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Analytic Urns

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General: BALLS and one or more URNS.

Two kinds of models

- **“Balls-and-bins”:** Throw balls at random into a number of urns.

  = Random allocations. Basic in the analysis of hashing algorithms; also SAT problem, cf V. Puyhaubert.


- **“Urn models”:** One urn contains balls whose nature may randomly change according to ball drawn and finite set of rules.
Here: URNS with BALLS of TWO COLOURS

<table>
<thead>
<tr>
<th>Type I</th>
<th>Type II</th>
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<tbody>
<tr>
<td>Black</td>
<td>White</td>
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RULES are given by a $2 \times 2$ Matrix

The composition of the urn at time 0 is fixed. At time $n$, a ball in the urn is randomly chosen and its colour is inspected (thus the ball is drawn, looked at and then placed back in the urn): if it is black, then $\alpha$ black and $\beta$ white balls are subsequently inserted; if it is white, then, $\gamma$ black balls and $\delta$ white balls are inserted.

\[
\begin{array}{c|cc}
\text{drawn} & \text{added} \\
\downarrow & B & W \\
B & \alpha & \beta \\
W & \gamma & \delta .
\end{array}
\]
• Drawing with replacement = \[
\begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}
\].

• Drawing without replacement = \[
\begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix}
\].

• Laplace’s “melancholic” model (1811): if a ball is drawn, it is repainted black no matter what its colour is.
\[
\begin{pmatrix}
0 & 0 \\
1 & -1
\end{pmatrix}
\]

• Ehrenfest & Ehrenfest = Über zwei bekannte Einwände gegen das Boltzmannsche II-Theorem, 1923. Irreversibility contradicts Ergodicity. Exchanges of basic balls (“atoms of heat”) between two urns, one cold and one hot:
\[
\begin{pmatrix}
-1 & 1 \\
1 & -1
\end{pmatrix}
\]

Bernoulli (1768), Laplace (1812).
• Polya Eggenberger model. A ball is drawn at random and then replaced, together with \( s \) balls of the same colour.

\[
\begin{pmatrix}
s & 0 \\
0 & s
\end{pmatrix}
\]

A model of positive influence. Closed form.

• “Adverse influence” model

\[
\begin{pmatrix}
0 & s \\
s & 0
\end{pmatrix}
\]

Used in epidemiology, etc.

• The special search tree model

\[
\begin{pmatrix}
-2 & 3 \\
4 & -3
\end{pmatrix}
\]

Here case of a $2 \times 2$-matrix

$$
\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix}
$$

with constant row-sum $\heartsuit \heartsuit$

$$
s := \alpha + \beta = \gamma + \delta.
$$

At time $n$ size $t_n$ satisfies $t_n = t_0 + sn$.

Constant increment $s$

A problem with three parameters + two initial conditions.

♠ Kotz, Mahmoud, Robert (2000) show “pathologies’ in some of the other cases.

Huge literature: Math. Reviews

TITLE=urn : Number of Matches=186’’
Lead to amazingly wide variety of behaviours, special functions, and limit distributions.

**Methods**

- Difference equations and explicit solutions.
- Same but with probability generating functions.
- Connection with branching processes.
- Stochastic differential equations (KMR)
- Martingales (Gouet)

**Here:** A frontal attack:
- PDE of snapshots at time $n$
- Usual solution for quasilinear PDE
- Bivariate GF and singularity perturbation

Conformal mapping argument, Abelian integrals over Fermat curves $z^h + y^h = 1$

+ Special solutions with elliptic functions
Part I

The $\mathcal{T}_{2,3}$ model—basic equations

\[
\begin{pmatrix}
-2 & 3 \\
4 & -3
\end{pmatrix}
\]

- Insertions in a 2–3 tree: 2–node $\mapsto$ 3–node; 3–node $\mapsto$ (2–node + 2–node).

- Fringe-balanced 2–3 tree analogous to median-of-three quicksort.

Mahmoud (1998); Panholzer–Prodinger (1998)
Evolution is:

\[ X_1 = 2; \quad X_n - X_{n-1} = \begin{cases} -2 & \text{with probability } \frac{X_{n-1}}{n} \\ +4 & \text{with probability } 1 - \frac{X_{n-1}}{n} \end{cases} \]

Let \( p_{n,k} = \mathbb{P}(X_n = k) \), \( p_n(u) := \sum_k p_{n,k} u^k \), and

\[ F(z, u) := \sum_{n \geq 1} p_n(u) u^n = \sum_{n,k} p_{n,k} u^k z^n, \]

whose elicitation is our main target.

**Lemma:** PDE satisfied by BGF of probabilities is

\[ (u^5 z - u) \frac{\partial F}{\partial z} + (1 - u^6) \frac{\partial F}{\partial u} + u^5 F + u^3 = 0. \]

**Proof.** Each \( p_n \) is determined from previous one by \( \partial_u \) = a differential recurrence. Gives PDE for bivariate generating function \( F \).

