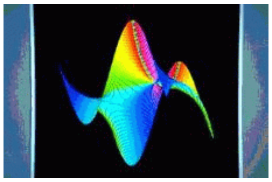

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# Singular Combinatorics

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*Combinatorics*: discrete structures by finitary rules
$\rightsquigarrow$ Enumerative & Quantitative aspects

- Counting and asymptotics

$$n! \sim n^n e^{-n} \sqrt{2\pi n}$$

- Asymptotic laws (e.g., Monkey and typewriter!)

$$\Omega_n \xrightarrow{\mathcal{D}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt.$$

*Approaches*
— Probabilistic, stochastic
— Analytic: Generating Functions

# 1. Introduction

**"Symbolic" Methods**

Rota-Stanley; Foata-Schutzenberger; Joyal and UQAM group; Jackson-Goulden, &c; ca $1980^{\pm}$.

*Basic combinatorial constructions admit of direct translations as operators over generating functions* (GF's).

$\mathcal{C}$ : class of comb. structures;

$C_n$ : # objects of size $n$

$\Downarrow$

(counting)
$$\begin{cases} C(z) := \sum C_n z^n \\ \widehat{C}(z) := \sum C_n \frac{z^n}{n!} \end{cases}$$

(params)
$$\begin{cases} C(z,u) := \sum C_{n,k} z^n u^k \\ \widehat{C}(z,u) := \sum C_{n,k} u^k \frac{z^n}{n!} \end{cases}$$

Ordinary GF's for unlabelled structures
Exponential GF's for labelled structures.

## “Dictionaries”

= Constructions viewed as Operators over GF’s.

| Constr. | Operations | |
|---|---|---|
| Union | $+$ | $+$ |
| Product | $\times$ | $\times$ |
| $\mathfrak{S}$equence | $(1-f)^{-1}$ | $(1-f)^{-1}$ |
| $\mathfrak{M}$ultiSet | Pólya Exp. | $e^f$ |
| $\mathfrak{C}$ycle | Pólya Log. | $\log(1-f)^{-1}$ |
| | (unlab.) | (lab.) |

Books: Goulden-Jackson, Bergeron-LL, Stanley, F-Sedgewick

$\Longrightarrow$ Formal power series w/ rich set of functional eqns!!

$\Longrightarrow$ How to extract coeff., especially, asymptotically??

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$$\mathrm{Exp}(f) := \exp\left(f(z) + \frac{1}{2}f(z^2) + \cdots\right); \quad \mathrm{Log}(f) := \log\frac{1}{1-f(z)} + \cdots$$

## "Complex–analytic Structures"

*Interpret:*

♡ *Counting GF as analytic transformation of $\mathbb{C}$;*

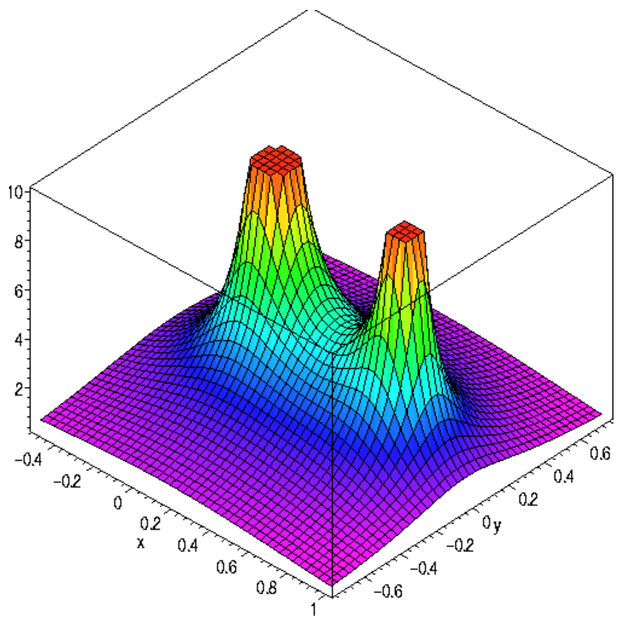
♡ *Comb. Construction as analytic functional.*

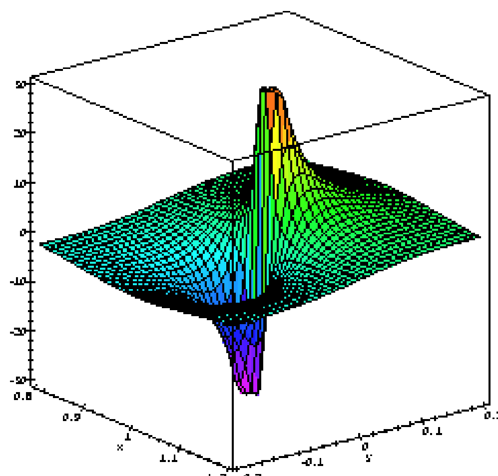
**Singularities are crucial to asymptotic prop's!**

(cf. analytic number theory, complex analysis, etc)

Asymptotic counting via Singularity Analysis (S.A.)

