SINGULAR COMBINATORICS

A. Symbolic Methods

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ANALYTIC COMBINATORICS

• Find quantitative properties of large discrete structures = random combinatorial structures.

• Identify the fundamental analytic structures ≠ probabilistic approaches.

Via complex analysis establish relationship

Combinatorics $\rightsquigarrow$ Analysis $\rightsquigarrow$ Asymptotics

• Organization into major schemas where chain can be worked out: “combinatorial processes” // stochastic processes.

Example: “bag” process (Set); “row” process” (Seq).
Universality: E.g. take a random tree of size \( n \) (large):
— **Height** is with high probability (w.h.p.) \( O(\sqrt{n}) \);
— Any designated pattern \( \varpi \) occurs on average \( C_\varpi \cdot n \), and distribution is asymptotically normal.

- Such properties hold for a very wide range of local **construction rules** (also Galton-Watson trees conditioned on size).
- Similar properties hold for “molecule trees”, random mappings, etc. But labelled trees based on order properties belong to a different **universality class**, with e.g., logarithmic height.
Figure 5.5  A random binary tree with 256 internal nodes

Figure 5.11  A binary search tree built from 256 randomly ordered keys
Analytic combinatorics

A. Counting Generating Function
B. Analytic properties of GF
Singularities + transfer to coefficients
C. Perturbation for distributions.

SYMBOLIC METHODS + COMPLEX ASYMPTOTICS + PERTURBATION.
Duality: Combinatorics versus probability

Brownian motion, continuum random tree, etc.
PART A. SYMBOLIC METHODS

**Goal:** develop generic tools to determine generating functions \( \equiv \text{GFs} \).

**Approach:** Formulate a programming language to specify combinatorial structures such that translation into GFs is automatic.

Abstraction:

Embed a fragment of elementary set theory into a **language of constructions**. Map to algebra(s) of special functions.
1 UNLABELLED STRUCTURES AND OGFS

Ordinary Generating Function (OGF)

\[(f_n) \longrightarrow f(z) := \sum_{n=0}^{\infty} f_n z^n.\]

\((f_n)\) is number sequence, e.g., counting sequence.

Later: Exponential Generating function (EGF): \((f_n) \longrightarrow f(z) := \sum_{n=0}^{\infty} \frac{f_n z^n}{n!}.\)
\( \mathcal{C} \) = a combinatorial class: at most denumerable set with size function.

\( \mathcal{C}_n \) = subclass of objects of size \( n \).

\( C_n \) = \# objects of size \( n \) = \( \text{card}(\mathcal{C}_n) \).

\[ C(z) = \text{OGF} := \sum_{n \geq 0} C_n z^n = \sum_{\gamma \in \mathcal{C}} z^{\mid \gamma \mid}. \]

Count up to combinatorial isomorphism: \( \mathcal{C} \cong \mathcal{D} \) iff \( \exists \) size-preserving bijection.

Atom: \( \mathcal{Z} \mapsto z \); neutral element: \( \mathcal{E} \mapsto 1 \).
How many binary trees $B_n$ with $n$ external nodes?

$B = □ + \bullet, (B \times B)$.

Euler-Segner (1743): Recurrence

$$B_n = \sum_{k=1}^{n-1} B_k B_{n-k}.$$ 

Form OGF: $B(z) = z + (B(z) \times B(z))$.

Solve equation (quadratic):

$B(z) = \frac{1}{2} (1 - \sqrt{1 - 4z}) = \frac{1}{2} - \frac{1}{2} (1 - 4z)^{1/2}.$

Expand:

$B_n = \frac{1}{n} \binom{2n-2}{n-1} \text{ (Catalan numbers)}$

Analogy:

$\begin{align*}
B = □ + (\bullet B \times B) 
\end{align*} \sim
\begin{align*}
B(z) &= z + (B(z) \times B(z))
\end{align*}$
Outline

Define a collection of constructions

union, product, sequence, set, cycle, \ldots

allowing for recursive definitions.

meta-THM1: OGFs are automatically computable (equations!)

meta-THM2: Counting sequences are automatically computable in time $O(n^2)$, and even $O(n^{1+\epsilon})$.

meta-THM3: Random generation is fast in $O(n \log n)$ arithmetic op’ns.
**Theorem.** There exists a dictionary:

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<tr>
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</tr>
<tr>
<td>$C = \text{CYC}(A)$</td>
<td>$C(z) = \text{Log} \frac{1}{1 - A(z)}$</td>
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$E$ or $1$: “neutral class” formed with element of size $0 \mapsto E(z) = 1$.

$Z$: “atomic class” formed with element of size $1 \mapsto E(z) = 1$.

$$\text{Exp}(g(z)) = \exp \left( \sum_{k \geq 1} \frac{1}{k} g(z^k) \right) ; \quad \widehat{\text{Exp}}(g(z)) = \exp \left( \sum_{k \geq 1} \frac{(-1)^k}{k} g(z^k) \right) ;$$

$$\text{Log}(g(z)) = \sum_{k \geq 1} \frac{\varphi(k)}{k} g(z^k) \text{ with } \varphi(k) = \text{Euler totient}.$$
Proofs. $A \mapsto A(z) = \sum A_n z^n = \sum_\alpha z^{|\alpha|}$.

— Union: $C = A + B; \sum_\gamma = \sum_\alpha + \sum_\beta$. 

$C(z) = A(z) + B(z)$

— Product: $C = A \times B; \sum_\gamma = \sum_\alpha \cdot \sum_\beta$. 

$C(z) = A(z) \cdot B(z)$

— Sequence: $C = \text{SEQ}(A)$ means $C = 1 + A + (A \times A) + \cdots$. 

$C(z) = \frac{1}{1 - A(z)}$

— Multiset: $C = \text{MSET}(A)$ means $C \cong \prod_\alpha (1 + \{\alpha\})$, so that

$$C(z) = \prod_\alpha \frac{1}{1 - z^{|\alpha|}} = \prod_{n \geq 1} \frac{1}{(1 - z^n) A_n} ,$$

and conclude by $C'(z) = \exp(\log C(z)) \ldots$ 

$C(z) = \text{Exp}(A(z))$

— Cycle: (omitted) $\varphi(k)$ is Euler’s totient function.
Example 1. Binary words

\[ W = \text{SEQ}(\{a, b\}) \implies W(z) = \frac{1}{1 - 2z}. \]

Get \( W_n = 2^n \) (!?). Words starting with \( b \) and < 4 consecutive \( a \)'s:

\[ W^* \cong \text{SEQ}(b \times (1+a+aa+aaa)) \implies W^*(z) = \frac{1}{1 - (z + z^2 + z^3 + z^4)}. \]

*Longest run statistics* lead to rational functions (Feller).

Example 2. Plane trees (“general” = all degrees allowed)
Example 3. Nonplane trees (all degrees allowed) $\mathcal{U} = \mathbb{Z} \times \text{MSET}(\mathcal{U})$. $U_1 = 1$, $U_2 = 1$, $U_3 = 2$, $U_4 = 5$.

$$U(z) = z \exp \left( \frac{1}{1} U(z) + \frac{1}{2} U(z^2) + \frac{1}{3} U(z^3) + \cdots \right).$$

Cayley: recurrences; Pólya: asymptotics of this infinite functional equation.