Take \( p_0(u) \) that satisfies PDE and write \( G := p_0(u) + F(z, u) \). then, we get a **homogeneous PDE**.

\[ (u^5 z - u) \frac{\partial G}{\partial z} + (1 - u^6) \frac{\partial G}{\partial u} + u^5 G = 0. \]

with

\[ p_0(u) = (1 - u^6)^{1/6} \int_0^u t^3 (1 - t^6)^{-7/6} \, dt. \]
Quasilinear first-order PDE's are reducible to ODEs.

\[ A(z, u, G) \frac{\partial G(z, u)}{\partial z} + B(z, u, G) \frac{\partial G(z, u)}{\partial u} + C(z, u, G) = 0 \]

1. Look for a solution in implicit form \( X(z, u, G) = 0 \).

\[ A(z, u, w) \frac{\partial X}{\partial z} + B(z, u, w) \frac{\partial X}{\partial u} - C(z, u, w) \frac{\partial X}{\partial w} = 0. \]

2. Consider the ordinary differential system

\[ \frac{dz}{A} = \frac{du}{B} = - \frac{dw}{C}. \]

The solution of two "independent" ordinary differential equations, e.g.,

\[ \frac{du}{B} = - \frac{dw}{C} \quad \text{and} \quad \frac{dz}{A} = \frac{du}{B}, \]

leads to two families of integral curves,

\[ U(u, z, w) = C_1 \quad \text{and} \quad V(u, z, w) = C_2. \]

3. The generic solution of the PDE is provided by

\[ X(z, u, w) = \Phi(U(u, z, w), V(u, z, w)), \]

for arbitrary bivariate \( \Phi \). Solving for \( w \) in \( X(z, u, w) = 0 \) provides a relation \( w = R_{\Phi}(z, u) \). General solution is

\[ G(z, u) := R_{\Phi}(z, u). \]
\[(u^5z - u)\frac{\partial G}{\partial z} + (1 - u^6)\frac{\partial G}{\partial u} + u^5G = 0.\]

Consider
\[
\frac{du}{1 - u^6} = \frac{dz}{u^5z - u} = -\frac{dw}{u^5w}.
\]

- \(du \leftrightarrow dw\) first integral by separation:
  \[w(1 - u^6)^{-1/6} = C_1.\]

- \(du \leftrightarrow dz\) variation of constant:
  \[z(1 - u^6)^{1/6} + \int_0^u \frac{t}{(1 - t^6)^{5/6}}\, dt = C_2.\]

Bind the two integrals by arbitrary \(\Phi\) & \(w \equiv G\)

\[
\Phi \left(\frac{G}{(1 - u^6)^{1/6}}, z(1 - u^6)^{1/6} + \int_0^u \frac{t}{(1 - t^6)^{5/6}}\, dt\right) = 0,
\]

Solve for \(G\), introducing arbitrary \(\psi\):

\[G(z, u) = \delta(u)\psi(\delta(u)z + I(u)), \quad I(u) := \int_0^u \frac{t}{(1 - t^6)^{5/6}}\, dt,
\]

with \(\delta(u) := (1 - u^6)^{1/6}\).
Initial conditions identify $\psi$. 

**Theorem 1.** Define the quantities

\[
\delta(u) = (1 - u^6)^{1/6}, \quad I(u) = \int_0^u \frac{t}{(1 - t^6)^{5/6}} \, dt, \quad J(u) = \int_0^u \frac{t^3}{(1 - t^6)^{7/6}} \, dt.
\]  

(1)

Then, the bivariate generating function of the probabilities is

\[
G(z, u) = \delta(u) \psi(z\delta(u) + I(u)),
\]

(2)

where $\psi$ is the function defined parametrically for $|u| < 1$ by

\[
\psi(I(u)) = J(u).
\]

(3)
Dominant singularities of $\psi$?

The diagram that summarizes $\psi$ is

$$
\begin{aligned}
\text{u} & \\
\text{z} = I(u) & \xrightarrow{\psi} & \psi(z) = J(u).
\end{aligned}
$$

The radius of analyticity of $\psi$ is

$$
\rho = I(1), \quad I(u) := \int_0^u \frac{t}{(1 - t^6)^{5/6}} \, dt.
$$

Proof: There is local (analytic) invertibility of $I(u)$ along $(0, \rho)$. Thus $\psi$ is analytic along $(0, \rho)$.

We have $\psi(z) = G(z, 0)$ which has nonnegative coeffs and is Pringsheim.

We have $I(1) < \infty$ while $J(1) = \infty$. Thus $\rho$ is a singularity.

By Eulerian Beta integrals:

$$
\rho \equiv I(1) = \frac{1}{6} B\left(\frac{1}{6}, \frac{1}{3}\right) = \frac{1}{6} \frac{\Gamma(1/3)\Gamma(1/6)}{\Gamma(1/2)} \approx 1.40218 21053 25454.
$$

$$
\left(\frac{\psi_{47}}{\psi_{50}}\right)^{1/3} \approx 1.40218 21053 25456.
$$
Local expansions near $u = 1$ plus symmetries of the problem are compatible with:

**Proposition 1.** There are no singularities of $\psi(z)$ on $|z| = \rho$ other than $\rho, \rho \omega, \rho \omega^2$ that are simple poles. Precisely, let

$$S(z) = \frac{1}{\rho - z} + \frac{1}{\rho \omega - z} + \frac{1}{\rho \omega^2 - z} = \frac{3z^2}{\rho^3 - z^3}.$$ 

The function

$$\psi(z) - S(z)$$

is analytic in a disc $|z| < R$ for some $R$ satisfying $R > \rho$. (One can take $R = 2\rho$.)