Asymptotic laws via Perturbation + S.A.



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Refs: F–Odlyzko, SIAM A&DM, 1990; Odlyzko's 1995 survey in *Handbook of Combinatorics*

+ Banderier, Fill, J. Gao, Gonnet, Gourdon, Kapur, G. Labelle, Laforest, T. Lafforgue, Noy, Odlyzko, Panario, Pouyanne, Prodinger, Puech, Richmond, Robson, Salvy, Schaeffer, Sipala, Soria, Steyaert, Szpankowski, B. Vallée.

♠ Location of singularity at $z = \rho$:

$$\text{coeff. } [z^n] f(z) = \rho^{-n} \cdot \text{coeff. } [z^n] f(\rho z),$$

where latter is singular at $z = 1$.

♠ Nature of singularity at $z = 1$:

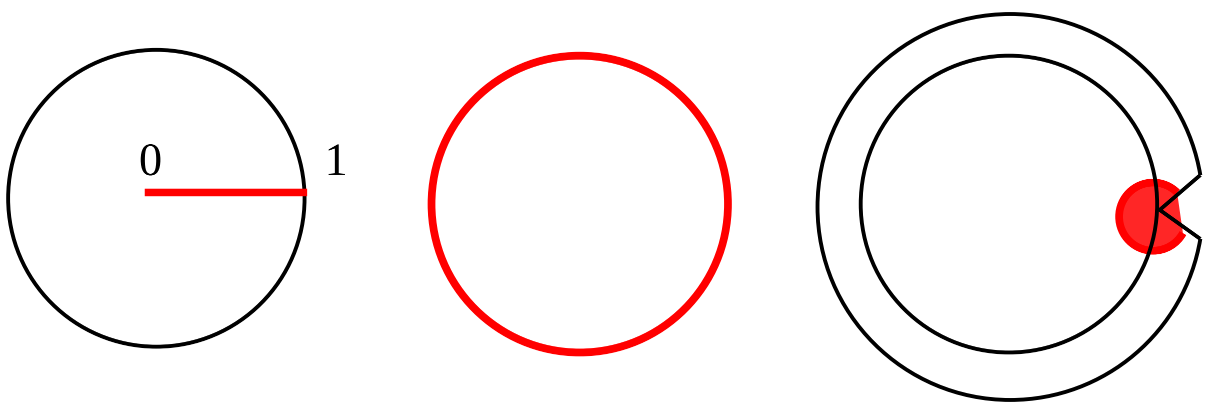
| | | | | |
|---|---|---|---|---|
| $\dfrac{1}{(1-z)^2}$ | $\longrightarrow$ | $n+1$ | $\sim$ | $n$ |
| $\dfrac{1}{1-z}\log\dfrac{1}{1-z}$ | $\longrightarrow$ | $H_n \equiv \frac{1}{1} + \ldots + \frac{1}{n}$ | $\sim$ | $\log n$ |
| $\dfrac{1}{1-z}$ | $\longrightarrow$ | $1$ | $\sim$ | $1$ |
| $\dfrac{1}{\sqrt{1-z}}$ | $\longrightarrow$ | $\dfrac{1}{2^{2n}}\dbinom{2n}{n}$ | $\sim$ | $\dfrac{1}{\sqrt{\pi n}}$ |

| | | |
|---|---|---|
| Location of sing's : | Exponential factor | $\rho^{-n}$ |
| Nature of sing's : | "Polynomial" factor | $\vartheta(n)$ |

## Generating Function ⇝ Coefficients

Solving a "Tauberian" problem

| $\mathbb{R}$eal–Tauberian | Darboux-Pólya | Singularity An. |
|---|---|---|




| (large $\Longrightarrow$ large) | (smooth $\Longrightarrow$ small) | (Full mappings) |
|---|---|---|

