LIPN

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Comptage probabiliste : entre mathématique et informatique

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Routers in the range of Terabits/sec (10^{14}b/s).

Google indexes 6 billion pages & prepares to index 100 Petabytes of data (10^{17}B).

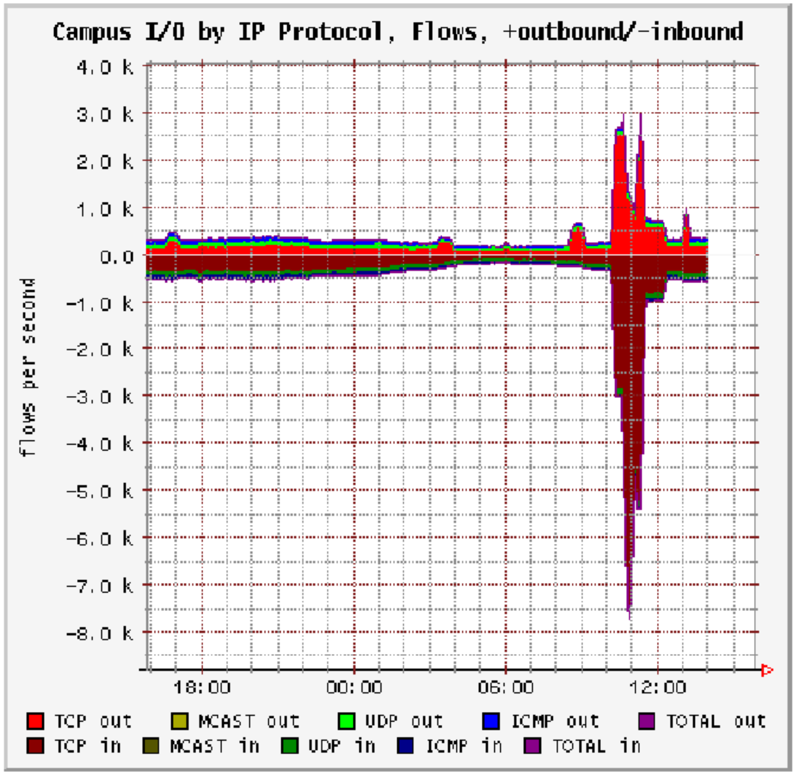
What about **content**?

Message: *Can get a few key characteristics, QUICK and EASY*

Combinatorics +algorithms +probabilities +analysis are useful!

From Estan-Varghese-Fisk: traces of attacks

Need number of active connections in time slices.



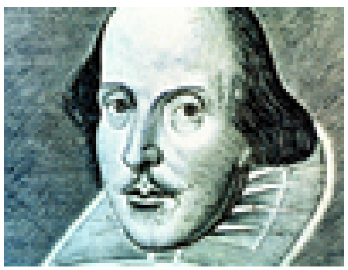
Incoming/Outgoing flows at 40Gbits/second.

Code Red Worm: 0.5GBytes of compressed data per hour (2001).

CISCO: in 11 minutes, a worm infected 500,000,000 machines.

The situation is like listening to a play of Shakespeare and at the end *estimate the number of different words*.

Rules: Very little computation per element scanned, very little auxiliary memory.



From Durand-Flajolet, LogLog Counting (ESA-2003):

