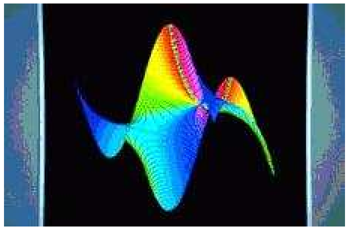

On the Comparison of Numbers:

An excursion into non-classical corners of computational mathematics

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Based on joint work with Brigitte Vallée, Caen

A very simple problem

Let X, Y be uniformly distributed over $[0, 1]$.

Cost of deciding $X \lessgtr Y$?

A very simple solution

If X, Y are given in some base, e.g., $B = 2$

— scan digits;
— stop when discrepant digits/bits are found.

Expected cost (number of comparisons) is $E[C] = 2$.
Probability distribution is **geometric**: $P(C = k) = 2^{-k}$.

THE END!?

What if X and Y are given as fractions?

$$X = \frac{a}{b}, \qquad Y = \frac{c}{d} \quad ?$$

E.g. Compare:

$$\frac{36}{113} \lessgtr \frac{113}{355}; \qquad \frac{97.8}{307.1} \lessgtr \frac{307.1}{964.9} \quad ?$$

Still mathematically trivial $\mathbf{sign}(ad - bc)$ ...

- Requires double precision
- Unsafe in fixed precision (floats)

Efficiency in symbolic manipulation systems and number-theory packages: $O(N^2)$ boolean complexity.

Robustness: Problems with floats.

Independent solutions:

- HAKMEM (Gosper 1972)
- Knuth's Metafont (late 1970s);
- computational geometry $\equiv$ orientation (Boissonnat, et al.)

Introduction.

HAKMEM 1972

Item 101A (Gosper): Numerical comparison of continued fractions is slightly harder than in decimal, but much easier than with rationals -- just invert the decision as to which is larger whenever the first discrepant terms are even-numbered. Contrast this with the problem of comparing the rationals 113/36 and 355/113.

$$\alpha = \frac{113}{36} = 3 + \frac{1}{7 + \frac{1}{\mathbf{5}}}, \qquad \beta = \frac{355}{113} = 3 + \frac{1}{7 + \frac{1}{\mathbf{16}}},$$

= two "lazy" parallel executions of the continued fraction algorithm.

Simulations (10^6): HAKMEM $\boxed{X \lessgtr Y}$ for $X, Y \in [0, 1]$, uniform.

	Basic CF-sign	Centered CF-sign
$P(L = 1)$	0.710050	0.918003
$P(L = 2)$	0.241275	0.075710
$P(L = 3)$	0.038339	0.000422
$P(L = 4)$	0.008424	0.000035
$P(L = 5)$	0.001608	0.000005
$E(L)$	**1.351612**	**1.088791**

Decay roughly geometrically with k: 5^{-k} for BCF; 13^{-k} for CCF.

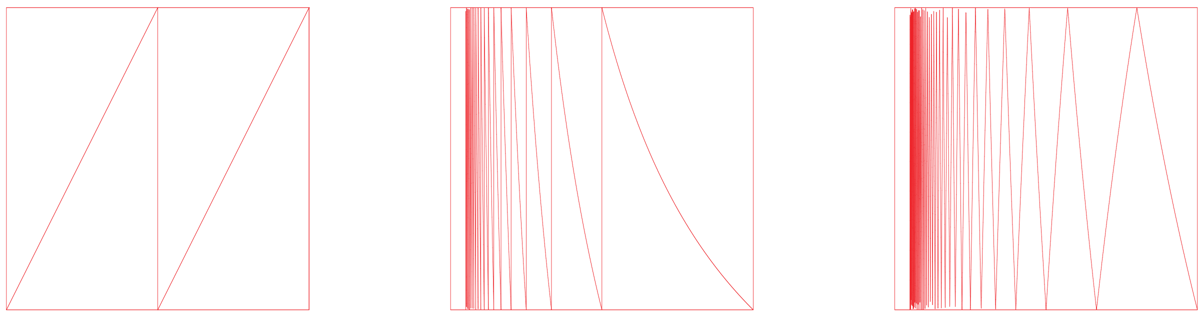
The expectations seem to be constants:
Anticipate $E(L) = 1.352 \pm 0.001$ and $E(\widetilde{L}) = 1.089 \pm 0.001$.