Why singularities of $\psi$, BTW?

$$G(z, u) = \delta(u)\psi(z\delta(u) + I(u)), \quad \psi(I(u)) = J(u).$$

Set $u = 0$ and estimate $[z^n]\psi(z)$: Get extremely large deviations, all balls of one colour.

Know approximately $[z^n]G(z, u) = \text{PGF of distribution} \sim \text{LIMIT LAW}.$

Set $u$ to value $\neq 1$ and get \text{LARGE DEVIATIONS}.

And a good deal more...
Part II
The $\mathcal{T}_{2,3}$ model—elliptic structure

Recall: an elliptic function is a doubly periodic meromorphic function in $\mathbb{C}$.

Historically: Integration over a conic $\int Q(z, y)$ where $y = \sqrt{P(x)}$ and $\deg P = 1, 2$, yields functions like $\arctan$, $\arcsin$ and hyperbolic counterparts. Such functions satisfy $\arctan(z) \equiv \arctan(z) + k\pi$ so that inverses are simply periodic. This is a way to (re)build trigonometry from integrals over conics.

Integration over a cubic or a quartic $y = \sqrt{P(x)}$ with $\deg P = 3, 4$, which are topologically “doughnuts” leads to double periodicity. Such things occur when rectifying the ellipse hence the name elliptic integrals and elliptic functions for inverses.

$$\sum 1/(z - \omega)^3$$
For a parameterized curve, \( \psi(I(u)) = J(u) \), examine all possible paths in the \( u \)-plane, and the corresponding determinations of \( I(u) \). Reflect on

\[
\psi \left( \int_{1}^{u} \frac{dt}{t} \right) = u,
\]

which defines \( \psi(\log u) = u \), that is, \( \psi(z) = e^z \).

Here:

\[
\psi(I(u)) = J(u)
\]

\[
I(u) = \int_{0}^{u} \frac{t \, dt}{(1-t^6)^{5/6}}, \quad I(u) = \int_{0}^{u} \frac{t^3 \, dt}{(1-t^6)^{7/6}}.
\]

The curve is \( t^6 + y^6 = 1 \) and has genus 10.

Go step by step.

- **The elementary triangle**
- **The fundamental triangle**
The region $R_0$ (left) and a rendering of the six-sheeted Riemann surface $\mathcal{R}$ of $\delta(u) \equiv (1 - u^6)^{1/6}$ for $u$ near 1 (right).

Because of double parameterization, taking $u$ in a half-plane suffices.
Lemma 1. The function $\psi$ maps the interior of $(R_0 \cap H)$ in a one-to-one manner on the interior of the equilateral triangle $T$ with vertices $\rho, \rho \omega, \rho \omega^2$, where $\omega := e^{2i\pi/3}$.

Proof. Folds angles in an appropriate way... Start with Elementary triangle.

The “elementary triangle” $T_0$ (right) is the image of the basic sector $S_0$ (left) via the mapping $u \mapsto I(u)$.

The “fundamental triangle” $T$ (right) is the image of the slit upperhalf plane $(R_0 \cap H)$ (left) via the mapping $u \mapsto I(u)$. 
Three **elementary triangles** assemble to form a fundamental triangle

![Diagram](image)

\[ \zeta^2, \zeta, 0, 1 \quad \Rightarrow \quad \rho \omega \equiv I(\zeta), \rho \equiv I(1), \rho \omega^2 \equiv I(\zeta^2) \]

Based on \( I(\zeta u) = \omega I(u) \), where \( \zeta^6 = 1, \omega^3 = 1 \).

Another view of the image of \((R_0 \cap H)\) by \( I(u) \) giving the fundamental triangle \( T \): a representation of the images of rays emenating from 0 and of circles centred at 0.
Lemma 2. The function $\psi$ is analytic (holomorphic) in the disk $|z| < 2\rho$ stripped of the points $\rho, \rho \omega, \rho \omega^2$. (The function admits these three points as simple poles, as asserted in Prop. 1.)

Rotated copies of the fundamental triangle around $\rho, \rho \omega, \rho \omega^2$ shown against the circle of convergence of $\psi(z)$.