Whole of Shakespeare, $m = 256$ small "bytes" of 4 bits each = 128bytes

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ghfffghfghgghggggghghheehfhfhhgghghghhfgffffhhhiigfhhffgfiihfhhh
igigighfgihfffghigihghigfhhgeegeghgghhhgghhfhidiigihighihehhhfgg
hfgighigffghdieghhhgggghhfghhfiiheffghghihifgggffihgihfggighgiiif
fjgfgjhhjiifhjgehgghfhhfhjhigggghghihigghhihihgiighgfhlgjfgjjjmfl
```

Estimate $n^\circ \approx 30,897$ vs $n = 28,239$ distinct words. Error: $+9.4\%$ w/ 128 bytes!

Uses:

— Routers: intrusion, flow monitoring & control

— Databases: Query optimization, cf $\mathfrak{M} \cup \mathfrak{M}'$ for *multisets*; Estimating the size of queries & "sketches".

— Statistics gathering: on the fly, fast and with little memory even on "unclean" data $\simeq$ layer 0 of "*data mining*".

This talk:

- *Estimating characteristics of large data streams*

— sampling; size & cardinality & nonuniformity index (F_1, F_0, F_2)

$\leadsto$ power of randomization via hashing

◇ Gains by a factor of $>$400 (Palmer *et al.*)

- *Analysis of algorithms*

— generating functions, complex asymptotics, Mellin transforms

◇ Nice problems for theoreticians.

- *Theory and Practice*

— Interplay of analysis and design $\leadsto$ super-optimized algorithms.

Problems on Streams

Given: S = a large *stream* $S = (r_1, r_2, \ldots, r_\ell)$ with duplicates
— $\|S\|$ = length or size: *total* # of records (ℓ)
— $|S|$ = cardinality: # of *distinct* records (c)

◇ How to estimate size, cardinality, etc?

More generally, if f_v is frequency of value v: $F_p := \sum_{v \in \mathbb{D}} (f_v)^p$.

Cardinality is F_0; size is F_1; F_2 is indicator of nonuniformity of distribution; "F_∞" is most frequent element (Alon, Matias, Szegedy, STOC96)

◇ How to sample?
— with or without multipicity

◇ How to find icebergs, mice, elephants?

1 ICEBERGS—COMBINATORICS HELPS!

Definition: *A k–iceberg is present in proportion $> 1/k$.*

One pass detection of icebergs for $k = 2$ using 1 registers is possible .

— Trigger a gang war: equip each individual with a gun.
— Each guy shoots a guy from a different gang, then commits suicide: Majority gang survives.
— Implement sequentially & adapt to $k \geq 2$ with $\mathbf{k-1}$ registers. (Karp et al. 2003)