1 Expanding maps & continued fractions

A piecewise monotonic function $U(x)$ (shift) from some interval $\mathcal{I}$ that is expanding: $|U'(x)| > 1$.

Branches are indexed by $\mathcal{M}$ = digits.

A coding of a real x: $m(x),\ m(U(x)),\ m(U^2(x)),\ \ldots$



procedure $(\mathcal{I}, U, m)$-**expansion**$(x : \mathcal{I})$
  for $k := 1$ **to** $+\infty$ **do**
    $m_k := m(x); x := U(x);$

Set $\mathcal{H}_1$ of branches of $U^{(-1)}$. Run backwards:

$$x_0 = h_{\mathbf{m}}(x_k) \quad \text{where} \quad h_{\mathbf{m}}(y) = h_{m_1} \circ h_{m_2} \circ \cdots \circ h_{m_k}(y).$$

The set $\mathcal{H}$ of all compositions $h_{\mathbf{m}}$ is set of *inverse branches*.

Equivalence principle.

— the stochastic behaviour of a numbering system

— the dynamics of $U(x)$ on the interval $\mathcal{I}$

— the dynamics of the semigroup of contractions $\mathcal{H}$.

Basic continued fraction expansions.

$$I = [0, 1], \quad U(x) = \left\{ \frac{1}{x} \right\}, \quad m(x) = \left\lfloor \frac{1}{x} \right\rfloor .$$

Inverse branches of depth 1: $\mathcal{H}_1 := \{h(z) = \dfrac{1}{m+z} \;\; m \geq 1\}$.

$$x_0 = h_{\mathbf{m}}(x_k) \qquad \text{where} \quad h(y) = \cfrac{1}{m_1 + \cfrac{1}{m_2 + \cfrac{1}{\ddots_{\,m_k + y}}}} .$$

Centered continued fraction expansions.

$$\widetilde{\mathcal{H}}_1 := \left\{ h(z) = \frac{1}{m+z} \;\; m \geq 2 \right\} \bigcup \left\{ h(z) = \frac{1}{m-z} \;\; m \geq 3 \right\} .$$

Sign algorithms.

Take two random points; follow their trajectories under the shift U. *Stop on discrepancy.*

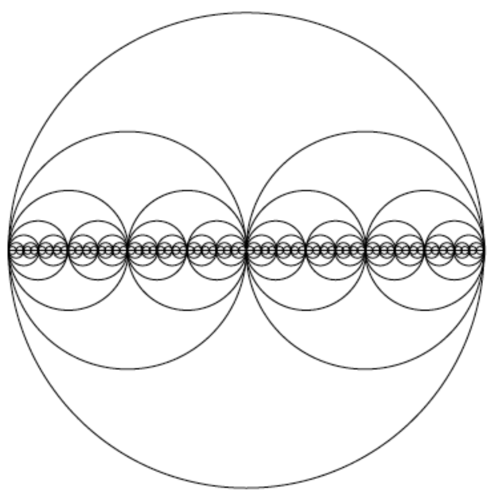
> **procedure** $(\mathcal{I}, U, m)$-**sign**$(x, x' \ : \ \mathcal{I})$
>
> **for** $k := 1$ **to** $+\infty$ **do**
>
> $m_k := m(x)$; $m'_k := m(x)$; $x := U(x)$; $x' = U(x')$;
>
> **if** $m_k \neq m'_k$ **then exit**.

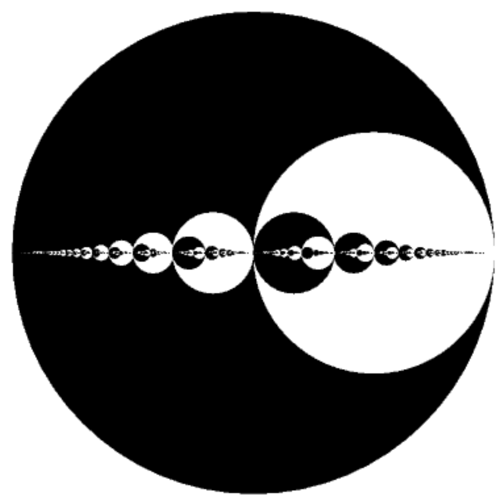
2 Fundamental intervals & signs

Uniform probability model (Lebesgue) on $\mathcal{I}$.