Combinatorial constructions ⇝ Analytic Functionals

$\Longrightarrow$ Analytic continuation prevails for comb. GF's

# 2. Basic Singularity Analysis

**Theorem 1.** *Basic scale translates:*

$$\sigma_{\alpha,\beta}(z) := (1-z)^{-\alpha}\left(\frac{1}{z}\log\frac{1}{1-z}\right)^{\beta}$$
$$\Longrightarrow \quad [z^n]\sigma_{\alpha,\beta} \underset{n\to\infty}{\sim} \frac{n^{\alpha-1}}{\Gamma(\alpha)}(\log n)^{\beta}.$$

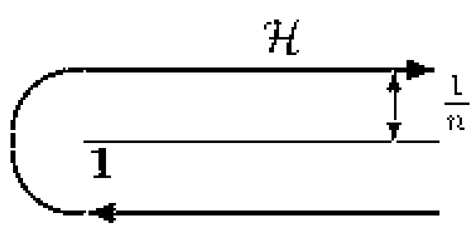
PROOF. Cauchy's coefficient integral, $f(z)=(1-z)^{-\alpha}$

$$[z^n]f(z) = \frac{1}{2i\pi}\int_{\gamma} f(z)\,\frac{dz}{z^{n+1}}$$

$$\Downarrow \quad (z = 1+\frac{t}{n}) \quad \Downarrow$$

$$\frac{1}{2i\pi}\int_{\mathcal{H}} \left(-\frac{t}{n}\right)^{-\alpha} e^{-t}\,\frac{dt}{n}$$

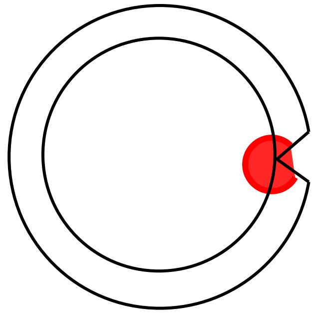
$$n^{\alpha-1} \times \frac{1}{\Gamma(\alpha)}.$$




♡ Slowly varying $\Longrightarrow$ slowly varying

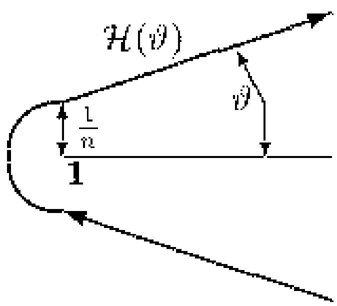
♡ Log-log $\Longrightarrow$ Log-Log

♡ Full asymptotic expansions



**Theorem 2.** *$\mathcal{O}$–transfers:*
*Under continuation in a $\Delta$-domain,*

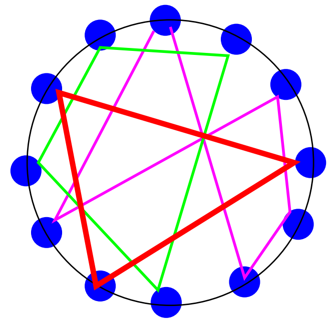
$$f(z) = O(\sigma_{\alpha,\beta}(z)) \quad \Longrightarrow \quad [z^n]f(z) = O\left([z^n]\sigma_{\alpha,\beta}(z)\right).$$

PROOF: 


Usage: $\begin{cases} f(z) = \lambda\sigma(z) + \mu\tau(z) + \ldots + O(\omega(z)) \\ \Longrightarrow \\ f_n = \lambda\sigma_n + \mu\tau_n + \ldots + O(\omega_n). \end{cases}$

Similarly: $o$-transfer.

- Dominant singularity at $\rho$ gives factor $\rho^{-n}$.
- Finitely many singularities work fine
- Some cases with $\infty$ of sing's.



EXAMPLE **1**. *2-regular graphs* [Comtet]

$$\mathcal{G} = \mathfrak{M}\left(\frac{1}{2}\mathfrak{C}_{\geq 3}(\mathcal{Z})\right)$$

$$\widehat{G}(z) = \exp\left(\frac{1}{2}\log\frac{1}{1-z} - \frac{z}{2} - \frac{z^2}{4}\right)$$

$$\widehat{G}(z) \underset{z\to 1}{\sim} \frac{e^{-3/4}}{\sqrt{1-z}}$$

$$\frac{G_n}{n!} \underset{n\to\infty}{\sim} \frac{e^{-3/4}}{\sqrt{\pi n}}.$$

(Originally by Darboux-Pólya.) □

EXAMPLE **2**. *Richness index of trees* [FSS,90] = Number of different terminal subtrees. Catalan case:

$$K(z) = \frac{1}{2z}\sum_{k\geq 0}\frac{1}{k+1}\binom{2k}{k}\left(\sqrt{1-4z-4z^{k+1}} - \sqrt{1-4z}\right)$$

$$K(z) \underset{z\to 1/4}{\approx} \frac{1}{\sqrt{X\log X}}, \qquad X := 1-4z$$

$$\text{Mean index} \underset{n\to\infty}{\sim} C\frac{n}{\sqrt{\log n}}, \qquad C \equiv \sqrt{\frac{8\log 2}{\pi}}.$$