Proof. Laces around $u = 1$ and changes of variables: $I(1) - I(u) \sim 6^{1/6}(1 - u)^{1/6}$. 
The full story and the elliptic connection

A lattice $\Lambda$ with generators $\xi, \eta \in \mathbb{C}$:

$$\Lambda(\xi, \eta) = \{ n_1 \xi + n_2 \eta \mid n_1, n_2 \in \mathbb{Z} \}.$$  

The Weierstraß zeta function relative to $\Lambda$ is classically defined as

$$\zeta(z; \Lambda) := \frac{1}{z} + \sum_{w \in \Lambda \setminus \{0\}} \left( \frac{1}{z - w} + \frac{1}{w} + \frac{z}{w^2} \right).$$

**Theorem 2.** The $\psi$-function of the $T_{2,3}$ model initialized with 2 balls of the first type ($a_0 = t_0 = 2$) is exactly

$$\psi(z) = \frac{1}{\rho \sqrt{3}} \left( -\zeta \left( \frac{z - \rho}{\rho \sqrt{3}} \right) + \zeta \left( -\frac{1}{\sqrt{3}} \right) \right), \quad \rho := \frac{1}{6} \frac{\Gamma(1/3)\Gamma(1/6)}{\Gamma(1/2)} (4),$$

where $\zeta(z) := \zeta(z; \Lambda_{\text{hex}})$ is the Weierstraß zeta function of the hexagonal lattice:

$$\Lambda_{\text{hex}} := \left\{ n_1 e^{i\pi/6} + n_2 e^{-i\pi/6} \mid n_1, n_2 \in \mathbb{Z} \right\}.$$
Proof.

- Follow all paths and examine $I(\gamma(u))$: any point $z \in \mathbb{C}$ is reachable.

- There is a pole of $\psi$ at lattice points and residue is $-1$ since determinations of $\delta(u)$ in $I, J$ are the same.

- By Liouville, $\psi(z)$ and $\zeta$ coincide (up to normalization).
A path in the \( z \)-plane from 0 to \( P \) and the contour \( \gamma \) above the \( u \)-plane that realizes it via \( u \mapsto z = I(u) \).
Part III

Probabilistic consequences

Extract coeffs in simple fractions:

**Corollary 1.** For the $\mathcal{T}_{2,3}$ model, the probability generating function $p_n(u) = \mathbb{E}(u^{X_n})$ admits an exact formula valid for all $n \geq 2$,

$$p_n(u) = \sum_{n_1,n_2=-\infty}^{+\infty} \left( K(u) + \frac{\rho \sqrt{3}}{\delta(u)} (n_1 e^{i \pi / 6} + n_2 e^{-i \pi / 6}) \right)^{-n-1},$$

where

$$K(u) := \frac{1}{\delta(u)} \int_u^1 \frac{t}{\delta(t)^5} dt, \quad \delta(u) = (1 - u^6)^{1/6}.$$

**Note:** when $u \approx 1$, this is like $K(u)^{-n-1}$. 
The Quasi-powers framework.

Classics are:

(Laplace) Given a random variable $X$, define its characteristic function aka Fourier transform as

$$
\phi_X(t) := \mathbb{E}(e^{itY}) = \sum_k \mathbb{P}(Y = k)e^{itk} = \varphi(e^{it}).
$$

If $S_n = X_1 + \cdots + X_n$ with i.i.d. $X_j$, then:

$$
\phi_{S_n}(t) = (\phi_X(t))^n.
$$

(Lévy et al.) Fourier inversion is continuous: convergence of F.T.’s

$$
\lim_{n \to \infty} \phi_{Y_n}(t) = \phi_Z(t) \quad \text{pointwise}
$$

implies $Y_n \Rightarrow Z$ in distribution.

(Berry-Esseen) Uniform distance on F.T. furthermore gives bounds on uniform distance on distribution functions.
A Sedgewick plot of $\{\mathbb{P}(x_n = k)\}_{k=0}^{n-1}$ for $n = 24 \ldots 96$ (the horizontal axis is normalized to $n + 1$).
Gaussian laws in analytic combinatorics

Classically $S_n = X_1 + \cdots + X_n$, where $X_j$ have mean and variance. Calculation shows that

$$\log \mathbb{E} \left[ \exp \left( it \frac{S_n - n\mu}{\sigma \sqrt{n}} \right) \right] \xrightarrow{n \to \infty} -\frac{t^2}{2}.$$

Hence Central Limit Theorem.

A “good” uniform approximation $p_n(u) \sim a(u) \cdot B(u)^n$ for $u \approx 1$ (complex neighbourhood) is called QuasiPowers approximation.

From Bender, F.-Soria, Hwang (1995), one has:
— Moments result from differentiation (complex an.)
— Convergence to Gaussian distribution (erf)
— Speed of convergence is $\frac{1}{\sqrt{n}}$.
— Some large deviation estimates: probability of being far from mean at $cn$ for $c \neq \mu$ is exponentially small.
Corollary 2 (Gaussian limit). For the $T_{2,3}$ model, the random variable $X_n$ representing the number of balls of the first type at time $n$ is asymptotically Gaussian with speed of convergence to the limit $O(n^{-1/2})$,

$$\mathbb{P}\left(\frac{X_n - \mathbb{E}(X_n)}{\sqrt{\operatorname{Var}(X_n)}} \leq x\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} \, dy + O\left(\frac{1}{\sqrt{n}}\right).$$

Proof. From lattice sum, in complex neighbourhood $u \approx 1$:

$$p_n(u) = K(u)^{-n-1}(1 + O(2^{-n})).$$

Note that $K(u)^{-1}$ plays the rôle of a probability characteristic function but it isn’t!

$$K(u)^{-1} \approx 0.713 + 0.254u^2 + 0.090u^4 - 0.086u^6 + 0.022u^8 + \cdots.$$
The shape of moments.