Exercise: computable in polynomial time ($O(n^2)$).
Example 4. Words containing a pattern \((abb)\)

\[ L_j := \text{language accepted from state } j. \]

\[ \{ L_0 = aL_1 + bL_0, L_1 = aL_1 + bL_2, L_2 = aL_1 + bL_3, \ldots \} \]

**Theorem.** Regular language (finite automaton) has rational GF.

\[ \text{Reg} \mapsto Q(z). \]

Patterns of all sorts in words. Applications in pattern matching algorithms and computational biology.

**Borges’ Theorem:** Large enough text contains any finite set of patterns w.h.p.
Example 5. Walks and excursions.

$\text{Walks in } \mathbb{Q}(\mathbb{Z}, \sqrt{1-4z^2})$

$\text{Excursion} = \text{Seq} \left( \uparrow \text{Excursion} \downarrow \right)$

Positive path $= \text{Excursion} \times \text{Seq} \left( \uparrow \text{Seq} \left( \uparrow \text{Excursion} \downarrow \right) \right)$

Draw game $= \text{Seq} \left( \downarrow \text{Excursion} \uparrow + \uparrow \text{Excursion} \downarrow \right)$ etc.
Exercise A. Integer compositions. Argue that \( C_n = 2^{n-1} \) since

\[
\mathcal{C} = \text{SEQ}(\mathcal{N}), \quad \mathcal{N} = \mathbb{Z} \times \text{SEQ}(\mathbb{Z}) \quad \implies \quad C(z) = \frac{1}{1 - \frac{z}{1-z}} = \frac{1 - z}{1 - 2z}.
\]

Exercise B. Denumerants. In how many ways can one give change for \( n \) cents, given coins of 1, 2, 5, 10c?

\[
D(z) = \frac{1}{(1 - z)(1 - z^2)(1 - z^5)(1 - z^{10})}.
\]

Exact form of coefficients? Asymptotics?

Exercise C. Unary binary trees. \( U = z(1 + U + U^2) \).

Exercise D. Binary trees, general plane trees, excursions, and polygonal triangulations are all enumerated by Catalan numbers \( C_n = \frac{1}{n+1} \binom{2n}{n} \). Why?
Simple families of plane trees.
Let \( \Omega \subseteq \mathbb{Z}_{\geq 0} \) be the set of allowed (out)degrees. Define
\[
\phi(y) := \sum_{w \in \Omega} y^w.
\]
Then the simple family \( \mathcal{Y} \) has OGF:
\[
Y(z) = z \phi(Y(z)).
\]
If \( \phi \) is finite, get an algebraic function.

Lagrange Inversion Theorem.
\[
[z^n] Y(z) = \frac{1}{n} \text{coeff}[w^n] \phi(w)^n.
\]
If \( \phi \) is finite, get multinomial sums.
2 LABELLED STRUCTURES AND EGFS

EGF = exponential generating function

\[(f_n) \quad \rightarrow \quad f(z) = \sum_{n \geq 0} f_n \frac{z^n}{n!}.\]

A labelled object has atoms that bear distinct integer labels (canonically numbered on \([1 \ldots n]\)).

Unlabelled: “anonymous atoms”. Labelled: distinguished atoms or colours.

**Example.** How many (undirected) graphs on \(n\) (distinguishable) vertices? \(G^n = 2^{n(n-1)/2}\).

Graphs: unlabelled problem is harder (Pólya theory). In general, can get unlabelled by identification of labelled.
PERMUTATIONS = typical labelled objects: write \( \sigma = \left( \begin{array}{c} 1 \\ 2 \\ \vdots \\ n \\ \sigma_1 \sigma_2 \cdots \sigma_n \end{array} \right) \)
as \( \sigma_1 \sigma_2 \cdots \sigma_n \) and view as linear digraph that is labelled:

EGF is \( \frac{1}{1-z} \) since \( P(z) = \sum_{n} n! \frac{z^n}{n!} \).
DISCONNECTED GRAPHS (labelled) = no edges aka “Urns”.

EGF is $U(z) = \exp(z) = e^z$.

CYCLIC GRAPHS (directed)

EGF $K(z) = \log \frac{1}{1 - z}$.
ROOTED TREES (graphs) nonplane and labelled

$T_n = ??$

$T_1 = 1, \quad T_2 = 2, \quad T_3 = 9, \quad (T_4 = 64...)$

$\Rightarrow$ Unlabelled:

$U_1 = 1, \quad U_2 = 1, \quad U_3 = 2, \quad U_4 = 4, ...$
Labelled product. Let $A$ and $B$ be labelled classes. Then the cartesian product $A \times B$ is not well-labelled (why?).

Given $(\beta, \gamma)$ form all possible relabellings that preserve the order structure within $\beta, \gamma$, while giving rise to well-labelled objects.

- **Labelled product of two objects.**

  $$(\alpha \star \beta) := \{ \gamma \mid \gamma = (\alpha', \beta') \},$$

  where $\gamma$ is well-labelled and $\alpha' \equiv_{\text{order}} \alpha$ and $\beta' \equiv_{\text{order}} \beta$.

- **Labelled product of two classes.**

  $$C := \bigcup_{\alpha \in A, \beta \in B} (\alpha \star \beta).$$
Example: \( U = \text{class of all disconnected graphs} \)

\[
\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1
\end{array}
\]

\[
U \ast U \ast \ldots \ast U = \underbrace{U \ast U \ast \ldots \ast U}_m \text{ times}
\]

\[\text{Allocation of } n \text{ elements into } m \text{ cells; functions from } [1..n] \to [1..m] \]

\[U = U \setminus \{\varepsilon\} \quad 0, 1, 1, 1, \ldots\]

\[U \ast U \ast \ldots \ast U = \text{allocation with no empty cell} \]

\[\text{surjections from } [1..n] \to [1..m] \]

GFs; Stirling numbers.
Sequences, Sets, Cycles

- \( \mathcal{E} \) (or 1): neutral class.
- \( \mathcal{Z} \): atomic class \( \equiv \{1\} \).

Define \( \text{SEQ}(\mathcal{A}), \text{SET}(\mathcal{A}), \text{CYC}(\mathcal{A}) \) by relabellings:

\[
\text{SEQ}(\mathcal{A}) = 1 + \mathcal{A} + (\mathcal{A} \star \mathcal{A}) + \cdots.
\]

Sets: quotient up to perms. Cyc: up to cyclic perms.