2 APPROXIMATE COUNTING—THE DYADIC PARADIGM

How to find an integer while posing few questions?

— Ask if in (1—2), (2—4), (4—8), (8–16), etc?
— Conclude by binary search: cost is $2 \log_2 n$.

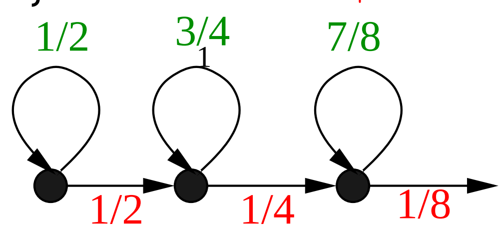
A paradigm for unbounded search:

• Ethernet proceeds by period doubling + randomization.
• Wake up process for mobile communication (OCAD: Lavault^{+})
• Adaptive data structures: e.g., extendible hashing tables.
♡ Approximate Counting

Approximate counting—probabilities help!

The oldest algorithm (Morris CACM:1977), analysis (F, 1985).

Maintain F_1, i.e., counter subject to $C := C + 1$.



ALG: Approximate Couting

Initialize: $X := 1$;

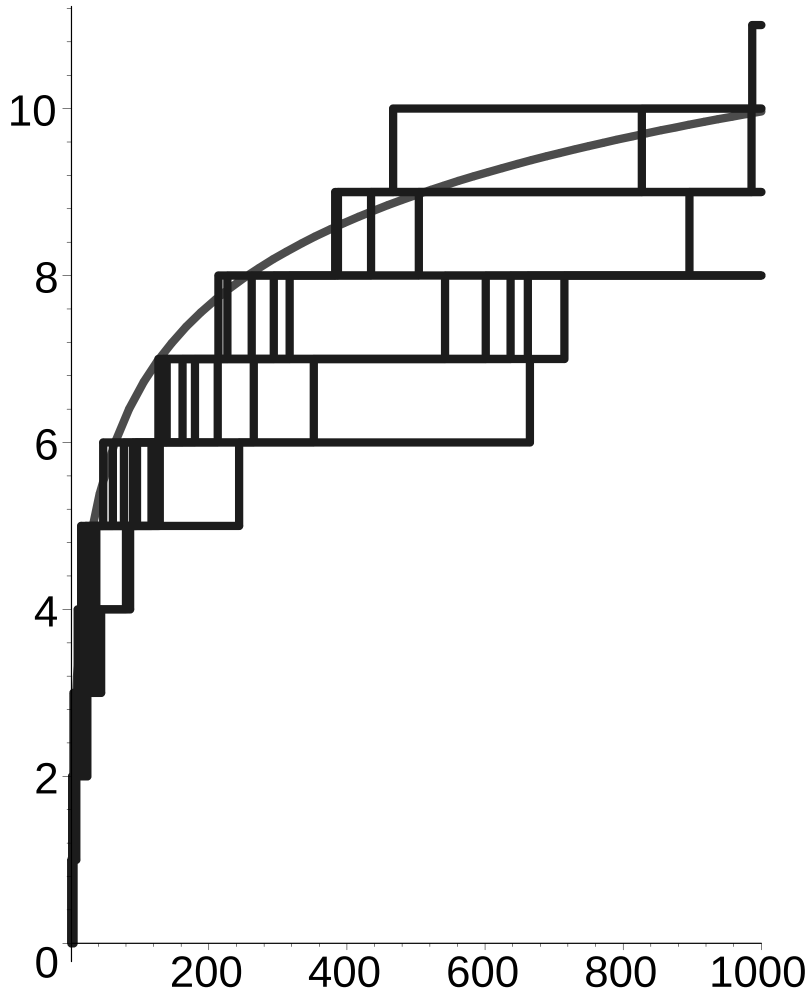
Increment: do $X := X + 1$ with probability 2^{-X};

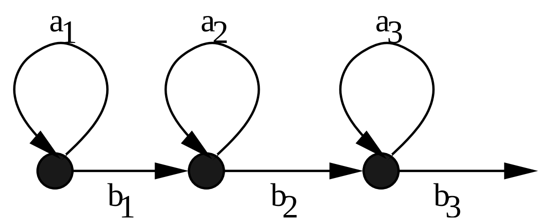
Output: $2^X - 2$.

Theorem: Count till n *probabilistically* using $\log_2 \log n + \delta$ bits, with accuracy about $0.59 \cdot 2^{-\delta/2}$.

Beats information theory(!?): 8 bits for counts $\leq 2^{16}$ w/ accuracy $\approx 15\%$.

10 runs of of APCO: value of X $(n = 10^3)$



Methodology: 

♡ Paths in graphs $\mapsto$ Generating Functions: $\boxed{(f_n) \mapsto f(z) := \sum_n f_n z^n.}$

Here: Symbolicly describe all paths:

$$\frac{(a_1)^\star b_1 (a_2)^\star b_2 (a_3)^\star}{\frac{1}{1-a_1} b_1 \frac{1}{1-a_2} b_2 \frac{1}{1-a_3}} \quad \text{since} \quad \frac{1}{1-f} = 1 + f + f^2 + \cdots \simeq (f)^\star.$$

Perform probabilistic valuation $a_j \mapsto q^j$; $b_j \mapsto 1 - q^j$:

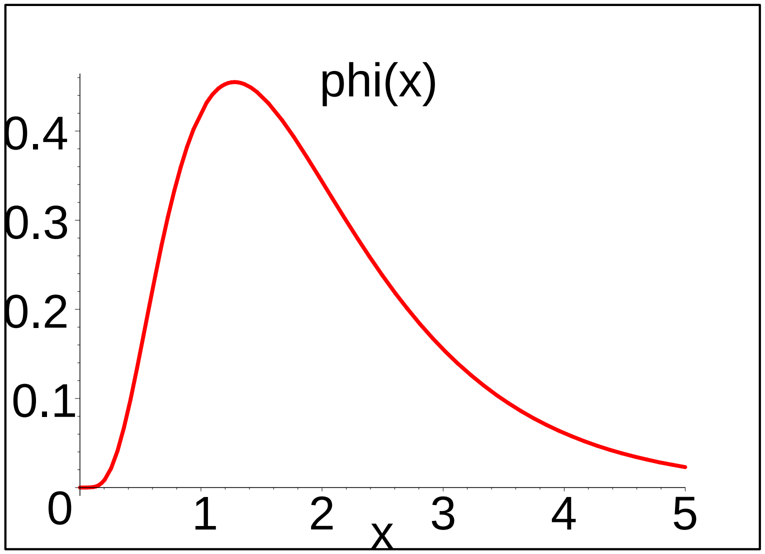
$$H_3(z) = \frac{q^{1+2} z^2}{(1-(1-q)z)(1-(1-q^2)z)(1-(1-q^3z))}.$$

♡ (Prodinger'94) Euler transform $\xi := z/(1-z)$: $zH_k(z) = \frac{q^{\binom{k}{2}} \xi^{k-1}}{(1-\xi q) \cdots (1-\xi q^k)}$.

Exact moments of X and estimate q^X via Heine's transformation of q-calculus: mean is unbiased, variance $\leadsto 0.59$.

♡ Partial fraction expansions $\rightsquigarrow$ asymptotic distribution
= quantify typical behaviour + risk! (*Exponential tails* $\gg$ Chebyshev ineq.)

We have $\mathbb{P}_n(X = \ell) \sim \phi(n/2^\ell)$, where ($(q)_j := (1-q)\cdots(1-q_j)$.)



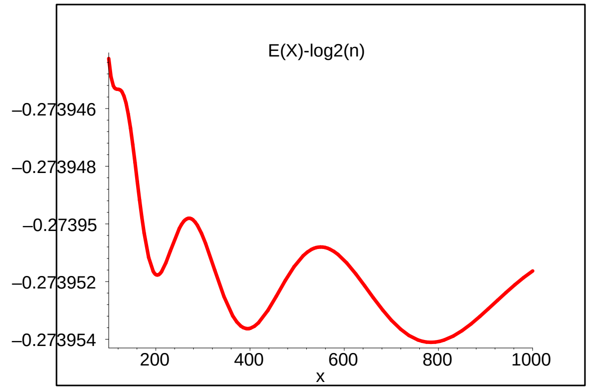
$$\boxed{\phi(x) := \sum_{j\geq 0} \frac{(-1)^j q^{\binom{j}{2}} e^{-xq^{-j}}}{(q)_\infty (q)_j}}$$