Related to compact tree representations as DAGs. □

# 3. Closure Properties

Function of S.A.–type = amenable to singularity analysis

- is continuable in a $\Delta$-domain,
- admits singular expansion in scale $\{\sigma_{\alpha,\beta}$.

**Theorem 3.** *Generalized polylogarithms*

$$\mathrm{Li}_{\alpha,k} := \sum (\log n)^k n^{-\alpha} z^n$$

*are of S.A.-type.*

PROOF. Cauchy-Lindelöf representations

$$\sum \varphi(n)(-z)^n = -\frac{1}{2i\pi}\int_{1/2-i\infty}^{1/2+i\infty} \varphi(s) z^s \frac{\pi}{\sin \pi s}\, ds.$$

+ Mellin transform techniques (Ford, Wong, F.).

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EXAMPLE 3. *Entropy of Bernoulli distribution*

$$H_n := -\sum_k \pi_{n,k} \log \pi_{n,k}, \qquad \pi_{n,k} \equiv \binom{n}{k} p^k (1-p)^{n-k}$$

involves $\sum \log(k!) z^k = (1-z)^{-1}\,\mathrm{Li}_{0,1}(z)$

$$\frac{1}{2}\log n + \frac{1}{2} + \log\sqrt{2\pi p(1-p)} + \cdots .$$

(Redundancy, coding, information th.; Jacquet-Szpankowski.) □

- Elements like $\log n$, $\sqrt{n}$ in combinatorial sums

**Theorem 4.** *Functions of S.A.-type are closed under integration and differentiation.*

PROOF. Adapt from Olver, Henrici, etc.

**Theorem 5.** *Functions of S.A.-type are closed under Hadamard product*

$$f(z) \odot g(z) := \sum_n (f_n g_n) z^n.$$

PROOF. Start from Hadamard's formula

$$f(z) \odot g(z) = \frac{1}{2i\pi} \int_\gamma f(t) g\left(\frac{w}{t}\right) \frac{dt}{t}.$$

+ adapt Hankel contours [H., Jungen, R. Wilson; Fill-F-Kapur]

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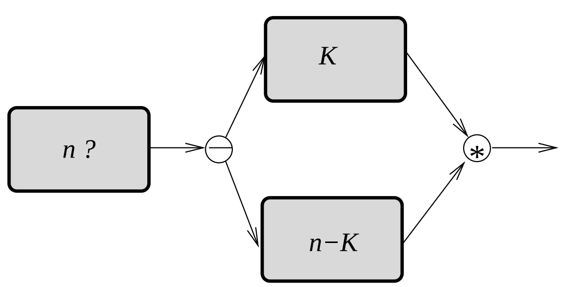
EXAMPLE 4. *Divide-and -conquer recurrences*

$$f_n = t_n + \sum \pi_{n,k}(f_k + f_{n-k})$$
$$\mathrm{Sing}(f(z)) = \Phi(\mathrm{Sing}(t(z)))$$
$$\mathrm{Asympt}[f_n] = \Psi(\mathrm{Sing}(t)).$$

E.g., Catalan statistics: need $\sum \binom{2n}{n} \log n \cdot z^n$.

Useful in computer science applications [FFK, 2002$^+$]. □



# 4. Functional Equations

• Rational functions. Combinatorics: linear system $\mathbb{Q}_{\geq 0}[z]$ implies polar singularities $(X^{-k})$:

$$[z^n]f(z) \approx \sum \omega^n n^k, \qquad \omega \in \overline{\mathbb{Q}}, \quad k \in \mathbb{Z}_{\geq 0}.$$

$+$ irreducibility: Perron-Frobenius $\Longrightarrow$ *simple dom. pole*.

• *Word problems* from regular language models;

• *Transfer matrices* [Bender-Richmond]: easy dimer coverings in $k \times n$ strip, knight tours, etc.