The probability that $x \in \mathcal{I}$ belongs to the fundamental interval determined by the branch h is, with $|I| = 1$,

$$u_h = |h(0) - h(1)|.$$

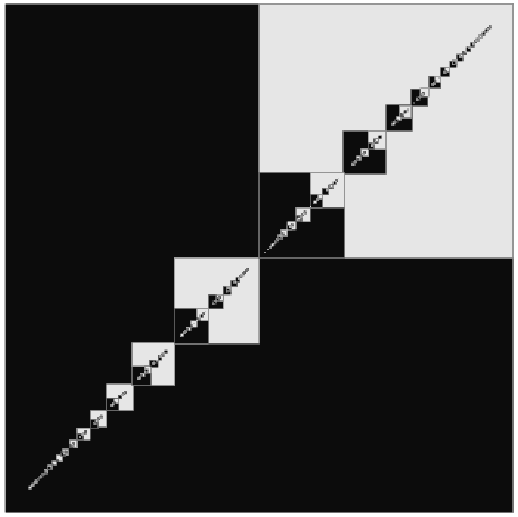




The square $\mathcal{I} \times \mathcal{I}$.

$$\text{Event } [L \geq k+1] = \bigcup_{|h|=k} h(\mathcal{I}) \times h(\mathcal{I}). \tag{1}$$

$$P([L \geq k+1]) = \sum_{|h|=k} u_h^2 = \sum_{|h|=k} |h(0) - h(1)|^2. \tag{2}$$



The average number of iterations is obtained by summation:

$$E[L] = \sum_h u_h^2 = \sum_h |h(0) - h(1)|^2. \tag{3}$$

More generally, we define the "moment sums"

$$\rho^{\langle \ell \rangle} := \sum_h u_h^\ell = \sum_h |h(0) - h(1)|^\ell$$

which are central in the n–sorting algorithm ($\leadsto$later).

Continued fraction sign-algorithms.

Theorem 1. *Expected costs of BCF/CCF-sign are* **lattice sums**,

$$E(L) = \rho^{\langle 2\rangle}, \qquad E(\widetilde{L}) = \widetilde{\rho}^{\langle 2\rangle},$$

where, the "moments" of index ℓ *are*

$$\rho^{\langle\ell\rangle} = 1 + \frac{1}{2^\ell} + \frac{2}{\zeta(2\ell)} \sum_{(d,c)\in\mathcal{C}(1,2)} \frac{1}{c^\ell d^\ell}$$

$$\widetilde{\rho}^{\langle\ell\rangle} = \frac{2^\ell}{\zeta(2\ell)} \sum_{(d,c)\in\mathcal{C}(\phi,\phi^2)} \frac{1}{c^\ell d^\ell}, \qquad (\phi = (1+\sqrt{5})/2),$$

w/ lattice sum : $\mathcal{C}(\beta,\gamma) := \{(m,n) \in \mathbf{N}^2 \mid m\beta < n < m\gamma\}.$

Proof (elementary!):

— The set $\mathcal{H}$ of all possible LFT's used by the BCF–algorithm is

$$\mathcal{H} := \left\{ \frac{az+b}{cz+d} \ \big| (a,b,c,d) \in \mathbf{N}^4, \ |ad-bc| = 1, \ c \le d, \ a \le c, \ b \le d \right\}.$$

= unimodular h + relatively prime coeffs. + inequalities.

— The set $\widetilde{\mathcal{H}}$ of all possible LFT's used by the CCF–algorithm is

$$\widetilde{\mathcal{H}} := \left\{ \frac{az+b}{cz+d} \ \big| (b,d) \in \mathbf{N}^2, \ (a,c) \in \mathbf{Z}^2, \ ac \ge 0, |ad-bc| = 1, \ \frac{-d}{\phi^2} < c < \frac{d}{\phi}, \ |a| \le \frac{|c|}{2}, b \le \frac{d}{2} \right\}.$$

3 The sorting algorithm

Given $X := \{x_1, x_2, \ldots, x_n\}$ with $x_j \in \mathcal{I}$. Build ***digital tree***, trie (X):

(R_1) If $X = \{x\}$ then trie (X) is single *leaf node* $\boxed{x}$.

