  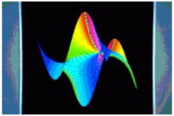  

Some Exactly Solvable Models of Urn Process Theory

Philippe Flajolet

Based on joint work with *Philippe Dumas and Vincent Puyhaubert*

MATHINFO'06, Nancy, September 15, 2006

Urn Models (1)

- An **urn** contains **balls** of m possible colours 
- A **fixed set of rules** governs the urn evolution:

$$\textit{Drawn}: \quad \begin{cases} \text{Red} \\ \text{Blue} \end{cases} \qquad \overbrace{\begin{array}{cc} \text{Red} & \text{Blue} \\ \hline \alpha & \beta \\ \gamma & \delta \end{array}}^{\textit{Place}}$$

Convention: The ball "drawn" is *not* withdrawn (not taken out)!

Urn Models (2): Examples

Balanced urns:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad : \qquad \alpha + \beta = \gamma + \delta =: \sigma.$$

Weight (size) of the urn is deterministic and equals $s_0 + n\sigma$ at time n.

- $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$: Selfish urn (Pólya); spread of epidemics/genes.

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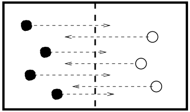
- $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$: Ehrenfest's two chambers model.



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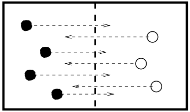
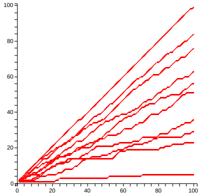
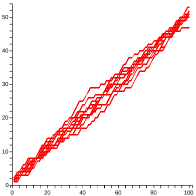
- $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$: Ehrenfest's two chambers model. 
- $\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$: Coupon collector: have N different items $\rightsquigarrow$blue and pick up at random with rule blue $\to$ red.

Urns exhibit different types of probabilistic behaviour

$\#$ balls of first type as a function of time (n)





$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Spread $+\mathcal{U}$niform — Concentrated $+\ \mathcal{N}$ormal

For coupon collector, scale is $N \log N$, etc.

Histories

Definition

Call history an unambigous description of an urn's evolution.

Write **x** for red balls and **y** for blue balls. Eg, with $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$:

$\underline{\mathbf{x}}\mathbf{x} \to \mathbf{xx}\underline{\mathbf{y}}\mathbf{x} \to \mathbf{xx}\underline{\mathbf{x}}\mathbf{xxx} \to \cdots$.

- Initial conditions: a_0 **x**-balls; b_0 **y**-balls.
- Set $s_0 := a_0 + b_0$. Have size @ time n: $s_n := s_0 + n\sigma$.

Number of histories is $H_n := s_0(s_0 + \sigma) \cdots (s_0 + (n-1)\sigma)$**.**

$$H_n = n!\sigma^n \cdot \binom{n + s_0/\sigma - 1}{n}.$$

Proposition

For balanced urns at time n, Probability $\Leftrightarrow$ Combinatorics: Histories are equiprobable.

Generating functions

- Operate with **exponential generating function (EGF)**

$$\boxed{H(z) = \sum_{n\geq 0} H_n \frac{z^n}{n!} = \frac{1}{(1-\sigma z)^{s_0/\sigma}},} \qquad s_0 = a_0 + b_0.$$

- **Want:** $H_{n,k,\ell} := \#$ histories with k (resp. ℓ) balls of type **x** (resp. **y**) in the end.

$$H(x,y;z) := \sum_{n,k,\ell} H_{n,k,\ell} x^k y^\ell \frac{z^n}{n!}.$$

$$\mathbb{P}(A_n = k, B_n = \ell) = \frac{H_{n,k,\ell}}{H_n}.$$

Note. For balanced urns, index ℓ and variable y are redundant (but convenient): consider $H_{n,k}$ and $H(x,1;z)$.

The Fundamental Isomorphism (1)

Given urn $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, define *associated differential system*:

$$\Sigma \ : \qquad \begin{cases} \dot{x} & = & x^{\alpha+1}y^{\beta} \\ \dot{y} & = & x^{\gamma}y^{\delta+1} \end{cases}, \quad \begin{cases} x(0) & = & x_0 \\ y(0) & = & y_0 \end{cases}.$$

Notations: t is the independent variable. $\dot{x}$ means $\frac{d}{dt}x(t)$.