♣ Fluctuations: $\ldots, \frac{n}{2^L}, \ldots, \frac{n}{4}, \ldots$ depend on $L = \lfloor \log_2 n \rfloor$.
cf. Szpankowski, Mahmoud, Fill, Prodinger, ...

Analyse storage utilization via Mellin transform

Approximate Counting

Mean$(X) - \log_2 n$:



The Mellin transform (F. Régnier Sedgewick 1985); (FlGoDu 1995)

$$f^\star(s) := \int_0^\infty f(x) x^{s-1}\, dx.$$

Mapping properties (complex analysis): Asympt$(f) \longleftrightarrow$ Singularities$(f^\star)$.

♣ dyadic superpositions of models: $F(x) = \sum \phi\left(\frac{x}{2^\ell}\right) \rightsquigarrow F^*(s) = \frac{f^\star(s)}{1-2^s}$.

$\Longrightarrow$ Standard asymptotic terms + (small) fluctuations.

Cultural flashes

— Morris (1977): Couting a large number of events in small memory.
— The power of probabilistic machines & approximation (Freivalds IFIP 1977)
— The FTP protocol: Additive Increase Multiplicative Decrease (AIMD) leads to similar functions (Robert et al, 2001)
— Probability theory: Exponentials of Poisson processes (Yor et al, 2001)

Randomization and hashing

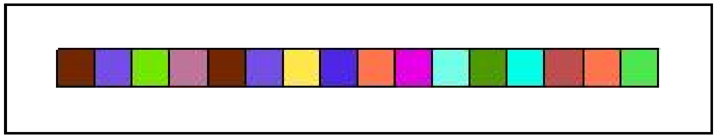
Theme: *randomization is a major algorithmic paradigm.*

• Cryptography (implementation, attacks)

• Combinatorial optimization (smoothing, random rounding).

• Hashing and direct access methods
— Produce (seemingly) uniform data from actual ones;
— Provide reproducible chance

Can get random bits from nonrandom data: **Works fine!**

$\boxed{\textit{To be or not to be...}}$ $\rightsquigarrow$ 

Justifies:

Angel

Daemon

The  —  Model

3 COUPON COLLECTOR COUNTING

Let a flow of people enter a room.

— *Birthday Paradox:* It takes on average 23 to get a **birthday collision**

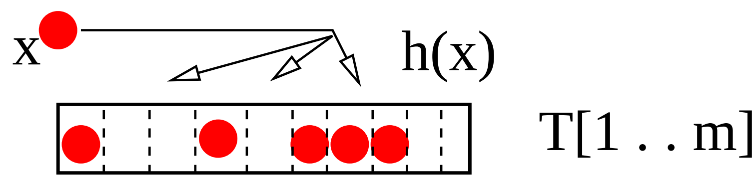
— *Coupon Collector*: After 365 persons have entered, expect a **partial collection** of ~ 231 different days in the year; it would take more than 2364 to reach a full collection.

B 1st birthday coll.	n	C complete coll.
$\mathbb{E}_n(B) \sim \sqrt{\frac{\pi n}{2}}$	$\approx ne^{-1}$	$\mathbb{E}_n(C) = nH_n \sim n \log n$

BP: Suppose we *didn't know* the number N of days in the year but could identify people with the same birthday. Could we *estimate* N?

Coupon Collector Counting