- Perms \( \mathcal{P} \cong \text{SEQ}(\mathcal{Z}) \)
- Urn \( \mathcal{U} \cong \text{SET}(\mathcal{Z}) \)
- Circulars graphs \( \mathcal{K} \cong \text{CYC}(\mathcal{Z}) \)
- \( m \)-functions: \( \mathcal{F}^m \cong \mathcal{U} \star \cdots \star \mathcal{U} \equiv \text{SEQ}_m(\mathcal{U}) \)
- \( m \)-surjections: \( \text{SEQ}(\mathcal{V}), \mathcal{V} = \text{SET}_{\geq 1}(\mathcal{Z}) \)
- Set partitions: \( \text{SET}(\text{SET}_{\geq 1}(\mathcal{Z})) \)
- Lab. trees: \( T = \mathcal{Z} \star \text{SET}(T) \).
**Theorem.** There exists a dictionary:

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$\mathcal{E}$ or 1: “neutral class” formed with element of size 0 $\leftrightarrow E(z) = 1$.

$\mathcal{Z}$: “atomic class” formed with element of size 1 $\leftrightarrow E(z) = 1$. 
Product lemma:

\[ C = A \times B \implies C(z) = A(z) \cdot B(z) \]

\[ C = (A \ast B) \text{ implies } C_n = \sum_{k=0}^{n} \binom{n}{k} A_k B_{n-k} \text{ (# possibilities \times # relabelings)} \]

Hence \[ \frac{C_n}{n!} = \sum_k \frac{A_k}{k!} \cdot \frac{B_{n-k}}{(n-k)!} \sim C(z) = A(z) \cdot B(z). \]

SEQ: \[ 1 + A + A^2 + \cdots = \frac{1}{1 - A}. \]

SET: \[ 1 + \frac{A}{1!} + \frac{A^2}{2!} + \cdots = \exp(A). \]

CYC: \[ 1 + \frac{A}{1} + \frac{A^2}{2} + \cdots = \log \frac{1}{1 - A}. \]
Example 0

- (labelled) linear digraphs

\[ \varepsilon, 1, 1, 2, \quad 1, 1, 2, \quad \varepsilon, 1, 2, 1, 1 \]

\[ P(n) = \text{Seq}(Z) \]

\[ P(n) = \frac{1}{1-z} \]

\[ P_n = n! \]

- (labelled) cycle digraphs

\[ 1, 1, 2, \ldots \]

\[ K(n) = \text{Cyc}(Z) \]

\[ K(n) = \frac{1}{1-z} \]

\[ K_n = (n-1)! \]

- (labelled) disconnected graphs

\[ 1, 1, 1, 1, 1 \]

\[ U = \text{Seq}(Z) \]

\[ U(n) = e^z \]

\[ U_n = 1 \]
Example 1. Permutations and cycles:

\[ P = \text{SET(CYC}(Z)) \quad \implies \quad P(z) = \exp \left( \log \frac{1}{1 - z} \right) = \frac{1}{1 - z}. \]

Derangements (no fixed point)

\[ D = \text{SET(CYC}(Z)\backslash Z) \quad \implies \quad D(z) = \exp \left( \log \frac{1}{1 - z} - z \right) \equiv \frac{e^{-z}}{1 - z}. \]

Thus \( \frac{D_n}{n!} = 1 - \frac{1}{1!} + \frac{2}{2!} - \cdots + \frac{(-1)^n}{n!} \sim e^{-1}. \)

Example 2. Labelled (Cayley) trees:

\[ T = Z \star \text{SET}(T) \quad \implies \quad T(z) = ze^{T(z)}. \]

Thus \( T_n = n^{n-1} \) by Lagrange Inversion Th.
Example 3. Set partitions:

\[ B = \text{SET}(\text{SET}_{\geq 1}(\mathbb{Z})) \implies B(z) = e^{ez} - 1. \]

Bell numbers:

\[ B_n = e^{-1} \sum_{k \geq 0} \frac{k^n}{k!}. \]
Example 4. Allocations to \([1 \ldots m]\):

— all: \(e^{mz} \sim F_n = m^n\).

— surjective: \((e^z-1)^m \sim\) Stirling numbers, \(m!\binom{m}{n} = \sum (\frac{m}{k})(-1)^{m-k}kn\).

— injective: \((1+z)^m \sim \binom{m}{n}n!\) (arrangement #).

Exercise: Birthday Problem and Coupon Collector.

\[
\mathbb{E}(B) = \int_0^\infty \left(1 + \frac{t}{m}\right)^m e^{-t} \, dt, \quad \mathbb{E}(C) = \int_0^\infty \left(e^t - (e^{t/m} - 1)^m\right) e^{-t} \, dt.
\]

Multiple birthdays, multiple collections. (Cf Poissonization.)

\[ T = z e^T, \quad K = \log(1-T)^{-1}, \quad M = e^K: \quad M_n = n^n. \quad P(\text{connected}) = O \left( \frac{1}{\sqrt{n}} \right). \]

Exercise: A binary functional graph is such that each \( x \) has either 0 or 2 preimages (cf \( x^2 + a \mod p \)). \textbf{Q1.} Construct; \textbf{Q2.} enumerate.

Exercise: All graphs \( G(z) = 1 + \sum_{n=1}^{\infty} 2^{n(n-1)/2} z^n / n! \). \textbf{Q1.} EGF \( K(z) \) of connected graphs? \textbf{Q2.} Probability of connectedness. \textbf{Q3*} Prove not constructible.
3 MULTIVARIATE GFS AND PARAMETERS

Bivariate GF (ordinary) \((E_{n,k})\) \(\leadsto\) \(E(z, u) = \sum_{n,k} E_{n,k} u^k z^n.\)

Bivariate GF (exponential) \((E_{n,k})\) \(\leadsto\) \(E(z, u) = \sum_{n,k} E_{n,k} u^k z^n / n!.\)

Figure 6.5 Distribution of trees, \(4 \leq N \leq 50\) (leaves scaled to \(N\)) (Eulerian numbers)
BGF encodes exact distributions. hence, moments.

\[ \mathbb{E}_{\chi} = \sum_{k} k \cdot \frac{E_{n,k}}{E_n} = \frac{1}{E_n} \text{coeff}[z^n] \frac{\partial}{\partial u} E(z, u) \bigg|_{u=1} \]

Variance & moment of order 2: second derivative, etc.

Chebyshev inequalities: \( \sigma_n / \mu_n \to 0 \) implies convergence in probability.
Bivariate GF (ordinary) \( E(z, u) = \sum_{n,k} E_{n,k} u^k z^n \equiv \sum_{\varepsilon \in E} z^{\mid \varepsilon \mid} u^{\chi(\varepsilon)}. \)

- BGF is reduction of combinatorial structure. Thus expect multivariate dictionaries.

**Definition.** Parameter is inherited if (i) it is compatible with unions; (ii) it is additive over products (also SEQ, SET, CYC).

**meta-THM** Previous dictionaries (U/L) work verbatim!

Proof (hint): \( C = A \times B \implies C(z, u) = \sum_{\gamma} = \sum_{(\alpha, \beta)} = A(z, u) \cdot B(z, u). \)

Same principles as counting, but with size now extended to \( \mathbb{N} \times \mathbb{N}. \)
Example 1. Permutations, counting $\#$ cycles:

$$P = \text{SET}(\text{CYC}(Z)) \implies P(z, u) = \exp \left[ \frac{uz}{1} + \frac{z^2}{2} + \cdots \right] = (1-z)^{-u}.$$  

Expand and get probability GF: $\frac{1}{n!} u(u+1) \cdots (u+n-1)$; mean is $H_n \sim \log n$; standard dev. is $\sim \sqrt{\log n}$; distribution is concentrated (by Chebyshev).