Ex: Fried. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rightsquigarrow \begin{cases} \dot{x} = xy \\ \dot{y} = xy \end{cases}$; Ehrenf. $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \rightsquigarrow \begin{cases} \dot{x} = y \\ \dot{y} = x \end{cases}$

2–3 tree: $\begin{pmatrix} -2 & 3 \\ 4 & -3 \end{pmatrix} \rightsquigarrow \begin{cases} \dot{x} = xy^3 \\ \dot{y} = x^4y^2 \end{cases}$

The Fundamental Isomorphism (2)

Assoc. system: $\begin{cases} \dot{x} = x^{\alpha+1}y^{\beta} \\ \dot{y} = x^{\gamma}y^{\delta+1} \end{cases}, \qquad \begin{cases} x(0) = x_0 \\ y(0) = y_0 \end{cases}.$

Theorem (Fundamental Isomorphism)

Solutions X, Y to associated systems determines GF of histories:

$$H(x_0, y_0; z) = X\left(z \mid x_0\right)^{a_0} Y\left(z \mid y_0\right)^{b_0}.$$

It suffices to solve differential system with "floating" initial conditions.

The Fundamental Isomorphism (3): Proof

Θ. *Solutions X, Y to associated systems determines GF of histories:*
$H(x_0, y_0; z) = X(z)^{a_0} Y(z)^{b_0}$.

Represent k balls **x** and ℓ balls **y** by monomial $x^k y^\ell$.

— Differentiation $\partial_x \equiv \frac{d}{dx}$ mean *pick up and erase*":
$\partial[xxx] = \cancel{x}xx + x\cancel{x}x + xx\cancel{x}$, so that $x\partial[xxx] = \underline{x}xx + x\underline{x}x + xx\underline{x}$

— One step transition of the urn is described by PD operator
$\boxed{\mathfrak{D} = x^{\alpha+1} y^{\beta} \partial_x + x^{\gamma} y^{\delta+1} \partial_y}$. Thus $\mathfrak{D}^n x^{a_0} y^{b_0}$ enumerates n–histories.

— Differentiation w.r.t. t on solution monomial $X(t)^{a_0} Y(t)^{b_0}$ mimicks
$\mathfrak{D}$: $\frac{d}{dt} X^{a_0} Y^{b_0} = \mathfrak{D}\left[x^{a_0} y^{b_0}\right]_{(x,y)\mapsto(X,Y)}$. Then by Taylor's formula:

$$\Longrightarrow \qquad H(X(t), Y(t); z) = X(t+z)^{a_0} Y(t+z)^{b_0}. \qquad \text{QED!}$$

Special 2–dim. Models

- Pólya $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \leadsto \begin{cases} \dot{x} = x^2 \\ \dot{y} = y^2 \end{cases}. \implies$ separation, etc.

$$H(x_0, y_0; z) = \frac{x_0^{a_0} y_0^{b_0}}{(1 - x_0 z)^{a_0} (1 - y_0 z)^{b_0}} \implies \mathbb{P}(A_n = a, B_n = b) = \frac{\binom{a-1}{a_0-1}\binom{b-1}{b_0-1}}{\binom{a+b-1}{a_0+b_0-1}}.$$

- Ehrenfest $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \leadsto \begin{cases} \dot{x} = y \\ \dot{y} = x \end{cases} \implies \ddot{x} = x \leadsto$ hyperbolic fns:

$$H(x_0, y_0, z) = (x_0 \cosh z + y_0 \sin z)^N.$$

$\neq$ Kac. Generalizes to 3^+ chambers . . .

- Get explicitly all 10 models of dimension two with entries in $\{0, \pm 1\}$.

Analysis of 2–dim Urns

Proposition (First integral)

Let $p := \gamma - \alpha \equiv \beta - \delta$ *(balanced urn). Then*

$$X^p - Y^p = \text{Constant}\,.$$

Proof:

$$\frac{d}{dt}(x^p - y^p) = px^{p-1}\dot{x} - py^{p-1}\dot{y} \quad \underset{\Sigma}{\leadsto} \quad \mathbf{0}.$$

E.g., Ehrenfest: $X^2 - Y^2 = 1$ is satisfied by $\cosh, \sinh$.