In the literature, only a few moments are computed via (unpleasant?) recurrence manipulations from probabilities and original rec. Here: everything is almost as though

\[ p_n(u) = K(u)^{-n-1}. \]

\[ P_1(\nu) = \frac{4\nu}{7}, \quad P_2(\nu) = \frac{4\nu}{637}(52\nu + 17), \]
\[ P_3(\nu) = \frac{8\nu}{84721}(1976\nu^2 + 1938\nu - 11063). \]

**Corollary 3 (Moments).** For the \( \mathcal{T}_{2,3} \) model, exact polynomial forms for moments of any order are available: the factorial moments satisfy

\[ \mathbb{E}((X_n)^r) = P_r(n + 1), \quad n \geq 6r, \]

where the \( P_r \) are polynomials generated by

\[ e^{vL(h)} = \sum_{r=0}^{\infty} \frac{h^r}{r!} P_r(\nu) \quad \text{and} \quad L(h) = -\log K(1 + h). \]

Rota: polynomials of “binomial type” satisfying various convolution relations.
Large deviations.

From dominant poles of $\psi$, corresponding to $u = 0$:

**Corollary 4 (Extreme large deviations).** The probability that, in the $\mathcal{T}_{2,3}$ model, all balls are of the first colour satisfies

$$[z^{3n+2}]\psi(z) \sim 3\rho^{-3n-3} \left(1 + O(A^{-n})\right),$$

for any $A < 8$.

Moreover:

**Corollary 5 (Large deviations).** Let $\alpha$ be a number of the open interval $(0, \frac{4}{7})$. One has

$$\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(X_n \leq \alpha \cdot n) = -\rho(\alpha),$$

where the rate function $\rho$ is determined by

$$\rho(\alpha) = \log (\lambda_0^\alpha K(\lambda_0)),$$

and $\lambda_0$ depending on $\alpha$ is the implicitly defined root $u \in (0, 1)$ of

$$\frac{uK'(u)}{K(u)} + \alpha = 0.$$
Proof is standard for probabilists. Assume
\[ p_n(u) \approx B(u)^n, \text{ where } B(u) \text{ increases from } c_0 \text{ to } 1 \text{ as } u \in (0, 1). \]

One has Cauchy aka saddle-point bounds:

\[ [u^k]p_n(u) \leq \frac{p_n(u_0)}{u_0^k} \approx \frac{B(u_0)^n}{u_0^k}. \]

Adopt the best \( u_0 \) (which must exist by some convexity prop.) and get an exponentially small upperbound. Cramér aka “shifting the mean”: apply a form of CLT near \( u_0 \) to conclude that the upperbound is also a lowerbound.
Left: a Sedgewick plot of $\{-\frac{1}{n} \log \mathbb{P}(X_n = k)\}_{k=0}^{n+1}$ for $n = 24 \ldots 96$ (the horizontal axis is normalized to $n + 1$); right: a comparison against the large deviation rate (thick line).
Related work

Gaussian law by moments: (Bagchi & Pal 1985). Here global expression for moment polynomials + speed of convergence.

Elliptic connection related to (Panholzer & Prodinger 1998) via specific approach $y''' = \cdot y'^2 + \cdot$. Here: much more general, for whole class.

Large deviations seem to be new.

Local limit laws? Probably true. Want to apply saddle point, need bounding technique.

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<th>Category</th>
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<tr>
<td>Counting</td>
<td>$u = 1$</td>
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<tr>
<td>Moments</td>
<td>$u = 1 \pm \frac{1}{\infty}$</td>
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<tr>
<td>Large deviations</td>
<td>$u = [1 - \eta, 1 + \eta]$</td>
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<tr>
<td>Central limit</td>
<td>$u = 1 + \Box$</td>
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<tr>
<td>Local limit</td>
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Consider general case of urns with replacement, i.e., $\alpha < 0$, $\beta < 0$.

\[
\begin{pmatrix}
-a & a + s \\
b + s & -b
\end{pmatrix}
\]

A 3-parameter family
Plus initially $a_0$ black; $b_0$ white.
Ideas:

- Look at the *enumerative* version.
- Set up PDE for bivariate generating function via operator $e^{z\Gamma}$.
- Get a \( \psi \)-function parameterized by Abelian integrals over Fermat curve \( x^h + y^h = 1 \).
- Determine singularities by looking at geometry of conformal maps of basic domains.
- Generally, non-elliptic solutions, but:
  - Gaussian limit with speed of convergence;
  - Extreme large deviations;
  - Large deviation rate

Recycles most of \( \mathcal{T}_{2,3} \) case but without double periodicity at this level of generality.
The operator approach

