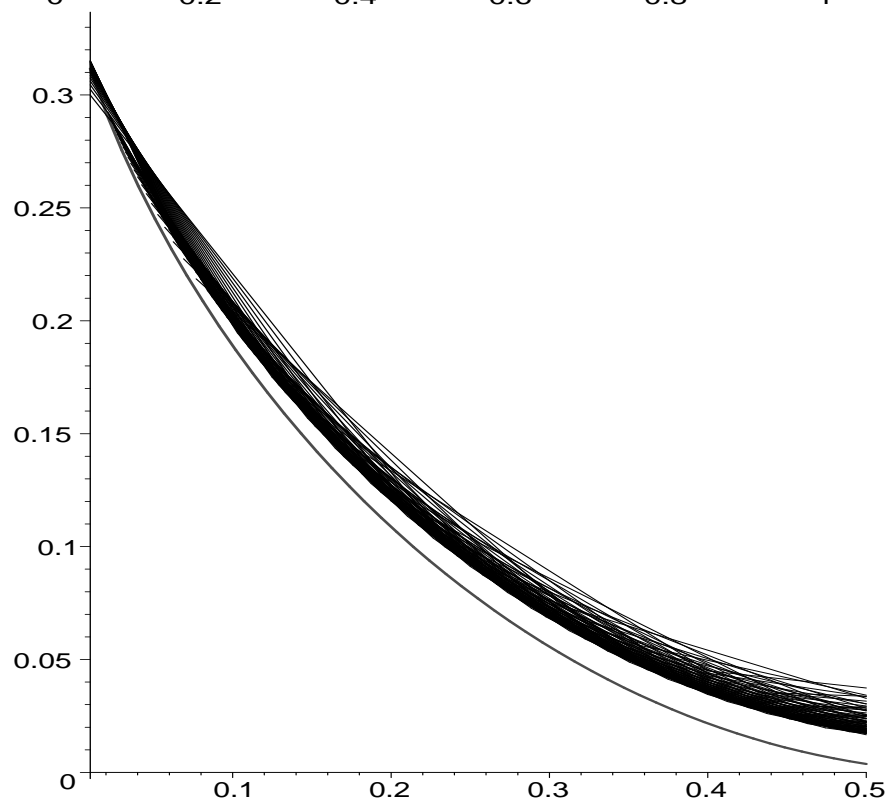
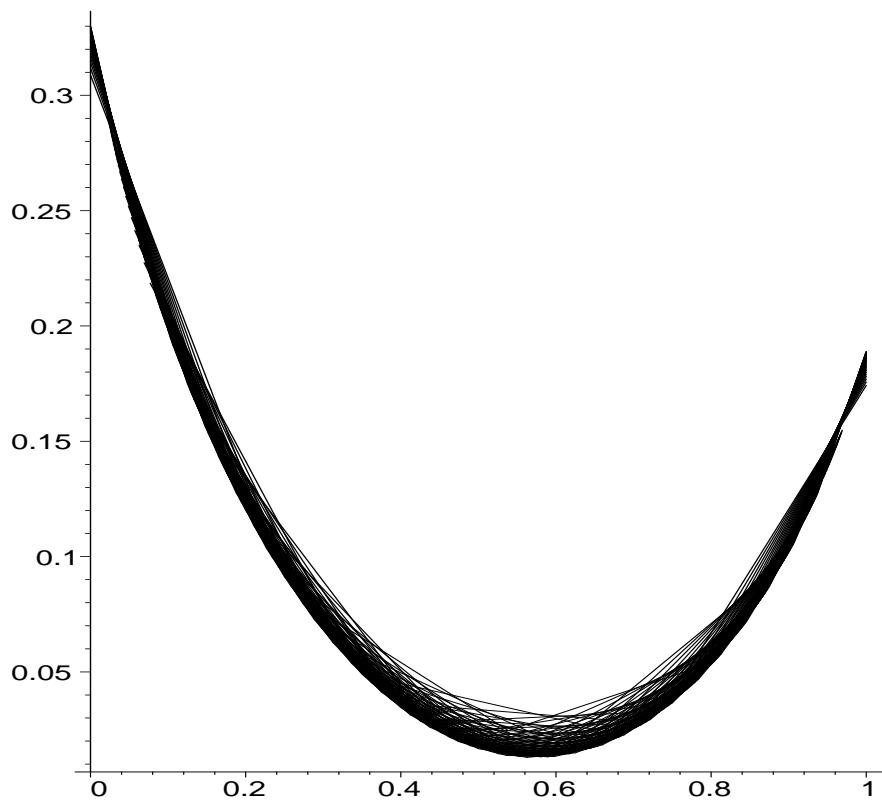
$$\frac{uK'(u)}{K(u)} + \alpha = 0. \quad (7)$$