Sacrificial Urns

Colour is called *sacrificial* if its diagonal entry is < 0.
— Semisacrificial urns: $\alpha < 0$, $\delta > 0$.
— Fully sacrificial urns: $\alpha < 0$, $\delta < 0$.

Theorem

For all sacrificial urns, the GF of urn histories $H(x, 1; z)$ is expressible from a fundamental hypergeometric function by inversion:

$$J_{\lambda,r}(u) := \int_0^u \frac{d\zeta}{(1+\zeta^r)^\lambda}.$$

There: $r = \frac{\alpha-\beta}{\alpha} \in \mathbb{Z}_{>0}$ *and* $\lambda = \frac{\beta}{\beta-\alpha} \in \mathbb{Q}_{>0}$.

Alternatively: view as *Abelian integral* over *Fermat curve* $Y^p - X^p = 1$, as well aa special hypergeometric function.

Proof: Standardize diff. system and use *first integral* to eliminate y.

The explicit form

With: $\boxed{r = \frac{\alpha-\beta}{\alpha},\ \lambda = \frac{\beta}{\beta-\alpha},\ s = \frac{\delta-\gamma}{\delta}}$.

- The fundamental integral J:

$$J(u) := \int_0^u \frac{d\zeta}{(1+\zeta^r)^\lambda}.$$

- The base functions S, C:

$$S := \mathrm{Inverse}[J]; \qquad C := (1+S^r)^{1/s},$$

- The GF of urn histories with $\Delta := (1-x^p)^{1/p}$:

$$\boxed{H(x,1,z) = \Delta^{s_0} S\left(-\alpha z \Delta^\sigma + J(x^{-\alpha}\Delta^\alpha)\right)^{-\frac{s_0}{\alpha}} C\left(-\alpha z \Delta^\sigma + J(x^{-\alpha}\Delta^\alpha)\right)^{-\frac{s_0}{\delta}}}$$

Examples

- Ehrenfest urn has $J(u) = \int_0^u \frac{d\zeta}{\sqrt{1+\zeta^2}} = \text{arcsinh}(u)$, etc.
- Algebraic urn: $\begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix}$ has $H = \frac{(1-x^2)^{1/2}}{(1-(z(1-x^2)+x)^2)^{1/2}}$.
- Generation-parity model: grow an increasing binary tree ($\cong$ BST):

$$\begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} \quad : \qquad S = \text{Inverse}\left[\int_0^u \frac{d\zeta}{(1+\zeta^3)^{2/3}}\right],$$

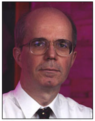
and get elliptic functions.

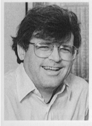
Probabilistic Properties

Theorem

For any semi-sacrificial urn, at time n:
— Urn composition is asymptotically Gaussian with speed $O(n^{-\epsilon})$.
— Extreme large deviation, $A_n = 0$, is exponentially small with Gamma value rate.
—Large deviation principle holds with rate a transform of J.
— All moments admit hypergeometric form.

Proof techniques inspired from F.-Gabarro-Pekari [Annals Prob. 2005] based on *Analytic Combinatorics* [FS07?], esp. singularity analysis [FlOd90].


Andrew M. Odlyzko to speak on "Cybersecurity, mathematics and limits on technology."



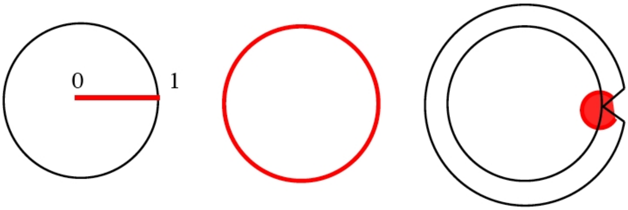
Proof and principles of analytic combinatorics

- Singularity analysis: locate singularities; expand locally; transfer to coefficents according to dictionary

$$[z^n](1-z/\rho)^{-\alpha} = \frac{n^{\alpha-1}}{\Gamma(\alpha)}\rho^{-n} + \cdots .$$

Needs *analytic continuation* in Δ–domain (“camembert”).