First Counting Algorithm: Estimate *cardinalities* $\equiv$ # of **distinct** elements.
Motivated by *query optimization in data bases.* (Whang$^+$, ACM TODS 1990)



ALG: Coupon Collector Counting
Set up a table $T[1\,..\,m]$ of m bit-cells.
— for x in S do mark cell $T[h(x)]$;
Return $-m \log V$, where V:=fraction of empty cells.

Alg. is indep. of replications.

Let n be sought cardinality. Then $\alpha := n/m$ is *filling ratio*.

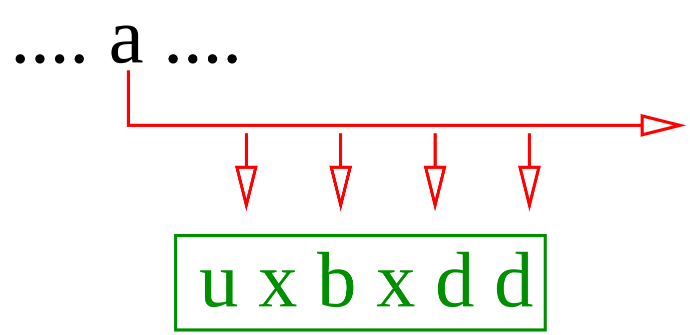
Expect $V \approx e^{-\alpha}$ empty cells by classical analysis of occupancy. Distribution is concentrated. Invert!

Count cardinalities till $N_{\max}$ using $\frac{1}{10}N_{\max}$ bits, for accuracy (standard error) = 2%.

Generating functions for occupancy; Stirling numbers; basic depoissonization.

4 SAMPLING

Classical sampling (Vitter, ACM TOMS 1985)



ALG: Reservoir Sampling (*with multiplicities*)

Sample m elements from $S = (s_1, \ldots, s_N)$; (N unknown a priori)

Maintain a cache (reservoir) of size m;

— for each coming s_{t+1}:

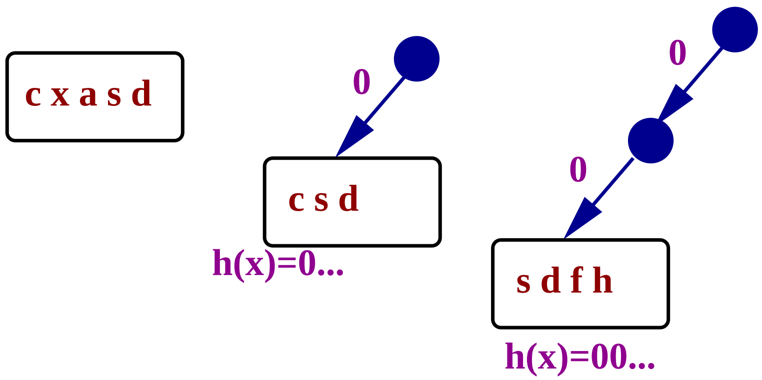
place it in cache with probability $m/(t+1)$; drop random element;

Can we sample *values* (i.e., without multiplicity)?

Algorithm due to (Wegman, ca 1984, unpub.), analysed by (F.1990).

Sample of size $\leq b$:

depth $d = 0, 1, 2, \ldots$



ALG: Adaptive Sampling (*without multiplicities*)
Get a sample of size m from S's values.

Set $b := 4m$ (bucket capacity);
— Oversample by adaptive method;
– Get sample of m elements from the $(b \equiv 4m)$ bucket.

Analysis.

View collection of records as a set of bitstrings.

Digital tree aka trie, paged version:

$$\left[\begin{array}{l} \mathrm{Trie}(\omega) \equiv \omega \text{ if } \mathrm{card}(\omega) \leq b \\[2ex] \mathrm{Trie}(\omega) = \overbrace{\mathrm{Trie}(\omega \setminus 0) \qquad \mathrm{Trie}(\omega \setminus 1)}^{\bullet} \quad \text{if } \mathrm{card}(\omega) > b \end{array}\right.$$

(Underlies dynamic and extendible hashing, paged DS, etc)

Depth in Adaptive Sampling is length of leftmost branch;
Bucket size is # of elements in leftmost page.

Bonus: Second Counting Algorithm for *cardinalities*.

Let d := sampling *depth*; ξ :=sample size.