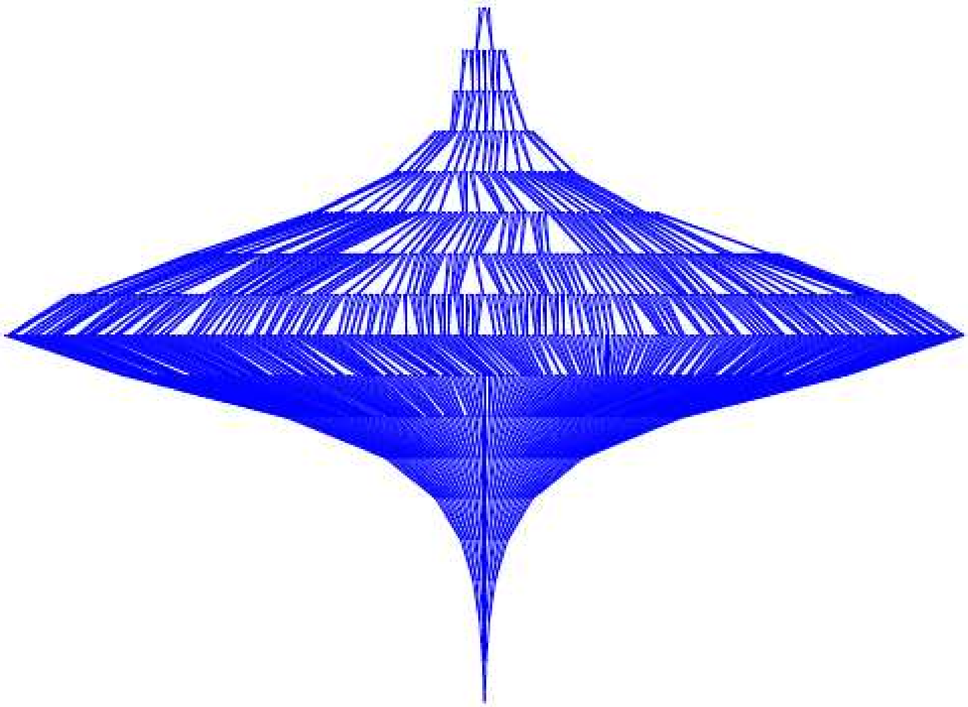
(R_2) If $\mathrm{card}(X) \geq 2$, then trie (X) is built recursively

$$\text{trie}\,(X) = \langle o, \text{trie}\,(X_1), \text{trie}\,(X_2), \ldots, \text{trie}\,(X_r)\rangle,$$

where $X_i = \{U(x) \mid m(x) = b_i,\ x \in X\}$.

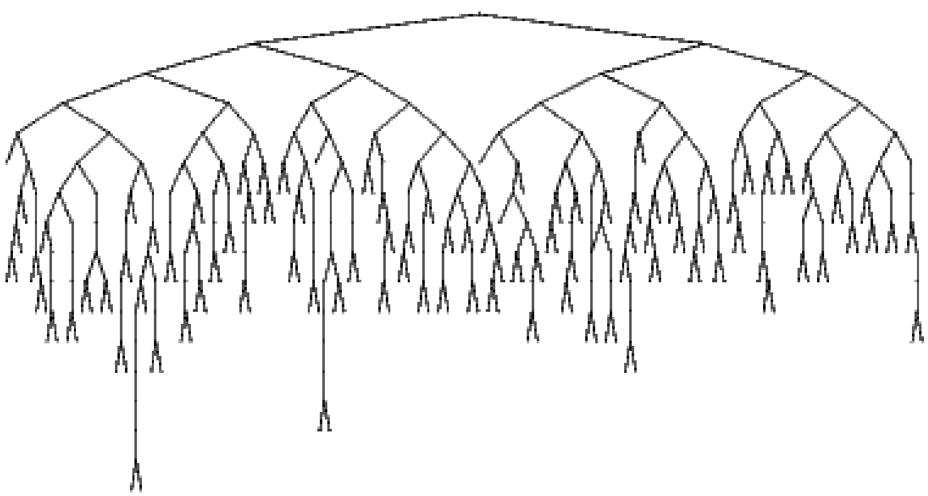
Follow n random trajectories and stop each when it “deviates”.

The binary trie



is another saga . . .