Number balls in order of appearance $1, 2, 3, \ldots$

choose 2 \hspace{1cm} choose 5 \hspace{1cm} choose 8 \hspace{1cm} choose 9
\[1_I, 2_I, 3_{II}, 4_{II}, 5_{II}, 6_I, 7_I, 8_I, 9_I, 7_I, 9_I, 10_{II}, 11_{II}, 12_{II} \]

$s = a + b$; at time $n$, after action, size is $t_n$;
$t_0 = a_0 + b_0$ is given;

\[ t_n = t_0 + sn. \]

Thus

\[ H_n = t_0(t_0 + s) \cdots (t_0 + ns). \]

Let $H_{n,k}$ be number of histories of length $n$ leading to $k$: Black (Type I) balls and

\[ H(z, u) := \sum_{n,k} H_{n,k} u^k \frac{z^n}{n!}. \]
Combinatorial marking $\equiv$ differentiation

Represent a particular history $h$ with $k$ black balls and $\ell$ white balls as $u^k v^\ell$.

Evolution chooses a black ball and acts; e.g., for $\mathcal{T}_{2,3}$:

$$u^k v^\ell \mapsto k u^k v^\ell \mapsto k u^{k-2} v^{\ell+3}.$$ 

Similarly for white balls. Cleverly introduce:

$$\Gamma := u^{-1} v^3 \frac{\partial}{\partial u} + u^4 v^{-2} \frac{\partial}{\partial v}.$$ 

Then $\Gamma u^k v^\ell$ describes all the successors of $h \cong u^k v^\ell$.

All evolutions of length $n$ are generated by $\Gamma^n u^{a_0} v^{b_0}$, and a trivariate version of $H$ is

$$\hat{H}(z, u, v) := e^{z\Gamma} \circ u^{a_0} v^{b_0}.$$
The basic PDE

- By general principles:

\[ \partial_z (e^{z\Gamma} f) = \Gamma e^{z\Gamma} f, \quad \partial_z \hat{H} = \Gamma \circ \hat{H}. \]

- By homogeneity, any term \( m = u^\alpha v^\beta z^n \) has \( \alpha + \beta = sn + t_0: (\theta_u + \theta_v - s\theta_z)m = t_0 m \), where \( \theta_u \equiv u\partial_u \).

In summary, system of PDEs:

\[ \left\{ \begin{array}{l}
\partial_z \hat{H} = \Gamma \circ \hat{H} \\
(\theta_u + \theta_v - s\theta_z)\hat{H} = t_0 \hat{H}.
\end{array} \right. \]

Eliminate \( \partial_v \) and get

\[ \partial_z \hat{H} = u^{-a} v^{1+a} \theta_u \hat{H} + u^{1+b} v^{1-b} \left( s\theta_z \hat{H} - \theta_u \hat{H} - t_0 \hat{H} \right). \]

One can set \( v = 1 \) and get \( H(z, u) = H(z; u, v \mapsto 1): \)

\[ [ (1 - szu^{b+s}) \partial_z + (u^{b+s+1} - u^{1-a}) \partial_u - t_0 u^{b+s} I ] \circ H(z, u) = 0. \]
Apply general technology for first-order PDEs.

**Theorem 3.** The probability of the urn defined by

matrix: \[
\begin{pmatrix}
-a & a + s \\
b + s & -b
\end{pmatrix}
\]

initial cond: \(a_0, t_0 := a_0 + b_0\),

assuming it is tenable, is

\[
p_n(u) = \frac{\Gamma(n + 1)\Gamma\left(\frac{t_0}{s}\right)}{s^n \Gamma\left(n + \frac{t_0}{s}\right)} [z^n] H(z, u).
\]

There \(H(z, u)\) is given by

\[
H(z, u) = \delta(u)^{t_0} \psi(z\delta(u)^s + I(u)),
\]

where \(h = s + a + b\),

\[
\delta(u) := (1 - u^h)^{1/h}, \quad I(u) := \int_0^u \frac{t^{a - 1}}{\delta(t)^{a + b}} \, dt
\]

and the function \(\psi\) is defined implicitly by

\[
\psi(I(u)) = \frac{u^{a_0}}{\delta(u)^{t_0}}.
\]
Analytic aspects.

Abelian integrals over Fermat curve

\[ x^h + y^h = 1. \]

In general, global structure is not “clear”, but dominant singularities are OK.

Consider the complex plane with \( h \) rays emanating from 0 and having directions given by all the \( h \)th roots of unity.

\[ S_j := \left\{ z, \quad z = R e^{i\theta}, \quad 0 < R < \infty, \quad \frac{2j\pi}{h} < \theta < \frac{2(j + 1)\pi}{h} \right\}. \]

The image of \( S_0 \) by \( I(u) \) is a quadrilateral, the elementary kite with vertices at the points

\[ 0, \quad I(1), \quad I(+\infty), \quad I(e^{2i\pi/h}). \]

\[ \begin{array}{c}
\text{I}(e^{2i\pi/h}) \\
\angle_{s\pi/h} \quad \angle_{2\pi/h} \quad \text{I}(+\infty) \\
\angle_{2a\pi/h} \quad \angle_{s\pi/h} \quad \text{I}(0) \quad \rho = I(1)
\end{array} \]

The elementary kite.
Definition 1. The fundamental polygon of an urn model is the (closure of) the union of $h$ regularly rotated versions of the elementary kite about the origin.