$\#$ singleton cycles:

$$P(z, u) = \exp \left[ \frac{uz}{1} + \frac{z^2}{2} + \cdots \right] = \frac{e^{z(u-1)}}{1-z}.$$  

$\#$ singleton/doubleton cycles (joint): use $u_1, u_2$, and so on.
Example 2. Number of summands in compositions.

\[ C = \text{SEG}(Z \times \text{SEG}(Z)) \implies C(z, u) = \frac{1}{1 - z u / (1 - z)}. \]

Example 3. Number of leaves in a general plane tree.

\[ G = Z u + Z \text{SEG}_{\geq 1}(Z) \implies G = z \boxed{u} + z \frac{G}{1 - G}. \]

Summary: Place marker at appropriate places and translate with usual dictionary.
Summary. In order to enumerate, it suffices to find a construction.
— Get the OGF/EGF automatically;
— Get parameters that are traceable to constructions.

Integer compositions and partitions; words; trees; lattice paths; set partitions; allocations and functions; mappings; permutations and cycles.

Also: associate families of special functions to families of combinatorial classes.
— Regular languages $\sim$ Rational functions
— Tree grammars & CF languages $\sim$ Algebraic functions
— Simple tree families $\sim$ Implicit functions

Other: Constrained mappings: implicit function $\circ$ modified exp and log functions. Etc.
**Exercise A.** A *record* in a permutation is an element $\sigma_j$ larger than all preceding $\sigma_k$. Explain why the distribution of # records is the same as # cycles (on $\mathcal{P}_n$).

**Hint:**

**Exercise B.** Throw $n$ balls into $m$ urns. 1. The statistics of empty bins is obtained from $(e^z - 1 + u)^m$. 2. Mean and variance? 3. Same for bins filled with $r$ elements. 4. Relation to Poisson?
SINGULAR COMBINATORICS

Complex Asymptotics

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— Asymptotic analysis is often very precise.
— Can be done from generating functions directly, even if no expression for coefficients is available.
— Works for functional equations

\[ U(z) = z \exp \left( U(z) + \frac{1}{2} U(z^2) + \cdots \right). \]

— Makes it possible to discuss universality via schemas.
4 ANALYTIC FUNCTIONS

GFs are (usually) analytic functions near 0.

- Analytic aka holomorphic functions
- Meromorphic functions
- Integrals and residues
- Singularities and exponential growth orders
Let \( f(z) \) be defined from \( D \) (open connected set) to \( E \):

**Definition.** • \( f(z) \) is analytic at \( z_0 \) iff locally:

\[
f(z) = \sum_{n \geq 0} c_n (z - z_0)^n.
\]

• \( f(z) \) is complex differentiable iff

\[
\exists \lim_{h \to 0, \ h \in \mathbb{C}} \frac{f(z_0 + h) - f(z_0)}{h} =: f'(z_0) \equiv \frac{d}{dz} f(z) \bigg|_{z=z_0}.
\]

\( \leadsto \) \( f \) analytic/ differentiable in \( \Omega \), etc.

**Theorem.** Equivalence between the two notions!

Combinatorialists love power series; analysts love differentiability!

\[
\frac{\Delta f}{\Delta z} \text{ gives closure under } +, -, \times, \div, \text{ composition, inversion, } \& \text{c.}
\]
**Examples.** The function $\sqrt{z}$, such that $\sqrt{\rho e^{i\theta}} = \sqrt{\rho} \cdot e^{i\theta/2}$, can only be made continuous in $\mathbb{C} \setminus [-1, 1]$. 

— Same for $\log z = \log \rho + i\theta$.

— Exponential function $\exp(z)$ is entire.

— $\frac{e^z}{\sqrt{1 - z}}$ is analytic in $\mathbb{C} \setminus [1, \infty)$.

— Catalan GF $\frac{1 - \sqrt{1 - 4z}}{2z}$ is analytic in slit plane $\mathbb{C} \setminus [\frac{1}{4}, +\infty[$.

— Rational GF is analytic except at poles.
Integration and residues

**Theorem.** Let $f$ be analytic in $\Omega$ and $\gamma$ be contractible to a single point in $\Omega$. Then

$$
\int_{\gamma} f(z) \, dz = 0.
$$

In particular $\int_{A}^{B} f(z) \, dz$ does not depend on path.
**Definition.** $g(z)$ is meromorphic in $\Omega$ iff near any $z_0$, one has $g(z) = \frac{A(z)}{B(z)}$, with $A, B$ analytic.

A point $z_0$ such that $B(z_0) = 0$ is a pole. Its order is the multiplicity of $z_0$ as root of $B$ (assume $A(z_0) \neq 0$).

Pole of order $m$: $g(z) = \frac{c_{-m}}{(z - z_0)^m} + \cdots + \frac{c_{-1}}{(z - z_0)} + c + 0 + \cdots$. $c_{-1}$ is called residue of $g(z)$ at $z_0$. 
Cauchy’s Residue Theorem. If \( f(z) \) has poles, then
\[
\frac{1}{2i\pi} \int_{\gamma} f(z) \, dz = \sum \text{Residues}.
\]

Proof: local integration +

Cauchy’s Coefficient Theorem. \[
\text{coeff}[z^n] f(z) = \frac{1}{2i\pi} \int_{\gamma} f(z) \frac{dz}{z^{n+1}}
\]

Proof: by residues:
Residues: local versus global

- Computing integrals: \( \int_{-\infty}^{+\infty} \frac{dx}{1+x^4} \) = \( \lim_{R \to \infty} \int_{-R}^{R} \frac{dx}{1+x^4} \) = \( \frac{\pi}{\sqrt{2}} \)

By only considering local properties at \( \zeta = e^{i\pi/4}, e^{3i\pi/4} \).

- Estimating coefficients: \( d_n := \mathbb{P}[\text{derangement}] \text{ over } \mathcal{P}_n \).

\[
d_n = [z^n] \frac{e^{-z}}{1-z} = \frac{1}{2i\pi} \int_{|z|=1/2} \frac{e^{-z}}{1-z} \frac{dz}{z^{n+1}}.
\]

Evaluate instead on \( |z| = 2 \):
\[
J_n = \frac{1}{2i\pi} \int_{|z|=2} \frac{e^{-z}}{1-z} \frac{dz}{z^{n+1}} = O(2^{-n})
\]
\[
= \text{Res}_{z=0} + \text{Res}_{z=1} = d_n - e^{-1}.
\]

Thus: \( d_n = e^{-1} + O(2^{-n}) \).

Exercise: Double derangement: \([z^n]e^{-z-z^2/2}/(1-z)\), Generalize!
Singularities.