Proof is standard for probabilists. Assume  $p_n(u) \approx B(u)^n$ , where  $B(u)$  increases from  $c_0$  to 1 as  $u \in (0, 1)$ .

One has **Cauchy** aka **saddle-point bounds**:

$$[u^k]p_n(u) \leq \frac{p_n(u_0)}{u_0^k} \approx \frac{B(u_0)^n}{u_0^k}.$$

Adopt the best  $u_0$  (which must exist by some convexity prop.) and get an exponentially small upperbound. Cramér aka “shifting the mean”: apply a form of CLT near  $u_0$  to conclude that the upperbound is also a lowerbound.



Left: a Sedgewick plot of  $\left\{-\frac{1}{n} \log \mathbb{P}(X_n = k)\right\}_{k=0}^{n+1}$  for  $n = 24 \dots 96$  (the horizontal axis is normalized to  $n + 1$ ); right: a comparison against the large deviation rate (thick line).

## Related work

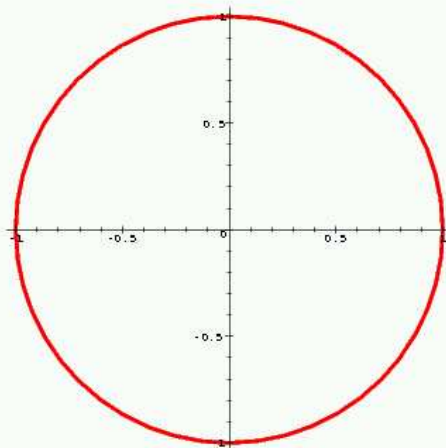
Gaussian law by moments: (Bagchi & Pal 1985). Here global expression for moment polynomials + speed of convergence

Elliptic connection related to (Panholzer & Prodinger 1998) via specific approach  $y''' = \cdot y'^2 + \cdot$ . Here: much more general, for whole class.

Large deviations seem to be new.

Local limit laws? Probably true. Want to apply saddle point, need bounding technique.

Counting	$u = 1$
Moments	$u = 1 \pm \frac{1}{\infty}$
Large deviations	$u = [1 - \eta, 1 + \eta]$
Central limit	$u = 1 + \square$
Local limit	$ u  = 1$



## Part IV

# General case

Matrix

drawn	added	
↓	<i>B</i>	<i>W</i>
<i>B</i>	$\alpha$	$\beta$
<i>W</i>	$\gamma$	$\delta$

$$\alpha + \beta = \gamma + \delta = s$$

Consider general case of urns with replacement,  
i.e.,  $\alpha < 0, \beta < 0$ .

$$\begin{pmatrix} -a & a + s \\ b + s & -b \end{pmatrix}$$

A 3-parameter family

Plus initially  $a_0$  black;  $b_0$  white.

Ideas:

- Look at the *enumerative* version.
- Set up PDE for bivariate generating function via operator  $e^{z\Gamma}$ .
- Get a  $\psi$ -function parameterized by **Abelian integrals** over **Fermat curve**  $x^h + y^h = 1$ .
- Determine **singularities** by looking at **geometry** of **conformal maps** of basic domains.
- Generally, non-elliptic solutions, but:
  - Gaussian limit with speed of convergence;
  - Extreme large deviations;
  - Large deviation rate

Recycles most of  $\mathcal{T}_{2,3}$  case but without double periodicity at this level of generality.