The elementary kite and the fundamental polygon of the urn

\[
\begin{pmatrix}
-1 & 4 \\
4 & -1
\end{pmatrix}
\]

$h = 5$, $s = 3$

\[
\psi(z) \asymp (\rho - z)^{-1/3}.
\]
Theorem 4. The $\psi$ function is analytic beyond its disc of convergence whose radius is

$$\rho = \frac{1}{h} B\left(\frac{a}{h}, \frac{s}{h}\right) = \frac{1}{h} \frac{\Gamma\left(\frac{a}{h}\right)\Gamma\left(\frac{s}{h}\right)}{\Gamma\left(\frac{a+s}{h}\right)}.$$  

It has an algebraic branch point at $z = \rho$, where

$$\psi(\rho - x) \asymp (\rho - x)^{-a/s}. \quad (8)$$

It is continuable beyond its circle of convergenc in a star-like domain.

Proof. Uses symmetries about origin, then rotations around vertices.

Suffices to apply singularity analysis.
Probabilistic consequences

**Corollary 6.** A quasipowers approximation holds but with weaker error terms than \( T_{2,3} \).
The limit law is Gaussian with speed of convergence \( O\left(\frac{1}{\sqrt{n}}\right) \).

**Corollary 7.** The large deviation rate exists and is expressible in terms of integrals over the Fermat curve.

**Corollary 8.** The extreme large deviation rate is given explicitly in terms of Gamma function values at rational points.
Part V

Special cases and explicitly solvable models

The urns

\[
\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}
\]

correspond to: sampling with replacement or without replacement, and Coupon Collector.

Solutions agree with basic combinatorics!

\[
H(z, u) = u^{a_0} e^{(a_0 + b_0)z}
\]

\[
H(z, u) = (z + u)^{a_0} (z + 1)^{b_0}.
\]

\[
H(z, u) = (e^z - 1 + u)^{a_0}.
\]
The Ehrenfest Urn • Initially: 2 urns with balls moving between them.
• A celebrated controversy: *irreversibility versus ergodicity.*

\[
\begin{pmatrix}
-1 & 1 \\
1 & -1
\end{pmatrix}
\]

Balance is \( s = 0 \). One has \( a = b = 1 \), hence \( h = 2 \).
Start with \( a_0 = m \).

One has \( \delta(u) = (1 - u^2)^{1/2} \), hence genus 0.

\[
I(u) = \int_0^u \frac{dt}{1 - t^2} = \frac{1}{2} \log \frac{1 + u}{1 - u} = \text{atanh}(u).
\]

The function \( \psi \) is defined implicitly by

\[
\psi(\text{atanh}(u)) = \left( \frac{u}{\sqrt{1 - u^2}} \right)^m,
\]

which is equivalent to \( \psi(w) = \sinh^m w \).

\[
H(z, u) = (1 - u^2)^{m/2} \sinh^m (z + \text{atanh} u) = (\sinh z + u \cosh z)^m.
\]

≡ *Combinatorics!*
Elliptic cases

\[ A = \begin{pmatrix} -2 & 3 \\ 4 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 2 \\ 3 & -2 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}. \]

**Corollary 9.** The three urn models \( A, B, C \) of balance 1 have solutions expressible in terms of elliptic functions. The corresponding lattices are the equilateral triangular lattice (cases \( A, C \)) and the square lattice tilted by \( \pi/4 \) (case \( B \)).

Like for \( T_{2,3} \): **TILINGS**.

\[ D = \begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix}. \]

**Corollary 10.** The urn model \( D \) admits an elliptic function solution of the lemniscatic type.
Urns without replacement

= the original models!

♡ Pólya–Eggenberger’s contagion urn.

\[
\begin{pmatrix}
a & 0 \\
0 & a
\end{pmatrix}.
\]

\[
H(z, u) = \frac{u^{a_0}}{(1 - az)^{b_0/a} (1 - au^{a}z)^{a_0/a}},
\]

With \(a = 1\) and \(a_0 = b_0 = 1\), the PGF of at time \(n\) is

\[
\frac{u}{n+1}(1 + u + \cdots + u^n),
\]

Cf. also M. Durand. In general:

\[
\mathbb{P}(\text{White}_n = a_0 + ja) = \frac{[z^n u^j] (1 - z)^{-b_0/a} (1 - uz)^{-a_0/a}}{[z^n] (1 - z)^{-t_0/a}}.
\]
The altruistic model.

\[ T = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}. \]

Friedman 1947: “Every time an accident occurs, the safety campaign is pushed harder. Whenever no accident occurs, the safety campaign slackens and the probability of an accident increases.”

\[ H(z, u) = u^{a_0} e^{a_0 z(1-u^a)} \frac{(1 - u^a) t_0/a}{(1 - u e^{a z(1-u^a)}) t_0/a}, \]

Smythe \( a = 1 \): stemma construction in philology as well as with recursive trees. Eulerian numbers and leaves in “recursive trees”.
The KMR urn: Kotz–Mahmoud–Robert!

\[
\begin{pmatrix}
  a + 1 & 0 \\
  1 & a
\end{pmatrix},
\]

Bagchi and Pal (1985): “present some curious technical problems”.