$\mathbb{R}$eal–Tauberian	Darboux-Pólya	Singularity An.



Extreme Large Deviations: $H(0,1,z) \equiv S(-\alpha z)^{-\frac{s_0}{\sigma}} \Longrightarrow \mathbb{P}(E.L.D.) \approx K^n$.

Singularity analysis preserves uniformity

- Analyse multivariate GF via singularity perturbation. E.g., movable singularity yields

$$[z^n]H(x,1,z) \approx \rho(x)^{-n},$$

- Use approximation of PGF of random variable to estimate of Limit Law of Gaussian type.

$\rightsquigarrow$ Quasi-Powers Theorem [Hwang]: *Analytically* like sum of RVs $\Longrightarrow\mathcal{N}$**ormal**, with speed $\prec$ Berry–Esseen.

- Adapt for Large Deviations combining with Cramér-type techniques [Hwang]. Singularities of S, C known from inversion + differential eqns.

Fully sacrificial urns

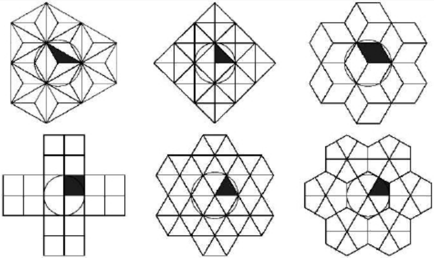
Theorem (FlGaPe05)

For any fully sacrificial urn, at time n:
Limit law + Speed + Large deviation + Extreme + Moments.

Proof via a PDE + method of characteristics + conformal mappings.

Theorem (FlGaPe05)

There are **six** *models solvable by elliptic functions assoc. tilings.*



Includes the fringe analysis of 2–3 trees [Yao, Aldous, **Panholzer-Prodinger**]

Triangular Urns (1)

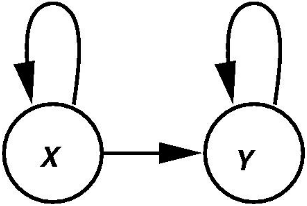
Classification:

— Special, e.g., Pólya $\equiv$ diagonal.

— Sacrificial, Semi or fully, $\rightsquigarrow$ Gaussian urns.

— Nonsacrificial: to be completed (algebra works fine!)

— Triangular: next!



$$\mathcal{M} = \begin{pmatrix} \alpha & \sigma - \alpha \\ 0 & \sigma \end{pmatrix}.$$

See [Janson06], even for nonbalanced.

Triangular Urns (2)

- Fundamental isomorphism first!

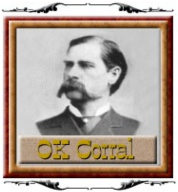
$$\begin{cases} \dot{x} = x^{\alpha+1}y^{\sigma} \\ \dot{y} = y^{\sigma+1} \end{cases} \Longrightarrow H(x,1,z) = \frac{x^{a_0}}{(1-\sigma z)^{-\frac{b_0}{\sigma}}} \left(1 - x^{\alpha}\left(1 - (1-\sigma z)^{\frac{\alpha}{\sigma}}\right)\right)^{-\frac{a_0}{\alpha}}.$$

- Singularity analysis strikes again!! (Hankel contours)

  Composition of singularities. Combinatorially: $\mathcal{H} = \mathcal{F} \circ \mathcal{G}$ implies $H(x,z) = F(xG(z))$. If F, G have singularities of algebraic type with critical composition, then get **stable laws**. [BaFlScSo01]

Universality: cores in maps, triangular urn models, forests-trees-mappings, + Gnedin-Pitman.

Intermezzo: OK Corral



Two gangs of m and n gunwomen face each other. They shoot at random. Which group survives? How many survivors?