- \( f(z) \) has a singularity at border point \( \sigma \) iff

**Theorem.** A series always has at least one singularity on its circle of convergence (but none inside).

Convergence radius \( \equiv \) Analyticity radius:

**Pringsheim’s Theorem.** If \( f_n \geq 0 \), one such singularity is positive.
Exponential growth of coefficients.

If \( f(z) \) has radius exactly \( R \), then \( \forall \epsilon > 0 \):

\[
    f_n(R - \epsilon)^n \to 0; \quad f_n(R + \epsilon)^n \text{ is unbounded.}
\]

That is \( \limsup |f_n|^{1/n} = \frac{1}{R} \), or

\[
    f_n = R^{-n} \vartheta(n), \quad \text{where } \vartheta(n) \text{ is “subexponential”}.
\]

Also write \( f_n \asymp R^{-n} \) with \( R := \text{distance to nearest sing(s)} \).

Find exponential growth by just “looking” at GF!!
**Examples** (singularities and growth)

- Binary words: \( W(z) = \frac{1}{1-2z} \sim W_n \propto 2^n \).

- Derangements: \( D(z) = \frac{e^{-z}}{1-z} \sim \frac{D_n}{n!} \propto 1^n \).

- General trees: \( G(z) = \frac{1}{2} \left( 1 - \sqrt{1 - 4z} \right) \sim G_n \propto 4^n \). By Stirling: \( G_n \sim \frac{4^{n-1}}{\sqrt{\pi n^3}} \).

- Unary-binary trees: \( U = z(1+U+U^2), U = \frac{1}{2z} \left( 1 - z - \sqrt{1 - 2z - 3z^2} \right) \), so that singularities are at \( z = -1, \frac{1}{3} \) and \( U_n \propto 3^n \).

Exponential order is computable(almost) automatically for GFs given by explicit expressions.

E.g.: \( \rho(f + g) = \min(\rho(f), \rho(g)); \rho\left( \frac{1}{1-f} \right) = \min(\rho(f), \{ z / f(z) = 1 \}) \), etc.
Find subexponential factors in

\[ f_n \propto R^{-n}, \quad \text{meaning} \quad f_n = R^{-n} \vartheta(n), \]

where \( \vartheta(n) \) is like \( n^\alpha, (\log n)^\beta, e^{\sqrt{n}}, \) etc.

Here: simple case of Rat & Mero.
**Coefficients of rational functions**

**Theorem.** Each pole $\zeta$ with multiplicity $r$ contributes to coefficients a term

$$\zeta^{-n} P(n),$$

where $P(n)$ is a polynomial of degree $r - 1$.

**Proof.** $[z^n] \frac{1}{(z - \zeta)^r} = (-\zeta)^{-r} \binom{n + r - 1}{r - 1} \zeta^{-m}$.

Poles are arranged in order of increasing modulus. Dominant ones matter for exponential growth. Multiplicities give polynomial factors.
Example 1. Denumerants.

- In how many ways can one give change with 1, 2, 5c coins?

\[ D_n = [z^n] \frac{1}{(1 - z)(1 - z^2)(1 - z^5)}. \]

One layer. Poles at 1, ±1, \( \zeta^5 = 1 \).

Observe the “transfer” \( D(z) \sim \frac{1}{10} (1 - z)^{-3} \) implies \( D_n \sim n^2/20 \).

- General case \( \Omega \)-denominations, \( m = \|\Omega\| \). Then (Schur)

\[ D_n \sim \frac{n^{m-1}}{(m - 1)!} \prod_{\omega \in \Omega} \frac{1}{\omega}. \]
Example 2. Longest $b$-runs in strings. (cf Feller)

\[
\begin{align*}
&\text{bbb abb ab a abbb} \\
&\text{SEQ}_{<m}(b) \times \text{SEQ}(a \text{ SEQ}_{<m}(b)) \\
&\frac{1 - z^m}{1 - z} \times \frac{1}{1 - z \frac{1-z^m}{1-z}} = \frac{1 - z}{1 - 2z + z^{m+1}}.
\end{align*}
\]

— Dominant pole is near $\frac{1}{2}$: $\rho_m \approx \frac{1}{2} (1 + 2^{-m-1})$.
— Dominant pole is separated by $|z| = \frac{3}{2}$; error is exp. small.
— Uniform estimates are obtained. Get

\[
P(\text{longest } b\text{-run } < m) \approx \left(\frac{1}{2\rho_m}\right)^n \approx e^{-n/2^{m+1}}.
\]

Threshold near $\log_2 n$.

Arbitrary patterns: similar with correlation polynomials of Guibas–Odlyzko. Quantitative normality of strings, Borges’ Theorem ,etc.
Coefficients of meromorphic functions

Assumption: $g(z)$ is meromorphic in $|z| < R$ and analytic on $|z| = R$.

**Theorem.** Each pole $\zeta$ with multiplicity $r$ contributes to coefficients a term

$$\zeta^{-n} P(n),$$

where $P(n)$ is a polynomial of degree $r-1$. Error term is $O(R^{-n})$.

**Proof.** (i) Subtracted singularities. Let $h(z)$ gather contributions of poles. Then $g(z) - h(z)$ is analytic in $|z| \leq R$. Use Cauchy with trivial bounds.

(ii) Estimate $\int g$ by residues.
Example 3. Derangements.

\[ D = \text{SET}(\text{CYC}_\geq 2(Z)) \implies D(z) = e^{-z}1 - z. \]

Get simple pole at \( z = 1 \) so that \( \frac{1}{n!}D_n = [z^n] \frac{e^{-1}}{1-z} + O(2^{-n}) = e^{-1} + O(2^{-n}). \)

Generalized derangement: all cycles of length \( > r \):

\[ \frac{1}{n!}D^*_n \sim e^{-H_r}, \quad H_r = 1 + \frac{1}{2} + \cdots + \frac{1}{r}. \]

Encapsulates finite automata and finite Markov chains. GFs are rational.

If the graph $\Gamma$ is strongly connected and aperiodic, then there is unicity and simplicity of dominant pole ($\ll$ Perron-Frobenius): $f_n \sim c\rho^{-n}$.

Generalized patterns in random strings (F, Nicodème, Régnier, Salvy, Szpankowski, Vallée, &c).
**Example 5.** Surjections and Supercritical \( \text{SEG} \) Schema.

Random surjection \( \equiv \) ordered partition (pref. arrangement)

\[
\mathcal{R} = \text{SEG}(\text{SET}_{\geq 1}(\mathcal{Z})) \quad \implies R(z) = \frac{1}{2 - e^z}.
\]

Pole at \( \zeta = \log 2 \); subdominant ones at \( \zeta = \log 2 \pm 2ik\pi \), etc.

\[
\frac{R_n}{n^!} \sim c(\log 2)^{-n}.
\]

Also, mean number of blocks via \( \frac{1}{1 - u(e^z - 1)} \) is \( O(n) \). There is concentration, etc.