## The operator approach

Number balls in order of appearance  $1, 2, 3, \dots$

choose 2      choose 5      choose 8      choose 9  
 $\underbrace{1_I, 2_I}$  ,    $\underbrace{3_{II}, 4_{II}, 5_{II}}$  ,    $\underbrace{6_I, 7_I, 8_I, 9_I}$  ,    $\underbrace{7_I, 9_I, 10_{II}, 11_{II}, 12_{II}}$  , ...

$s = a + b$ ; at time  $n$ , after action, size is  $t_n$ ;

$t_0 = a_0 + b_0$  is given;

$$t_n = t_0 + sn.$$

Thus

$$H_n = t_0(t_0 + s) \cdots (t_0 + ns).$$

Let  $H_{n,k}$  be number of histories of length  $n$  leading to  $k$  Black (Type I) balls and

$$H(z, u) := \sum_{n,k} H_{n,k} u^k \frac{z^n}{n!}.$$

## Combinatorial marking $\equiv$ differentiation

Represent a particular history  $h$  with  $k$  black balls and  $\ell$  white balls as  $u^k v^\ell$ .

Evolution chooses a black ball and acts; e.g., for  $\mathcal{T}_{2,3}$ :

$$u^k v^\ell \xrightarrow{u \partial_u} k u^k v^\ell \xrightarrow{u^{-2} v^3} k u^{k-2} v^{\ell+3}.$$

Similarly for white balls. Cleverly introduce:

$$\Gamma := u^{-1} v^3 \frac{\partial}{\partial u} + u^4 v^{-2} \frac{\partial}{\partial v}.$$

Then  $\Gamma u^k v^\ell$  describes all the successors of  $h \cong u^k v^\ell$ .

All evolutions of length  $n$  are generated by  $\Gamma^n u^{a_0} v^{b_0}$ , and a trivariate version of  $H$  is

$$\hat{H}(z, u, v) := e^{z\Gamma} \circ u^{a_0} v^{b_0}.$$

## The basic PDE

- By general principles:

$$\partial_z (e^{z\Gamma} f) = \Gamma e^{z\Gamma} f, \quad \partial_z \hat{H} = \Gamma \circ \hat{H}.$$

- By **homogeneity**, any term  $\mathfrak{m} = u^\alpha v^\beta z^n$  has  $\alpha + \beta = sn + t_0$ :  $(\theta_u + \theta_v - s\theta_z)\mathfrak{m} = t_0\mathfrak{m}$ , where  $\theta_u \equiv u\partial_u$ .

In summary, system of PDEs:

$$\begin{cases} \partial_z \hat{H} = \Gamma \circ \hat{H} \\ (\theta_u + \theta_v - s\theta_z)\hat{H} = t_0\hat{H}. \end{cases}$$

Eliminate  $\partial_v$  and get

$$\partial_z \hat{H} = u^{-a} v^{1+a} \theta_u \hat{H} + u^{1+b} v^{1-b} (s\theta_z \hat{H} - \theta_u \hat{H} - t_0 \hat{H}).$$

One can set  $v = 1$  and get  $H(z, u) = H(z; u, v \mapsto 1)$ :

$$\left[ (1 - szu^{b+s})\partial_z + (u^{b+s+1} - u^{1-a})\partial_u - t_0 u^{b+s} I \right] \circ H(z, u) = 0.$$

Apply general technology for first-order PDEs.

**Theorem 3.** *The probability of the urn defined by*

*matrix:* 
$$\begin{pmatrix} -a & a + s \\ b + s & -b \end{pmatrix},$$
 *initial cond :*  $a_0, t_0 := a_0 + b_0,$

*assuming it is tenable, is*

$$p_n(u) = \frac{\Gamma(n+1)\Gamma\left(\frac{t_0}{s}\right)}{s^n \Gamma\left(n + \frac{t_0}{s}\right)} [z^n] H(z, u).$$

*There*  $H(z, u)$  *is given by*

$$H(z, u) = \delta(u)^{t_0} \psi(z\delta(u)^s + I(u)),$$

*where*  $h = s + a + b,$

$$\delta(u) := (1 - u^h)^{1/h}, \quad I(u) := \int_0^u \frac{t^{a-1}}{\delta(t)^{a+b}} dt$$