Theorem [F90]: $X := 2^d \xi$ estimates the cardinality of S using b words of memory, in a way that is unbiased and with standard error $\approx 1.20/\sqrt{b}$.

• $1.20 \doteq 1/\sqrt{\log 2}$: with $b = 1,000$W, get 4% accuracy.

• Distributional analysis by (Louchard RSA 1997).

• Related to folk algorithm for leader election on channel: "*Talk, flip coin if noisy; sleep if Tails; repeat!*

• Related to "tree protocols with counting"
$\gg$ Ethernet. Cf (Greenberg-F-Ladner JACM 1987).

Cardinality Estimators

F_0 = *Number of different values*

— 1983–1985: (F-Martin, FOCS+JCSS) Probabilistic Counting

— 1987–1990: (Whang et al) Coupon Collector Counting

— 1984–1990: (Wegner) (F90 COMP.) Adaptive Sampling

— 1996: (Alon et al, STOC) F_p statistics $\leadsto$ *later*

— 2000: (Indyk FOCS) Stable Law Counting $\leadsto$ *later*

— 2001: (Estan-Varghese SIGCOMM) Multiresolution Bitmap

— 2003: (Durand-F ESA) Loglog Counting

— 2005: (Giroire) MinCount

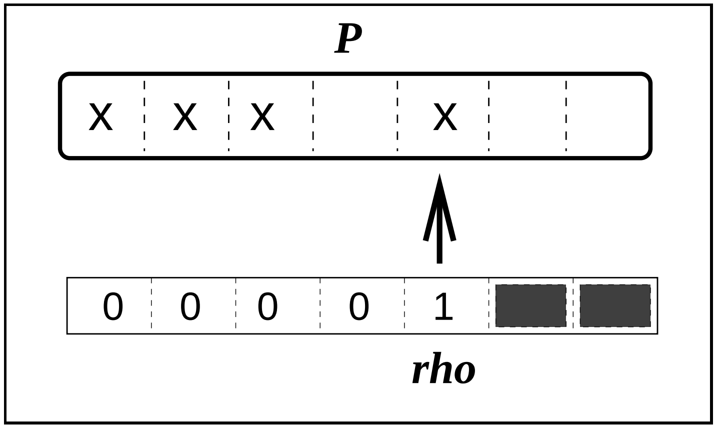
— 2006: (DFFM) HyperLoglog

Note: *suboptimal* algorithms can be **useful!**

5 PROBABILISTIC COUNTING

Third Counting Algorithm

for *cardinalities* :



ALG: Probabilistic Counting

Input: a stream S; Output: cardinality $|S|$

For each $x \in S$ do /* $\rho \equiv$ position of leftmost 1-bit */

Set BITMAP$[\rho(\mathrm{hash}(w))] := 1$; od;

Return P where P is position of first 0.

— P estimates $\log_2(\varphi n)$ for $\varphi \doteq 0.77351$

— Average over m trials $A = \frac{1}{m}[A_1 + \cdots + A_m]$; return $\frac{1}{\varphi}2^A$.

— In fact, use stochastic averaging, which needs only *one* hash function: $S \mapsto (S_{000}, \ldots, S_{111})$.

— Analysis provides

$$\varphi = \frac{e^\gamma}{\sqrt{2}} \prod_{m \geq 2}^{\star} m^{\epsilon(m)}, \qquad \epsilon(m) := (-1)^{\sum \text{bits}(m)}.$$

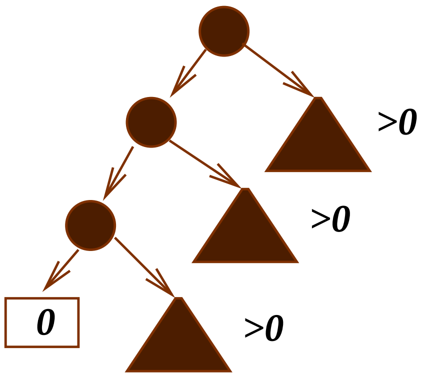
$\epsilon(19) = \epsilon(\langle 10011 \rangle_2) = (-1)^3 = -1$. Standard error is $0.78/\sqrt{m}$ for m Words of $\log_2 N$ bits. + Exponential Tails $\gg$ Chebyshev.

(AMS96) and subsequent literature claim wrongly that *several* hash functions are needed!

Theorem [FM85]: Prob. Count. is *asymptotically unbiased*. Accuracy is $\frac{0.78}{\sqrt{m}}$ for m Words of size $\log_2 N$. E.g. 1,000W = 4kbytes $\rightsquigarrow$ 2.5% accuracy.