Any supercritical sequence should similarly behave \( \sim \) schema.
6 SINGULARITY ANALYSIS

- Singularities more general than poles.
- Subexponential factors more general than polynomials:

\[ f_n \sim R^{-n} \vartheta(n), \]

with \( \vartheta(n) \) of the form \( n^\alpha (\log n)^\beta \).

Note: May assume singularity at 1 by scaling \([z^n] f(\lambda z) = \lambda^n [z^n] f(z)\).
Regular point

\[ f(z) \approx f(z_0) + f'(z_0)(z - z_0) \]

\[ \exp(z) \]

Singular point

\[ -\sqrt{1 - z} \]

\[ - (1 - z)^{3/2} \]

Coefficients: \[ n^{-3/2} \quad n^{-5/2} \]
From functions to coefficients:

\[
\begin{align*}
\frac{1}{(1 - z)^2} & \quad \rightarrow \quad n + 1 \quad \sim \quad n \\
\frac{1}{1 - z} \log \frac{1}{1 - z} & \quad \rightarrow \quad H_n \equiv \frac{1}{1} + \ldots + \frac{1}{n} \quad \sim \quad \log n \\
\frac{1}{1 - z} & \quad \rightarrow \quad 1 \quad \sim \quad 1 \\
\frac{1}{\sqrt{1 - z}} & \quad \rightarrow \quad \frac{1}{2^{2n}} \binom{2n}{n} \quad \sim \quad \frac{1}{\sqrt{\pi n}}
\end{align*}
\]

\[
\left\{
\begin{array}{ll}
\text{Location of sing’s} : & \text{Exponential factor} \quad \rho^{-n} \\
\text{Nature of sing’s} : & \text{“Polynomial” factor} \quad \vartheta(n)
\end{array}
\right.
\]
Principles of Singularity Analysis

Larger functions tend to have larger coefficients.

— Establish this for basic scales $(1 - z)^{-\alpha}$. Enrich with $\log$’s, $\log \log$’s, etc.

— Prove transfer theorems. If $f$ “resembles” $g$ via $O(\cdot)$, $o(\cdot)$, then $f_n$ resembles $g_n$. 
Theorem 1. Coefficients of basic scale:

\[ [z^n](1 - z)^{-\alpha} \sim \frac{1}{\Gamma(\alpha)} n^{\alpha - 1}. \]

Also: full expansion, log’s log-log’s, etc.

Gamma function: \( \Gamma(s) := \int_0^\infty e^{-t} t^{s-1} \, dt \), with analytic continuation by \( \Gamma(s + 1) = s\Gamma(s) \).

Idea:
**Theorem 1.** Basic scale translates:

\[
\sigma_{\alpha, \beta}(z) := (1 - z)^{-\alpha} \left( \frac{1}{z} \log \frac{1}{1-z} \right)^\beta \\
\Rightarrow [z^n] \sigma_{\alpha, \beta} \sim n^{\alpha-1} \frac{\log n}{\Gamma(\alpha)}.
\]

**Proof.** Cauchy’s coefficient integral, \( f(z) = (1 - z)^{-\alpha} \)

\[
[z^n] f(z) = \frac{1}{2i\pi} \int_\gamma f(z) \frac{dz}{z^{n+1}} \\
\downarrow \quad (z = 1 + \frac{t}{n}) \quad \downarrow \\
\frac{1}{2i\pi} \int_{\mathcal{H}} \left( -\frac{t}{n} \right)^{-\alpha} e^{-t} \frac{dt}{n} \\
n^{\alpha-1} \times \frac{1}{\Gamma(\alpha)}.
\]
**Theorem 2.** Transfer of asymptotic properties.

If

\[ f(z) = O((1-z)^{-\gamma}) \text{ as } z \to 1 \text{ in a Camembert region} \]

Then

\[ \text{Coef } [z^n] f(z) = O(n^{\delta-1}) \]

*Same for } \Theta(-) \text{ ; * Same for log's, etc.}

Proof: similarly by **Hankel contours**.
Example 1. 2–regular graphs.

\[ R = \text{Set (Unordered Cycle } (Z, \text{card } z \geq 3)) \]

\[ R(z) = \exp \left( \frac{1}{z} \log \frac{1}{1-z} - \frac{z}{2} - \frac{z^2}{4} \right) \]

\[ R(z) = \frac{e^{-\frac{z^2}{2} - \frac{z^2}{4}}}{\sqrt{1-z}} \]

By singularity analysis,

\[ R(z) \sim \frac{e^{-3/4}}{\sqrt{1-z}} \]

\[ \frac{R_n}{n!} \sim e^{-3/4} \sqrt{\pi n}. \]

Comtet’s clouds. Also full asymptotics.
**Example 2.** Some trees.

- **Catalan trees** have GF $\frac{1}{2} (1 - \sqrt{1 - 4z}) \sim c \frac{4^n}{\sqrt{\pi n^3}}$.

- **Unary binary trees.**

\[
\begin{align*}
T &= z + zT + zT^2 \\
\Rightarrow T &= \frac{1 - 2z - \sqrt{1 - 2z - 3z^2}}{2z} \\
1 - 2z - 3z^2 &= (1 - 3z)(1 + z) \\
\Rightarrow \sqrt{-}\text{singularity } \& \frac{1}{3},
\end{align*}
\]

\[T_n \sim c \cdot 3^n n^{-3/2}\]

In fact: *universality of* $n^{-3/2}$ *law* (later).
Example 3. Cycles in Perms.

Mean number of cycles in a random perm is $\text{coeff}[z^n]$ in

$$M(z) = \frac{\partial}{\partial u} \exp \left( u \log \frac{1}{1 - z} \right) \bigg|_{u \to 1} = \frac{1}{1 - z} \log \frac{1}{1 - z}.$$ 

Thus $[z^n]M(z) \sim \log n$.

Exercise: Holds for perms with finitely many excluded cycle lengths.

In fact: universality for the “exp-log” schema.
Closures

**Theorem 3.** Generalized polylogarithms

\[ \text{Li}_{\alpha,k} := \sum (\log n)^k n^{-\alpha} z^n \]

are of S.A.-type.

**Theorem 4.** Functions of S.A.-type are closed under integration and differentiation.

**Theorem 5.** Functions of S.A.-type are closed under Hadamard product

\[ f(z) \odot g(z) := \sum_n (f_n g_n) z^n. \]

(F) (Fill-F-Kapur 2005).
Solving a “Tauberian” problem

<table>
<thead>
<tr>
<th>Real-Tauberian</th>
<th>Darboux-Pólya</th>
<th>Singularity An.</th>
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</thead>
<tbody>
<tr>
<td>(large $\Rightarrow$ large)</td>
<td>(smooth $\Rightarrow$ small)</td>
<td>(Full mappings)</td>
</tr>
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</table>

+ Singularity analysis preserves uniformity $\sim$ distributions.
Focus on recursive structures including trees, mappings.

- Universality of $\sqrt{n}$–law for generating functions;
- Universality of $\rho^{-n} n^{-3/2}$–law for counts;
- Universal behaviour for major parameters (e.g., height).
Inversion:

Inversion Theorem: $\phi$ is analytically invertible iff $\phi'(y_0) \neq 0$.