$\phi - 1$	=	/**1, 1, 1**, *1, 1, 1,. . ./*
γ	=	/**1, 1, 2**, *1, 2, 1,. . ./*
$\exp(1) - 2$	=	/**1, 2, 1**, *1, 4, 1,. . ./*
$\log 2$	=	/**1, 2, 3**, *1, 6, 3,. . ./*
$\{\exp(\pi\sqrt{163})\}$	=	/**1, 1333462407511**, *1, 8, 1, 1, . . ./*
$2^{1/3} - 1$	=	/**3**, *1, 5, 1, 1, , 4,. . ./*
$\pi - 3$	=	/**7**, *15, 1, 292, 1, 1,. . ./*



Typical shape??

Theorem 2. *The expectation of the number of digit inspections*

$$P(n) = n \sum_{\ell=2}^{n} (-1)^{\ell} \binom{n-1}{\ell-1} \rho^{\langle \ell \rangle},$$

where $\rho^{\langle \ell \rangle}$ *is a moment sum.* *[Similarly for CCF.]*

= **finite differences** of moment sums!

$P(2) = 2\,\rho^{(2)}, \quad P(3) = 6\,\rho^{(2)} - 3\,\rho^{(3)}, \quad P(4) = 12\,\rho^{(2)} - 12\,\rho^{(3)} + 4\,\rho^{(4)}.$

Proof: (i) Poissonize and get independence;

(ii) de-Poissonize "algebraically".

Cf. Clément-Fl-Vallée, *Algorithmica* 2001.

4 Multiple zeta values [Euler–Zagier-Borwein^2]

With $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$:

$$\sum_{n=1}^{\infty} \frac{(H_{n-1})^4}{n^3} = \frac{185}{8}\zeta(7) - \frac{43}{2}\zeta(3)\zeta(4) + 5\zeta(2)\zeta(5)$$

$$\zeta^{+}(s) \equiv \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad \zeta^{-}(s) = \sum_{n=1}^{\infty} \frac{(-1)}{n^s} = (2^{1-s} - 1)\zeta(s),$$

$$\zeta^{-+}(s,t)) = \sum_{n=1}^{\infty} \sum_{q=1}^{n-1} \frac{(-1)^n}{n^s q^t} = \sum_{n=1}^{\infty} (-1)^n \frac{H_{n-1}^{(t)}}{n^s}.$$

A complete evaluation means a reduction to a polynomial form in zeta values $\zeta(2), \zeta(3), \ldots, \overline{\zeta}(1) = -\log 2$, and possibly other constants.

———- (Zagier 1994), (Borwein*), (Fl-Salvy, *Exp. Math.* 1998) . . .

$$\mathrm{Li}_m(z) = \frac{z}{1^m} + \frac{z^2}{2^m} + \frac{z^3}{3^m} + \cdots = \sum_{n=1}^{\infty} \frac{z^n}{n^m}. \qquad (4)$$

Theorem 3. *The mean number* $\rho^{\langle 2 \rangle}$ *of comparisons in* BCF–*Sign can be expressed in terms of double zeta values as*

$$\rho^{\langle 2 \rangle} = \frac{17}{4} + \frac{360}{\pi^4} \sum_{d=1}^{\infty} \frac{(-1)^d}{d^2} \sum_{c=1}^{d} \frac{1}{c^2} = \frac{15}{2} - \frac{720}{\pi^4} \sum_{d=1}^{\infty} \frac{(-1)^d}{d^3} \sum_{c=1}^{d} \frac{1}{c},$$

or with $\zeta(3)$ *and the tetralogarithm* $\mathrm{Li}_4(\frac{1}{2})$,

$$\rho^{\langle 2 \rangle} = -\frac{60}{\pi^4} \left(24\mathrm{Li}_4(\frac{1}{2}) - \pi^2 (\log 2)^2 + 21\zeta(3) \log 2 + (\log 2)^4 \right) + 17. \qquad (5)$$

Thus, $\rho^{\langle 2 \rangle}$ *is in class* **P** = *polynomial time computable:*

$\rho^{\langle 2 \rangle} = 1.35113\,15744\,91659\,00179\,38680\,05256\,46466\,84404\,78970\,85087 \pm 10^{-50}$.

Moment sums for BCF

♠ Original sum are **very slowly convergent**.