Bivariate algebraic solution, genus 0:

\[
u_t \left( 1 - \frac{1 - u^a}{(1 - (a + 1)u^{a+1}z)^a/(a+1)} \right)^{(a_0 - t_0)/\alpha} (1 - (a + 1)u^{a+1}z)^{-t_0}\]

Mean and variance at time \(n\) (\(a = 3\)):

\[
4n + 1 - \frac{1}{(n^{-3/4})} \sim 4n - \frac{\pi \sqrt{2}}{\Gamma(3/4)} n^{3/4} + 1 + O(n^{-1/4}).
\]

\[
\frac{2}{3} \frac{8\sqrt{2} - 3\pi}{\Gamma(3/4)^2} n^{3/2} - \frac{3\pi \sqrt{2}}{\Gamma(3/4)} n^{3/4} + O(\sqrt{n}).
\]

Distribution: prototype is

\[
\hat{H}(z, u) = \left( 1 - u(1 - (1 - z)^a/(a+1)) \right)^{-1/a}
\]

Singular exponent is discontinuous at \(u = 1\).
Banderier, F., Schaeffer, Soria: within analytic combinatorics such changes are associated to stable laws. (Modify singularity analysis techniques.)
— e.g., cores of random maps.

**Corollary 11.** Model with matrix \((a + 1, 0, 1, a)\) and \(t_0 = 1, a_0 = 0\):

\[
\mathbb{P}\left(\frac{X_n}{n^{a/(a+1)}} = x\right) \sim \frac{1}{n^{a/(a+1)}} \frac{\Gamma(1/(a + 1))}{\Gamma(1/a)} x^{1/a - 1} G\left(x; \frac{a}{a + 1}\right),
\]

\[
G(x; \lambda) = -\frac{1}{\pi} \sum_{j \geq 1} \frac{(-x)^j}{j!} \Gamma(1 + \lambda k) \sin(k\pi \lambda),
\]

the quantity \(x^{-1} G(x^{-\lambda}; \lambda)\) is exactly the density of a stable law of index \(\lambda\) when \(0 < \lambda < 1\).

Conclusions

A unified analytic framework

All $2 \times 2$ urns with constant balance admit of analytic model.

Some interesting special function solutions: algebraic, elliptic, etc.

Some new probability laws.

Work still in progress!
> op(magic);
proc(a, b, c, d, a0, b0)
local eq1, eq2, s, h, eq, t0;
  s := a + b;
  if s = c + d then
    lprint("Error, non constant
trow sum");
    RETURN(NULL)
  end if;
  h := s - a - d;
  δ := (1 - u^h)^((1 / h);
  t0 := a0 + b0;
  lprint("Matrix is:", [a, b, c, d], ", with s,h:", s, h);
  dsolve(diff(w(u), u) = t0*w(u)*u^((h - 1) / (u^h - 1), w(u));
  eq1 := subs(w(u) = H, solve(%));
  diff(z(u), u) = -s*z(u)*u^((h - 1) / (u^h - 1) + u^(-s - 1) / (u^h - 1);
  dsolve(%);
  eq2 := subs(z(u) = z, solve(%));
  lprint("ratio of eigenvals is", \(2\*s - s) / s);
  factor(solve(eq1 = ψ(eq2), H));
  eq := map(factor, %);
  [eq, "where", normal(subs(z = 0, eq)*δ^t0 / u^a0 = 1)]
end proc

> magic(-1, 3, 1, 1, 1, 0);
"Matrix is:", [-1, 3, 1, 1], "with s,h:", 2, 2
"ratio of eigenvals is", -2
\[
\begin{bmatrix}
\psi(zu^2 - z - u)\sqrt{u + 1}\sqrt{u - 1}, "where", \frac{\psi(-u)\sqrt{u + 1}\sqrt{u - 1}\sqrt{1 - u^2}}{u} - 1
\end{bmatrix}
\]

> magic(3, 0, 1, 2, 1, 0);
"Matrix is:", [3, 0, 1, 2], "with s,h:", 3, -2
"ratio of eigenvals is", 1
\[
\begin{bmatrix}
\psi\left(\frac{3zu^3 - 1}{3(u + 1)^{(3/2)}(u-1)^{(3/2)}}\right)u\quad \psi\left(\frac{1}{3(u + 1)^{(3/2)}(u-1)^{(3/2)}}\right)u \quad \sqrt{u + 1}\sqrt{u - 1}\sqrt{u^2 - 1} = 1
\end{bmatrix}
\]