*and the function*  $\psi$  *is defined implicitly by*

$$\psi(I(u)) = \frac{u^{a_0}}{\delta(u)^{t_0}}.$$

## Analytic aspects.

### Abelian integrals over Fermat curve

$$x^h + y^h = 1.$$

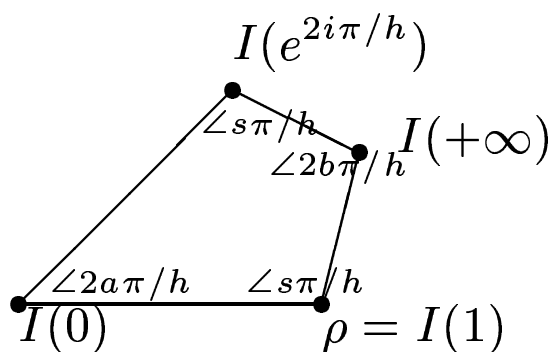
In general, global structure is not “clear”, but **dominant singularities** are Okay.

Consider the complex plane with  $h$  rays emanating from 0 and having directions given by all the  $h$ th roots of unity.

$$S_j := \left\{ z, \quad z = Re^{i\theta}, \quad 0 < R < \infty, \quad \frac{2j\pi}{h} < \theta < \frac{2(j+1)\pi}{h} \right\}.$$

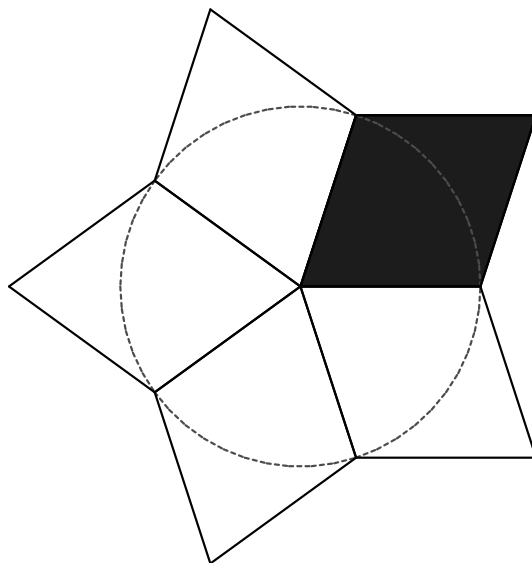
The image of  $S_0$  by  $I(u)$  is a quadrilateral, the **elementary kite** with vertices at the points

$$0, \quad I(1), \quad I(+\infty), \quad I(e^{2i\pi/h}).$$



The **elementary kite**.

**Definition 1.** *The fundamental polygon of an urn model is the (closure of) the union of  $h$  regularly rotated versions of the elementary kite about the origin.*



The elementary kite and the fundamental polygon of the urn

$$\begin{pmatrix} -1 & 4 \\ 4 & -1 \end{pmatrix}$$

$$h = 5, s = 3$$

$$\psi(z) \asymp (\rho - z)^{-1/3}.$$

**Theorem 4.** *The  $\psi$  function is analytic beyond its disc of convergence whose radius is*

$$\rho = \frac{1}{h} B\left(\frac{a}{h}, \frac{s}{h}\right) = \frac{1}{h} \frac{\Gamma\left(\frac{a}{h}\right)\Gamma\left(\frac{s}{h}\right)}{\Gamma\left(\frac{a+s}{h}\right)}.$$

*It has an algebraic branch point at  $z = \rho$ , where*

$$\psi(\rho - x) \asymp (\rho - x)^{-a/s}. \quad (8)$$

*It is continuable beyond its circle of convergence in a star-like domain.*

**Proof.** Uses symmetries about origin, then rotations around vertices.

Suffices to apply [singularity analysis](#).

## Probabilistic consequences

**Corollary 6.** *A quasipowers approximation holds but with weaker error terms than  $\mathcal{T}_{2,3}$ .*

*The limit law is Gaussian with speed of convergence  $O(\frac{1}{\sqrt{n}})$ .*

**Corollary 7.** *The large deviation rate exists and is expressible in terms of integrals over the Fermat curve.*

**Corollary 8.** *The extreme large deviation rate is given explicitly in terms of Gamma function values at rational points.*

## Part V

# Special cases and explicitly solvable models

The urns

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$$

correspond to: sampling with replacement or without replacement, and Coupon Collector.