♡ Following Zeilberger, define $f(z)$ to be *holonomic* if its satisfies a linear differential equation with coefficients in $\mathbb{Q}(z)$.
A constant that is the value $f(z_0)$ of a holonomic function at an algebraic point z_0 is called a *holonomic constant*.
Such holonomic constants are polynomial-time computable (**P**).

♡ Usual series-acceleration (Cohen-Villegas-Zagier; Pari, Maple) works just fine in practice!

BCF–Sorting: Get $P(n)$ to 16D for $n = 0, \ldots, 200$ via $\rho^{(\ell)}$ to 200D.

What about Centred Continued Fractions (CCF)?

$$\widetilde{\rho}^{\langle 2\rangle} = \frac{2^2}{\zeta(4)} \sum_{c\varphi<d<c\varphi^2} \frac{1}{c^2 d^2}, \qquad (\varphi = (1+\sqrt{5})/2)$$

Quiz:

— Can you compute to 100D : $\displaystyle\sum_{n\geq 1} \frac{1}{\lfloor n\phi \rfloor^2}$?

(= 1.29106 03681 14387 48950 47876 . . .)

— Do you believe this? (Borwein^2 *AMM* 1992)

$$\sum_{n=1}^{\infty} \frac{\lfloor n e^{\pi\sqrt{163}} \rfloor}{2^n} \doteq 1280640 \qquad \text{with error} < 10^{500,000,000}.$$

Theorem 4. *The expectation of the cost of the CCF-sign algorithm is in* **P***:*

$$\widetilde{\rho}^{(2)} = 1.08922\,14740\,95380\ldots\,.$$

Two ingredients:

Lemma 1 (Mahler, B2, FV). *Let $\theta < 1$ be an irrational number with convergents $\{p_n/q_n\}_{n=0}^{\infty}$. The generating function of the lattice cone $\mathcal{C}(0,\theta)$ is given by*

$$\sum_{(m,n)\in\mathcal{C}(0,\theta)} x^m y^n = \sum_{k=0}^{\infty} (-1)^k \frac{x^{q_k+q_{k+1}} y^{p_k+p_{k+1}}}{(1-x^{q_k}y^{p_k})(1-x^{q_{k+1}}y^{p_k+1})}.$$

Proof: Cut a lattice by a Q–slanted line; do inclusion-exclusion.

+ **Mellin trick**: for seq. (a_n) and $A(z) := sum_n z^{a_n}$, we have

$$\sum_n \frac{1}{(a_n)^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} A(e^{-t}) t^{s-1}\, dt = \frac{1}{\Gamma(s)} \int_0^{\infty} A(w) \left(\log \frac{1}{w}\right)^{s-1} \frac{dt}{t}.$$

5 Transfer operators and sign algorithms

If X is a random variable with density $f(x)$, then $U(X)$ has density

$$g(x) = \mathcal{G}[f](x), \quad \text{where} \quad \mathcal{G}[f](x) := \sum_{h \in U^{(-1)}} |h'(x)| f \circ h(x).$$

This is the density transformer aka PF–operator:

The Ruelle **transfer operator** is the generalization:

$$\mathcal{G}_s[f](x) := \sum_{h \in U^{(-1)}} |h'(x)|^s f \circ h(x).$$

Get the dynamics of U from analysis of $\mathcal{G}_s$?

For general sign/sorting algorithms, Vallée introduces secant generalization. . .

An aside: Euclid's algorithm (1-dim.)

$$\mathcal{G}_s[f](x) := \sum_{m \geq 0} \frac{1}{(m+x)^{2s}} f\left(\frac{1}{m+x}\right).$$

Theorem (Hensley–Baladi–Vallée) *Euclid's algorithm is Gaussian!!*

Basic continued fractions (BCF) & sign algorithm

The transfer operator is

$$\mathcal{G}_s[f](x) := \sum_{m \geq 0} \frac{1}{(m+x)^{2s}} f\left(\frac{1}{m+x}\right).$$

• Gauß conjectured in 1800 the stationary distribution of CFs as $\gamma(x) \propto \frac{1}{1+x}$. Certainly $\gamma(x)$ is eigenvalue of $\mathcal{G}_1$, but more was needed (Lévy, Kuzmin, Wirsing).