If not invertible $\phi''(y_0) \neq 0$.

$y = \sqrt{z}$

Square-root singularity is expected for inverse functions.
**Theorem 1.** Let \( \phi \) have nonnegative coefffs and be entire. Then the function that solves

\[ Y(z) = z \phi(Y(z)) \]

has a square-root singularity, so that

\[ [z^n]Y(z) \sim C \rho^{-n} n^{-3/2} \].

— Characteristic equation (singular value of \( Y \)) is \( \tau : \frac{d}{dy} \frac{y}{\phi(y)} = 0 \), i.e., \( \tau \phi'(\tau) - \phi(\tau) = 0 \). Then \( \rho = \frac{\tau}{\phi(\tau)} \). All is computable.

— \( \sqrt{-} \)-singularity propagates via suitable compositions, so that parameters can be analysed.

— Phenomena are robust.
**Example 1.** Cayley trees. $T = ze^T$ or $z = Te^{-T}$ is not invertible if
\[ \frac{d}{dT}(Te^{-T}) \equiv (1 - T)e^{-T} = 0, \]
that is, $T = 1, z = e^{-1}$. Find:

\[ T(z) \bigg|_{z \to e^{-1}} = 1 - \sqrt{2}\sqrt{1 - ez} + O((1 - ez)). \]

Implies $[z^n]T(z) \sim \frac{e^n}{\sqrt{2\pi n^3}}$; we redervive Stirling’s f. (since $T_n = n^{n-1}$ by Lagrange).

**Example 2.** Unlabelled trees. Recall

\[ U(z) = ze^{U(z) + \frac{1}{2} U(z^2) + \cdots}. \]

Express as $T$ composed with an analytic function and get SQRT sing: $U = \zeta e^U$, where $\zeta := z \exp(\frac{1}{2} U(z^2) + \cdots)$.

*Height is universally $O(\sqrt{n})$ wth local and integral limit laws (of theta type). Similarly for width (Marckert et al.). Leaves are universally normally distributed, etc.*
**Example 3.** Mappings (cyclic points).

Develop a theory of degree-constrained mappings: (Arney-Bender), (F.-Odlyzko).
Algebraic functions

Singularity analysis applies to any algebraic function

\[ f(z) = \sum_{j=0}^{\infty} c_j (z - z_0)^j \]

\( z_0 \) is a singularity.

**Newton-Puiseux Theorem**

Around any point \( z_0 \), \( f(z) \) admits a fractional power expansion.

\[ f(z) = \sum_{j=0}^{\infty} c_j (z - z_0)^{\alpha_j} \quad \alpha = \frac{p}{q} \in \mathbb{Q} \]

Algebraic function \( \iff \) Fractional exponents @ singularities.
Define an "algebraic" element to be of the form
\[ \omega^n \sum_{j=1}^{n} d_j \omega_j^m \beta \] (\beta \in \mathbb{D})

**Theorem:** If \( y(z) \) is an algebraic function, then there exists a finite collection of algebraic elements \( A_1 (a_1 \omega_1), A_2 (a_2 \omega_2), \ldots, A_n (a_n \omega_n) \) s.t.
\[
y_n = A_1 + \ldots + A_n + O(\xi^{-n})
\]
\[ |w_1| = |w_2| = \ldots = |w_n| = \rho \; \quad \xi > \rho.
\]
Singularity analysis applies to

- Trees with a finite number of node degrees
  (we know already √-singularity, \( p^{-n^{1/2}} \))

- Excursions defined by a discrete set of steps \( \Omega \)
  that is finite [Banderier, Ky 2003]

- **MAPS** = graphs embedded into the plane

  → Gimenez, Noy: counting of planar graphs by gen. function + complex analysis.
Singularity analysis applies to many non-linear ordinary differential equations, especially of order 1.

- Models of "logarithmic trees" : increasing trees, binary search trees, m-ary search, ...

- The whole class of linear ordinary differential equations with so-called "regular singularities" [generic case].

- The holonomic frame work = functions such that coefficients of the linear ODE are in $\mathbb{C}(z)$. 
• **“Holonomic” functions.** Defined as solutions of linear ODE’s with coeffs in \( \mathbb{C}(z) \) [Zeilberger] \( \equiv D \)-finite.

\[
\mathcal{L}[f(z)] = 0, \quad \mathcal{L} \in \mathbb{C}(z)[\partial_z].
\]

• Stanley, Zeilberger, Gessel: Young tableaux and permutation statistics; regular graphs, constrained matrices, etc.

**Fuchsian case (or “regular” singularity) \((Z^\beta \log^k Z)\):**

\[
[z^n] f(z) \approx \sum \omega^n n^\beta (\log n)^k, \quad \omega, \beta \in \overline{\mathbb{Q}}, \quad k \in \mathbb{Z}_{\geq 0}.
\]

S.A. applies automatically to classical classification.

Asymptotics of coeff is decidable
— general case: modulo oracle for connection problem;
— strictly positive case: “usually” OKay.
**Example 6.** *Quadtrees—Partial Match* [FGPR'92]

Divide-and-conquer recurrence with coeff. in $Q(n)$

Fuchsian equation of order $d$ (dimension) for GF

$$Q_n^{(d=2)} \prec n^{(\sqrt{17}-3)/2}.$$

E.g., $d = 2$: Hypergeom $\, _2F_1$ with algebraic arguments.

Extended by Hwang et al. Cf also Hwang’s *Cauchy ODE cases.*
Panholzer-Prodinger+Martinez, . . .
8 SADDLE POINT METHODS

- For functions with violent growth at singularities, including entire functions.

$$[z^n] f(z) = \frac{1}{2i\pi} \oint f(z) \frac{dz}{z^{n+1}}.$$  

Integer partitions, set partitions, involutions, ...
SINGULAR COMBINATORICS

C. Random Structures

Philippe Flajolet, INRIA, Rocquencourt
http://algo.inria.fr/flajolet

Large random combinatorial structures exhibit are (often) predictable!

Concentration? Limit law?

Relation to Bivariate GFs $C(z, u)$ and singularities?
Why is the binomial distribution asymptotically normal?

- **De Moivre**: approximation of \( \frac{1}{2^n} \binom{n}{k} \).

- **Laplace/Gauss**: as sum of many RV’s + Lévy: . . . : because of characteristic functions \( \rightarrow e^{-t^2/2} \).

- **Analytic combinatorics**: because of bivariate GF \( \frac{1}{1-z(1+u)} \) and smoothly varying singularity!
Classical Central Limit Theorem (CLT): \( \sum \) RV’s to Normal.
Proof: Levy’s continuity theorem \( \phi_n(t) \to \phi(t) \) implies \( F_n(x) \to F(x) \).
+ calculation of PGF \( f_n(u) = g(u)^n \) + normalization and \( u \to it \).