Solutions agree with basic combinatorics!

$$H(z, u) = u^{a_0} e^{(a_0 + b_0)z}$$

$$H(z, u) = (z + u)^{a_0} (z + 1)^{b_0}.$$

$$H(z, u) = (e^z - 1 + u)^{a_0}.$$

**The Ehrenfest Urn** • Initially: 2 urns with balls moving between them.

• A celebrated controversy: *irreversibility versus ergodicity*.

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

Balance is  $s = 0$ . One has  $a = b = 1$ , hence  $h = 2$ . Start with  $a_0 = m$ .

One has  $\delta(u) = (1 - u^2)^{1/2}$ , hence **genus 0**.

$$I(u) = \int_0^u \frac{dt}{1-t^2} = \frac{1}{2} \log \frac{1+u}{1-u} = \operatorname{atanh}(u).$$

The function  $\psi$  is defined implicitly by

$$\psi(\operatorname{atanh}(u)) = \left( \frac{u}{\sqrt{1-u^2}} \right)^m,$$

which is equivalent to  $\psi(w) = \sinh^m w$ .

$$H(z, u) = (1-u^2)^{m/2} \sinh^m(z + \operatorname{atanh} u) = (\sinh z + u \cosh z)^m.$$

**≡ Combinatorics!**

## Elliptic cases

$$A = \begin{pmatrix} -2 & 3 \\ 4 & -3 \end{pmatrix}, B = \begin{pmatrix} -1 & 2 \\ 3 & -2 \end{pmatrix}, C = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}.$$

**Corollary 9.** *The three urn models  $A, B, C$  of balance 1 have solutions expressible in terms of elliptic functions. The corresponding lattices are the equilateral triangular lattice (cases  $A, C$ ) and the square lattice tilted by  $\pi/4$  (case  $B$ ).*

Like for  $\mathcal{T}_{2,3}$ : **TILINGS.**

$$D = \begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix}.$$

**Corollary 10.** *The urn model  $D$  admits an elliptic function solution of the lemniscatic type.*

## Urns without replacement

= the original models!

♥ Pólya–Eggenberger's contagion urn.

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

$$H(z, u) = \frac{u^{a_0}}{(1 - az)^{b_0/a} (1 - au^a z)^{a_0/a}},$$

With  $a = 1$  and  $a_0 = b_0 = 1$ , the PGF of at time  $n$  is

$$\frac{u}{n+1} (1 + u + \cdots + u^n),$$

Cf. also M. Durand. In general:

$$\mathbb{P}(\text{White}_n = a_0 + ja) = \frac{[z^n u^j] (1 - z)^{-b_0/a} (1 - uz)^{-a_0/a}}{[z^n] (1 - z)^{-t_0/a}}.$$

## The altruistic model.

$$T = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}.$$

Friedman 1947: "Every time an accident occurs, the safety campaign is pushed harder. Whenever no accident occurs, the safety campaign slackens and the probability of an accident increases."

$$H(z, u) = u^{a_0} e^{a_0 z(1-u^a)} \frac{(1-u^a)^{t_0/a}}{(1-ue^{az(1-u^a)})^{t_0/a}},$$

Smythe  $a = 1$ : stemma construction in philology as well as with recursive trees. Eulerian numbers and leaves in "recursive trees".

## The KMR urn: Kotz–Mahmoud–Robert!

$$\begin{pmatrix} a + 1 & 0 \\ 1 & a \end{pmatrix},$$

Bagchi and Pal (1985): “present some curious technical problems”.