• Around 1975, Babenko proposed numerical estimates of subdominant eigenvalues of $\mathcal{G}_1$ to be found in (Knuth 1981)

• In 1994+, FV needed dominant eigenvalues of $\mathcal{G}_2$ for sign algorithm and on the way suspected some spurious values; extensive tests led to conjecture first 37 eigenvalues of $\mathcal{G}_1$ to 25D with "almost certainty".

• In 2005, Loïc Lhote (Caen) could provide certified algorithms for Wirsing's constant $\lambda_2(1)$, Vallée's constant $\lambda_1(2)$, and Hensley's constant $\lambda''(1)$.

Theorem 5. *The probability that the sign algorithm performs* $k+1$ *iterations satisfies*

$$P(L \geq k+1) = C\lambda_1(2)^k + \textit{exp. small},$$

where $\lambda_1(2) \doteq 0.19945\,88183\,43767\ldots$.

Proof. (i) For continued fraction need $\sum u_h^2$, where the quantity u_h is $|h(0) - h(1)| = h'(1)^2$. Here can express everything with $\mathcal{G}_2^k$ at depth k.
(ii) Need numerical acess to Spec$(\mathcal{G}_2)$?

Ideas: (a) Use projection on space of polynomials ($\rightsquigarrow$ below)
(b) Expect spectrum of $\mathcal{G}_2$ to be exponentially decreasing, simple, with sign alternations. Filter based on conjecture.
(c) Use a technique of "test functions" based on positivity to certify dominant eigenvalue.
(d) Use trace formulae of Mayer–Roepstorff for Trace$(\mathcal{G}_2^2)$ to check filtering; evaluation based on Fl-Vardi's "zeta trick".

Projection on polynomials

Fact: Transform of x^m is a Hurwitz zeta whose Taylor coefficients involve zeta values:

We wind up inverting matrices of zeta values $\otimes$ binomials, e.g.

$$\begin{pmatrix} \zeta(4) & \zeta(5) & \zeta(6) & \zeta(7) \\ -4\zeta(5) & -5\zeta(6) & -6\zeta(7) & -7\zeta(8) \\ 10\zeta(6) & 15\zeta(7) & 21\zeta(8) & 28\zeta(9) \\ -20\zeta(7) & -35\zeta(8) & -56\zeta(9) & -84\zeta(10) \end{pmatrix}.$$

This passes all consistency tests and it works (Lhote 2005)!

6 Sorting and RH!?

The problem is analytically well posed

$$P(n) = n \sum_{\ell=1}^{n-1} (-1)^{\ell-1} \binom{n-1}{\ell} \rho^{(\ell+1)},$$

$$\rho^{(s)} = 2^{-s} - 2^{1-s} + \left(2^{s-1} - 1\right) \frac{\zeta(s)^2}{\zeta(2\,s)} + 2^s \frac{\zeta^{-+}(s)}{\zeta(2\,s)},$$

$$\zeta^{-+}(s) =:= \sum_{m<n}^{\infty} \frac{(-1)^n}{m^s n^s},$$

but there are huge cancellations.

Theorem 6. *The expected cost of sorting* n *uniform real numbers given by their basic continued fraction representations is*

$$P(n) = K_0 n \log n + K_1 n + Q(n) + K_2 + o(1),$$

where K_0 *is Lévy's entropic constant,* K_1 *is Porter-like*

$$K_0 = \frac{6 \log 2}{\pi^2}, \qquad K_1 = 18 \frac{\gamma \log 2}{\pi^2} + 9 \frac{(\log 2)^2}{\pi^2} - 72 \frac{\log 2 \zeta'(2)}{\pi^4} - \frac{1}{2}.$$

The function $Q(u)$ *is an oscillating function with mean value* 0*:*

$$Q(n) = O(u^{\delta/2}),$$

where δ *is any number such that*

$$\delta > \sup \left\{ \Re(s) \;\middle|\; \zeta(s) = 0 \right\}.$$

The Nörlund–Rice trick: If (f_k) admits an analytic lifting $\phi(s)$:

$$\sum_{k=1}^{n} \binom{n}{k} f_k = \frac{n!}{2i\pi} \int_L \phi(s) \frac{ds}{s(s-1)\cdots(s-n)}.$$

Singularities of $\phi(s)$ matter!

Here, we get Riemann Hypothesis into the game...
(but it is also exponentially offset).