— Williams–McIlroy, Kingman–Volkov

Urn model is “non-standard”:

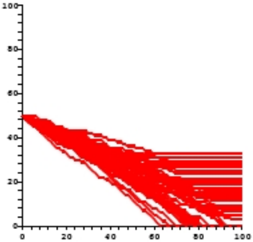
$$\mathcal{M} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

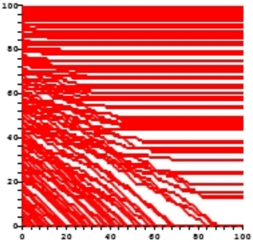
Use **time reversal** [KiVo] to reduce to Friedman's urn $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

$$\begin{cases} \dot{x} = xy \\ \dot{y} = xy \end{cases} \implies H(x,y,z) = \left(\frac{x(x-y)}{x - ye^{z(x-y)}} \right)^{a_0} \left(\frac{x(y-x)}{y - xe^{z(y-x)}} \right)^{b_0}.$$

OK Corral (2)

Fair / Unfair





Theorem

$\mathbb{P}(\#\ survivors{=}s) = \frac{s!}{(m+n)!} \sum_{k=1}^{m} (-1)^{m-k} \binom{k-1}{s-1} \binom{m+n}{n+k} k^{m+n-s}.$

$\mathbb{P}(Group\ I\ wins) = \frac{1}{(m+n)!} \sum_{k=1}^{n} (-1)^{m-k} \binom{m+n}{n+k} k^{m+n}.$

Unfair fights: $\frac{1}{n} \log \mathbb{P}_I(\alpha n, n) = W(\alpha) + o(1).$

Mean number of survivors (fair fight) is $\approx n^{3/4}$. Connections with Eulerian numbers, rises in perms. Asymptotics of moments, speeds, etc.

Higher dimensions

- The Fundamental Isomorphism theorem still holds. Get 3×3 nonlinear system.
- $\exists$ no First Integrals: [Jouanolou] $\dot{x} = y^2, \dot{y} = u^2, \dot{u} = x^2$.
- But many "natural" systems with structure can be solved.

Pólya, Friedman, & Pelican

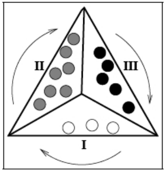
Solvable cases for $m = 3$ and beyond

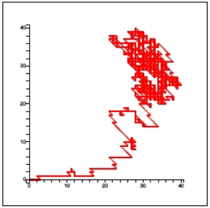
— Pólya's autistic model: diagonal matrix $\Longrightarrow \dot{x} = x^2$, etc.
— Generalized coupon collector: solved to get known solution.
— Pelican's urn: $\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$.

$$\begin{cases} \dot{x} = yu, & \dot{y} = ux, & \dot{u} = xy, \\ \frac{d}{dt}\operatorname{sn} t = \operatorname{cn} t \operatorname{dn} t, & \frac{d}{dt}\operatorname{cn} t = -\operatorname{dn} t \operatorname{sn} t, & \frac{d}{dt}\operatorname{dn} t = -k^2 \operatorname{sn} t \operatorname{cn} t, \end{cases}$$

where $\displaystyle\int_0^{\operatorname{sn} t} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = t$ is **Jacobian elliptic function**.
Also get hyperelliptic for dimension $m \geq 4$.

Ehrenfest $m \times m$

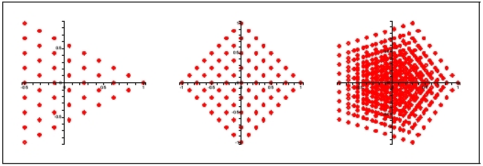
Cyclic Chambers: 



System: $\begin{cases} \dot{x} = y \\ \dot{y} = u \\ \dot{u} = x \end{cases}.$

Thus, $X(t)$ satisfies $\frac{d^3}{dt^3} X(t) = X(t)$. This is a trisection of the exponential. GF with everybody back to base is $\left(\sum_{n \equiv 0 \bmod 3} \frac{z^n}{n!} \right)^N$.

Read off spectrum of associated Markov matrices as

pseudo-lattices: 

Triangular 3×3

Triangular urns $m \times m$ are integrable.
Work out distribution for $m = 3$. Characteristic function is a double integral. Etc.

Cf also: Janson, Pouyanne [= moments approach]

Conclusions

From Eilbeck *et al.*:

"Integrable systems have a rich mathematical structure. [...]
These systems form an archipelago of solvable models in a sea of unknown,
and can be used to investigate properties of 'nearby' non-integrable systems."

- Here: get new distributions, local limit laws, large deviation rates, speeds of convergence, elliptic function solutions, etc.
- Hope to approach nonintegrable models via perturbation and singularities?