Bivariate algebraic solution, **genus 0**:

$$u^{t_0} \left( 1 - \frac{1 - u^a}{(1 - (a + 1)u^{a+1}z)^{a/(a+1)}} \right)^{(a_0 - t_0)/a} (1 - (a + 1)u^{a+1}z)^{-t_0}$$

Mean and variance at time  $n$  ( $a = 3$ ):

$$4n + 1 - \frac{1}{\binom{n-3/4}{n}} \sim 4n - \frac{\pi\sqrt{2}}{\Gamma(3/4)} n^{3/4} + 1 + O(n^{-1/4}).$$

$$\frac{2}{3} \frac{8\sqrt{2} - 3\pi}{\Gamma(3/4)^2} n^{3/2} - \frac{3\pi\sqrt{2}}{\Gamma(3/4)} n^{3/4} + O(\sqrt{n}).$$

Distribution: prototype is

$$\hat{H}(z, u) = \left( 1 - u(1 - (1 - z)^{a/(a+1)}) \right)^{-1/a}$$

Singular exponent is discontinuous at  $u = 1$ .

Banderier, F., Schaeffer, Soria: within **analytic combinatorics** such changes are associated to **stable laws**. (Modify singularity analysis techniques.)

— e.g., **cores of random maps**.

**Corollary 11.** *Model with matrix  $(a + 1, 0, 1, a)$  and  $t_0 = 1, a_0 = 0$ :*

$$\mathbb{P} \left( \frac{X_n}{n^{a/(a+1)}} = x \right) \sim \frac{1}{n^{a/(a+1)}} \frac{\Gamma(1/(a+1))}{\Gamma(1/a)} x^{1/a-1} G \left( x; \frac{a}{a+1} \right),$$

$$G(x; \lambda) = -\frac{1}{\pi} \sum_{j \geq 1} \frac{(-x)^j}{j!} \Gamma(1 + \lambda j) \sin(j\pi\lambda),$$

*the quantity  $x^{-1} G(x^{-\lambda}; \lambda)$  is exactly the density of a stable law of index  $\lambda$  when  $0 < \lambda < 1$ .*

Supplements **martingale arguments** of Gouet (1993) = **nonconstructive**.

## Conclusions

A unified analytic framework

All  $2 \times 2$  urns with constant balance admit of analytic model.

Some interesting special function solutions: algebraic, elliptic, etc.

Some new probability laws.

Work still in progress!

```

> op(magic);
proc(a, b, c, d, a0, b0)
local eq1, eq2, s, h, eq, delta, t0;
    s := a + b;
    if s # c + d then
        lprint("Error, non constant row sum!");
        RETURN(NULL);
    end if;
    h := s - a - d;
    delta := (1 - u^h)^(1/h);
    t0 := a0 + b0;
    lprint("Matrix is:", [a, b, c, d], "with s,h:", s, h);
    dsolve(diff(w(u), u) = t0*w(u)*u^(h-1)/(u^h-1), w(u));
    eq1 := subs(w(u) = H, solve(%, _C1));
    diff(z(u), u) = -s*z(u)*u^(h-1)/(u^h-1) + u^(-a-1)/(u^h-1);
    dsolve(%, z(u));
    eq2 := subs(z(u) = z, solve(%, _C1));
    lprint("ratio of eigenvals is", (2*a - s) / s);
    factor(solve(eq1 = psi(eq2), H));
    eq := map(factor, %);
    [eq, "where", normal(subs(z = 0, eq)*delta^t0 / u^a0 = 1)]
end proc
> magic(-1, 3, 1, 1, 1, 0);
"Matrix is:", [-1, 3, 1, 1], "with s,h:", 2, 2
"ratio of eigenvals is", -2
[psi(z*u^2 - z - u)*sqrt(u+1)*sqrt(u-1), "where", psi(-u)*sqrt(u+1)*sqrt(u-1)*sqrt(1-u^2)/u = 1]
> magic(3, 0, 1, 2, 1, 0);
"Matrix is:", [3, 0, 1, 2], "with s,h:", 3, -2
"ratio of eigenvals is", 1
[psi((3*z*u^3 - 1) / (3*(u+1)^(3/2)*(u-1)^(3/2)) * u) / (sqrt(u+1)*sqrt(u-1)), "where", psi(-1 / (3*(u+1)^(3/2)*(u-1)^(3/2))) / (sqrt(u+1)*sqrt(u-1)*sqrt((u^2-1)/u^2)) = 1]
> |

```