Proof:

trie analysis



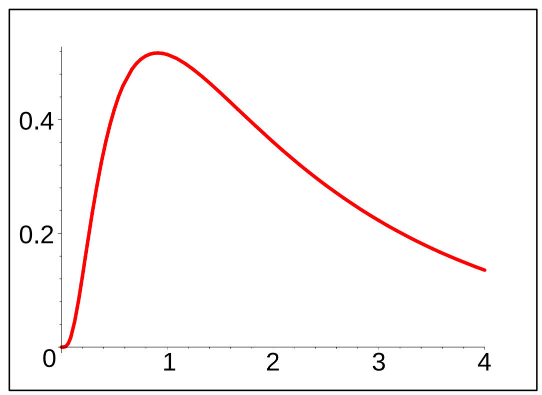
$$\mathbf{1} \cdot (e^{x/8} - 1)(e^{x/4} - 1)(e^{x/2} - 1)$$

$$(1-q)(1-q^2)(1-q^4)\cdots = \sum_n \overbrace{(-1)^{\sum \mathrm{bits}(n)}}^{\epsilon(n)} q^n.$$

Distribution:

$$Q(x) := e^{-x/2} \prod_{j=0}^{\infty} (1 - e^{-x2^j})$$

$$\mathbb{P}_n(X = \ell) \sim Q\left(\frac{n}{2^\ell}\right)$$



+ Mellin requires $N(s) := \sum_{n \geq 1} \frac{\epsilon(n)}{n^s}$. One finds $\log_2 \varphi \equiv -\Gamma'(1) - N'(0) + \frac{1}{2}$, *&c.*

Data mining of the Internet graph

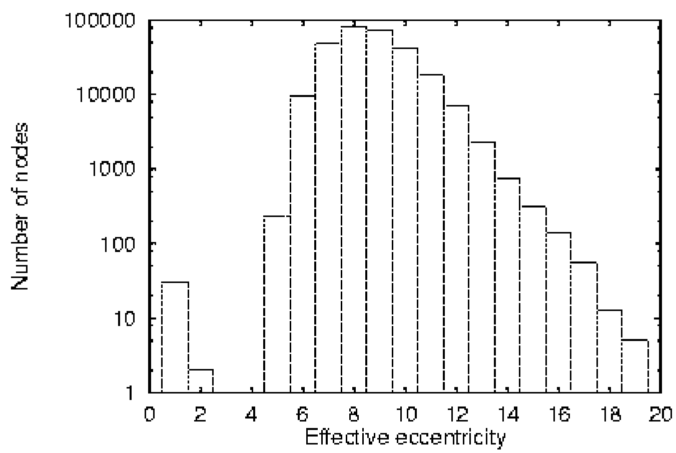
(Palmer, Gibbons, Faloutsos^2, Siganos 2001)

Internet graph: 285k nodes, 430kedges.

For each vertex v, define ball $B(v;R)$ of radius R.

Want: histograms of $|B(v,R)|$ $R = 1\,..\,20$

Get it in minutes of CPU rather than a day (400$\times$ speedup)



b) Histogram of diameters

+ Sliding window usage (Motwani et al) Fusy & Giroire for MinCount.

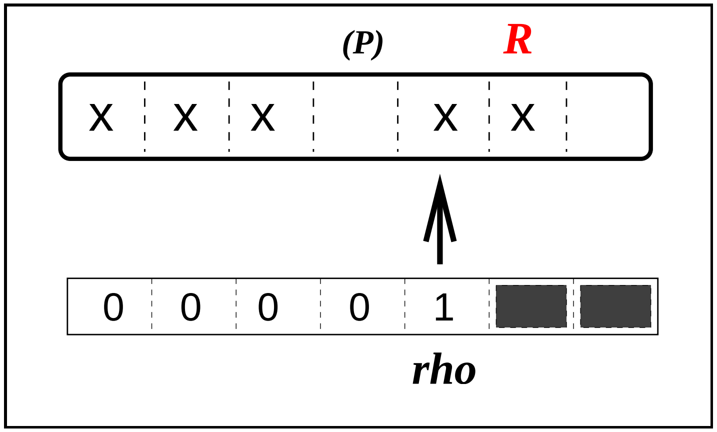
6 LOGLOG COUNTING

Fourth Counting Algorithm for *cardinalities*: (Durand-F, 2003/DFFM, 2006)

Claim: the best algorithm on the market!

• Hash values and get $\rho(h(x))$ = position of leftmost 1-bit = a geometric RV $G(x)$.

• To set S associate $R(S) := \max_{v \in S} G(v)$.



• Max of geometric RVs are well-known (Prodinger*).

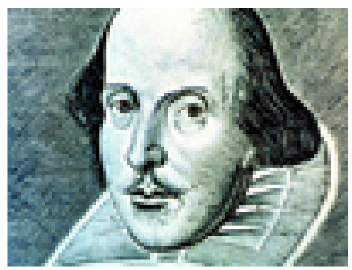
$R(s)$ estimates $\sim \log(\widehat{\varphi}\,\mathrm{card}(S))$, with $\widehat{\varphi} := e^{-\gamma}\sqrt{2}$.

• Do stochastic averaging with $m = 2^{\ell}$:
E.g., $S \cong \langle S_{00}, S_{01}, S_{10}, S_{11} \rangle$: count separately.
Return $\boxed{\dfrac{m}{\widehat{\varphi}} 2^{\mathrm{Average}}}$.

$++$ Switch to Coupon Collector Counting for small and large cardinalities.
$++$ Optimize by pruning discrepant values $\leadsto$ superLogLog or better by harmonic means $\leadsto$ superLogLog ($\ll$ Chassaing-Gerin, 2006)

Theorem. LogLog needs m "bytes", each of length $\log_2 \log N$.
Accuracy is: $\dfrac{1.30}{\sqrt{m}}$.

PROOF: Generating Functions + Saddle-point depoissonization (Jacquet-Szpankowski) + Mellin. $1.30 \doteq \sqrt{\frac{1}{12}\log^2 2 + \frac{1}{6}\pi^2}$.

Whole of Shakespeare: 

$m = 256$ small "bytes" of 4 bits each = 128bytes