Quasi-Powers Theorem (HK Hwang, circa 1995).
Assume \( (X_n) \) are RV’s with probability GF (PGF) \( f_n(u) = E(u^{X_n}) \) and
for \( A(u), B(u) \) analytic at 1:

\[
f_n(u) = A(u)B(u)^{\beta_n} \left( 1 + O\left( \frac{1}{\kappa_n} \right) \right),
\]

for \( u \approx 1 \), with \( \beta_n, \kappa_n \to \infty \), and \( \text{Var}(B(u)) > 0 \). Then

- mean: \( \mu_n = E(X_n) \sim \beta_n B'(1) \); s-dev.: \( \sigma_n^2 \sim \beta_n \text{Var}(B) \).
- normal limit: \( \Pr(X_n \leq \mu_n + x\sigma_n) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-w^2/2} dw \)
- Speed of convergence is \( O(\kappa_n^{-1} + \beta_n^{-1/2}) \).
Quasi-Powers Theorem: “If you resemble a power, then your limit law is normal”.

**Proof.** “Analytic expansions are differentiable”: this gives moments. Limit law results from Lévy’s continuity theorem. Speed results from Berry-Esseen. 
$\leq$Bender, Richmond$^+$. 
Example 1. Supercritical sequence schema.

Let $F = \text{SEQ}(G)$, so that number of components has BGF

$$F(z, u) = \frac{1}{1 - uG'(z)}.$$ 

Assume that $G(r) > 1$ where $r$ := radius of conv. of $G(z)$.

**Theorem.** The number of $G$–components in a random $F$–structure is asymptotically normal.

**Proof.**

A. Let $\rho \in (0, r)$ be such that $G(\rho) = 1$. This is r.o.c. of $F(z) \equiv F(z, 1)$. There is a simple pole.

B. Equation $1 - uG(z) = 0$ has root $\rho(u)$, where $\rho(u)$ depends analytically on $u$ for $u \approx 1$.

C. Function $F(z, u)$, with $u$ parameter, has simple pole at $\rho(u)$ and

$$[z^n]F(z, u) \sim c(u)\rho(u)^{-n}.$$

D. **Uniformity** is granted (by integral representations), so that Quasi-Powers Theorem applies.

QED
Example 1. Supercritical sequences (continued)

— Compositions: arbitrary; with \( \Omega \)–excluded or \( \Omega \)–forced summands. Compositions into prime summands, \( G(z) = z^2 + z^3 + z^5 + \cdots \). Same for twin primes (!!).

— Surjections aka ordered set partitions, \( G(z) = e^z - 1 \). Same with \( \Omega \)–constraints.

— \( k \)–components in compositions, surjections, etc.
Example 2. Cycles in permutations.

\[ F(z, u) = \exp \left( u \log \frac{1}{1 - z} \right) = (1 - z)^{-u}. \]

A. By singularity analysis, get main approximation: \( \left[ z^n \right] F(z, u) \sim \frac{n^{u-1}}{\Gamma(u)} \).

B. Approximation is uniform by nature of singularity analysis process (contour integration).

C. Rewrite as Quasi-Powers approximation:

\[ \left[ z^n \right] F(z, u) \sim \frac{1}{\Gamma(u)} \cdot \left( e^{(u-1)} \right)^{\log n}. \]

Thus, scale is now \( \beta_n \sim \log n \).

D. Quasi-Powers Theorem applies. QED
**Example 3.** Exp-Log schema.

Let $\mathcal{F} = \text{SET}(G)$, so that number of components has BGF

$$F(z, u) = e^{uG(z)}.$$

Assume that $G(z)$ is logarithmic: $G(z) \sim \lambda \log \frac{1}{1-z/\rho}$.

**Theorem.** The number of $G$–components in a random $\mathcal{F}$–structure is asymptotically normal, with logarithmic mean and variance.

Application: Random mappings, etc. $\Rightarrow$ Arratia-Barbour-Tavaré.
Example 4. Polynomials over finite fields.

> Factor\((x^7 + x + 1) \mod 29;\)
\[\begin{align*}
3^2 \quad 2^2 \\
(x^2 + x + 3)(x^2 + 25x + 25)(x^2 + 3x + 14)
\end{align*}\]

- Polynomial is a \textit{Sequence} of coeffs: \(\mathcal{P}\) has Polar singularity.
- By unique factorization, \(\mathcal{P}\) is also \textit{Multiset of Irreducibles}:
  \(\mathcal{I}\) has log singularity.

\[\implies\] Prime Number Theorem for Polynomials \(I_n \sim \frac{q^n}{n}\).

- Marking number of \(\mathcal{I}\)-factors is approx \(u\)th power:
  \[P(z, u) \approx \left(e^{I(z)}\right)^u.\]

\textit{Variable Exponent} \(\implies\) \textit{Normality} of \# of irred. factors.
(cf Erdős-Kac for integers.)

— Useful for analysis of \textit{polynomial factorization algorithms}. 

---

10
For a large collection of combinatorial classes
& parameters, we have a functional equation

$$\Phi(z, y, u) = 0$$

In the counting case ($u = 1$) get a singular expansion

$$y(z, 1) = \ldots (1 - z/p)^\alpha + \ldots$$

A perturbation of $u$ near 1 will often induce
a smooth perturbation of the expansion of $y(z, u)$, e.g.,

- Moveable singularity
  $$y(z, u) = \ldots (1 - z/p(1))^\alpha + \ldots$$
- Moveable exponent
  $$y(z, u) = \ldots (1 - z/p)^\alpha(u) + \ldots$$

With $f(u)$ or $\alpha(u)$ analytic at 1

$$\Rightarrow$$ Asymptotic normality

by singularity analysis + Quasi-Powers
Perturbation of rational functions
\( \approx \) Extends CLT for finite Markov chains.
Perturbation of algebraic functions: for irreducible systems, the Drmota-Lalley-Woods Theorem implies $\sqrt{-}\text{singularity}$.

**Example 5.** Non-crossing graphs (Noy-F.)

\[
\begin{align*}
G^3 + (2z^2 - 3z - 2)G^2 + (3z + 1)G &= 0 \\
G^3 + (2u^3z^2 - 3u^2z + u - 3)G^2 + (3u^2 - 2u + 3)G + u - 1 &= 0
\end{align*}
\]

Movable singularity scheme applies: Normality.

+ Patterns in context-free languages, in combinatorial tree models, in functional graphs: Drmota’s version of Drmota-Lalley-Woods.
Perturbation of differential equations.

**Example 6.** Profile of Quadtrees.

\[
F(z, u) = 1 + 2^3 u \int_0^z \frac{dx_1}{x_1(1 - x_1)} \int_0^{x_1} \frac{dx_2}{1 - x_2} \int_0^{x_2} F(x_3, u) \frac{dx_3}{1 - x_3}.
\]

Solution is of the form \((1 - z)^{-\alpha(u)}\) for algebraic branch \(\alpha(u)\); Variable Exponent \(\implies\) Normality of search costs.

Applies to many linear differential models that behave like *cycles-in-perms*.
Combinatorial world

Symbolic methods

Univariate GF

Bivariate GFs

Singularities

Asymptotic counts

+ perturbation \( u \approx 1 \)

Limit laws

That’s All, Folks!