```
ghfffghfghgghggggghghheehfhfhhgghghghhfgffffhhhiigfhhffgfiihfhhh
igigighfgihfffghigihghigfhhgeegeghgghhhgghhfhidiigihighihehhhfgg
hfgighigffghdieghhhggghhfghhfiiheffghghihifgggffihgihfggighgiiif
fjgfgjhhjiifhjgehgghfhhfhjhigggghghihigghhihihgiighgfhlgjfgjjjmfl
```

Estimate $n^\circ \approx 30,897$ against $n = 28,239$ distict words

Error is $+9.4\%$ for 128 bytes(!!)

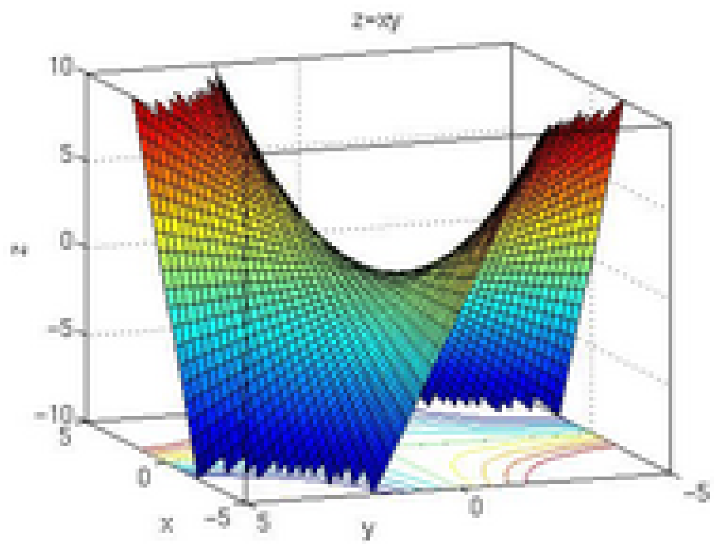
An aside: **Analytic depoissonization** (JaSz95$^+$)

- PROBLEM: Recover asympt. f_n from $f(z) = \sum_n f_n \frac{z^n}{n!}$?

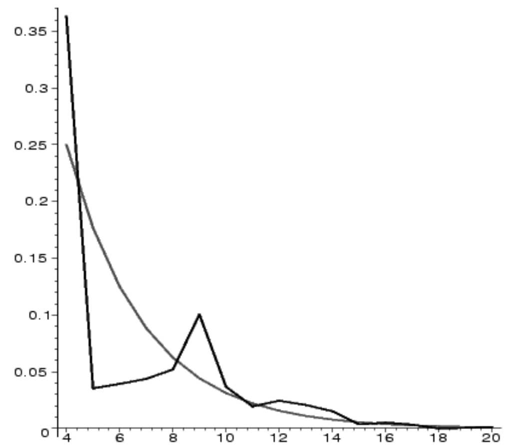
- Intuition: "with luck" $f_n \sim \phi(n)$ where $\phi(z) := e^{-z} f(z)$ is Poisson g.f..

(Here: "Luck" means good lifting of $\phi(z)$ to $\mathbb{C}$ ≡ Poisson flow of complex rate!)

$$f_n = \frac{n!}{2i\pi} \oint f(z) \frac{dz}{z^{n+1}} \qquad \approx \phi(n)$$



Features: Errors $\approx$ *Gaussian*, seldom more than $2\times$ standard error. Algorithm *scales down* (for small cardinalities) and Algorithm *scales up* (large memory size): **HYBRIDIZE with Collision Counting**.



Mahābhārata: 8MB, 1M words, 177601 diff.

HTTP server: 400Mb log pages 1.8 M distinct req.

m	2^6 (50by)	2^{10} (0.8kby)	2^{14} (12kb)	2^{18} (200kb)
Obs:	8.9%	2.6%	1.2%	0.32%
σ:	11%	2.8%	0.7%	0.36%

Summary

Analytic results ($\lg \equiv \log_2$): Alg/Mem/Accuracy

CouponCC	AdSamp	ProbC	LogLog
$\approx \frac{N}{10}$ bits	$m \cdot \lg N$ Words	$m \cdot \lg \frac{N}{m}$ Words	$m \cdot \lg \lg \frac{N}{m}$ Bytes
$\approx$ 2%	$\frac{1.20}{\sqrt{m}}$ W	$\frac{0.78}{\sqrt{m}}$ W	$\approx \frac{1.30--1.05}{\sqrt{m}}$ By

F_0 statistics, $N = 10^8$ & 2% error

— Coupon Collector Counting = 1 Mbyte + used for corrections
— Adaptive Sampling = 16 kbytes + sampling, unbiased
— Probabilistic Counting: = 8 kbytes + sliding window
— Multiresolution bitmap (analysis?) = 5 kbytes?
— MinCount ©Giroire = 4 kbytes + sliding window
— Loglog Counting = 2 kbytes + mice/elephants

+ Sliding window, Mice/elephant problem (A. Jean-Marie, O. Gandouët)

7 FREQUENCY MOMENTS: F_2

Recall: Alon, Matias, Szegedy (STOC 1996)***

$$F_2 := \sum_v (f_v)^2,$$

where f_v is frequency of value v.

An elegant idea: flip$(x) \equiv \epsilon(x)$= ± 1 based on hash(x).

ALG: F2;
Initialize Z:=0;
For each x in S do $Z := Z + \text{flip}(x)$.
Return Z^2.
Collect m Z-values and average, with T-transform.

$$\mathbb{E}(Z^2) = \mathbb{E}\left(\sum_{x \in S} \epsilon(v)\right)^2 = \mathbb{E}\left(\sum_j f_j \cdot \epsilon(j)\right)^2 = \sum_j (f_j)^2.$$

(Actually, AMS prove stronger complexity result by complicated (impractical?) algorithm.) (What about stochastic averaging?)

Indyk's F_p algorithm

A beautiful idea of Piotr Indyk (FOCS 2000)*** for F_p, $p \in (0,2)$.

• Stable law of parameter $p \in (0,2)$: $\mathbb{E}(e^{itX}) = e^{-|t|^p}$.
No second moment; no 1st moment if $p \in (0,1)$.
$c_1 X_1 + c_2 X_2 \overset{\mathcal{L}}{\cong} \mu X$, with $\mu := (c_1^p + c_2^p)^{1/p}$.

Alg: Fp;
Initialize Z:=0;
For each x in S do $Z := Z + \mathrm{Stable}_\alpha(x)$.
Return Z.
Estimate F_p parameter from m copies of Z-values.

Remark: Use of $\log(|Z|)$ to estimate seems better than median(?)

8 CONCLUSIONS

For streams, using practically $O(1)$ storage, one can:
— Sample positions and even distinct values;
— Estimate F_0, F_1, F_2, F_p $(0 < p \leq 2)$ even for *huge* data sets;
— Need **no assumption on nature of data**.

♡♡♡♡♡♡

The algorithms are based on randomization $\mapsto$ Analysis fully applies
— They work **exactly** as predicted on real-life data;
— They often have a wonderfully elegant structure;
— Their analysis involves beautiful methods for AofA: "Symbolic modelling by generating functions, Singularity analysis, Saddle Point and analytic depoissonization, Mellin transforms, stable laws and Mittag-Leffler functions, etc."

That's All, Folks!