• Algebraic functions, by Puiseux expansions $(X^{p/q})$:

$$[z^n]f(z) \approx \sum \sum \omega^n n^{p/q}, \qquad \omega \in \overline{\mathbb{Q}}, \quad p/q \in \mathbb{Q},$$

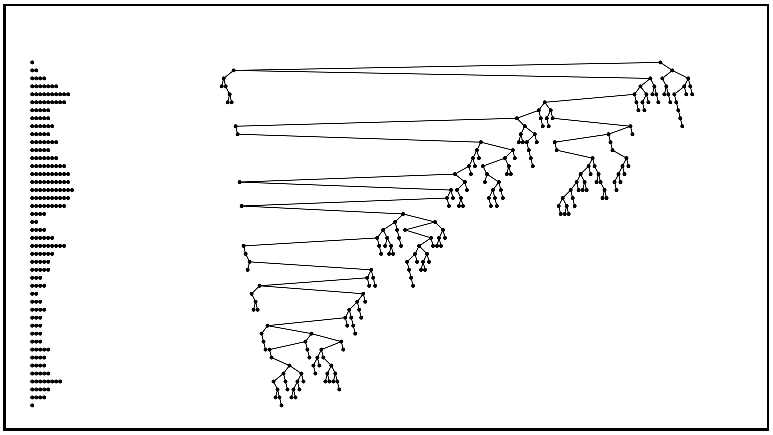
results from S.A. (or Darboux!)

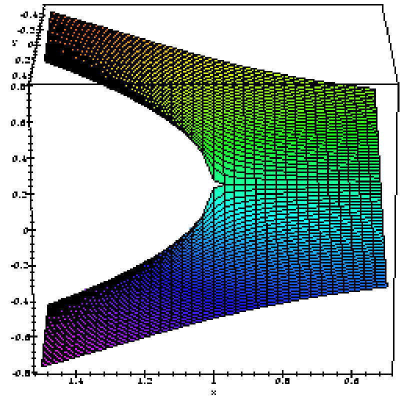
Asymptotics of coeff. is decidable [Chabaud-F-Salvy].

• Word problems and context-free models;

• *Trees*; geom. configurations (non-crossing graphs, polygonal triangs.); *planar maps* [Tutte].

$(1 - \sqrt{1-4z})/(2z)$





Square-root singularity is "universal" for many recursive classes (irreducibility). Owes to controlled "failure" of Implicit Function Theorem where quadratic dependency replaces linear (=analytic) dependency.
Entails coeff. asymptotic $\approx \omega^n n^{-3/2}$ with critical exponent $-3/2$ that is universal.

E.g., 2–3 trees (Meir-Moon):

$$f = z\phi(f), \qquad \phi(u) = 1 + u^2 + u^3.$$

Pólya's combinatorial chemistry programme:

$$f(z) = z\,\mathrm{Exp}(f(z)) \equiv z e^{f(z)+\frac{1}{2}f(z^2)+\frac{1}{3}f(z^3)+\cdots}$$

Starting with Pólya 1937; Otter 1949; Harary-Robinson et al. 1970's; Meir-Moon 1978; Drmota-Lalley-Woods thm.$1990^{+}$

• "Holonomic" functions. Defined as solutions of linear ODE's with coeffs in $\mathbb{C}(z)$ [Zeilberger] $\equiv \mathcal{D}$-finite.

$$\mathcal{L}[f(z)] = 0, \qquad \mathcal{L} \in \mathbb{C}(z)[\partial_z].$$

• Stanley, Zeilberger, Gessel: great importance for combinatorial enumeration: Young tableaux and permutation statistics; regular graphs, constrained matrices, etc.

Fuchsian case (or "regular" singularity) $(X^\beta \log^k X)$:

$$[z^n]f(z) \approx \sum \omega^n n^\beta (\log n)^k, \qquad \omega, \beta \in \overline{\mathbb{Q}}, \quad k \in \mathbb{Z}_{\geq 0}.$$

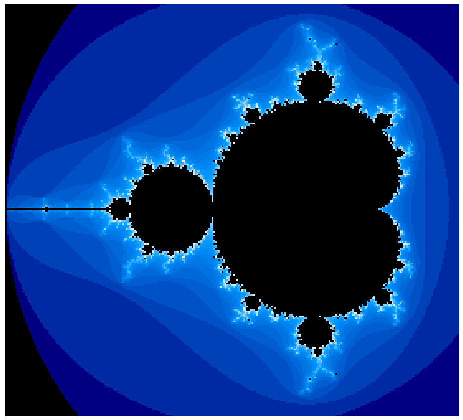
S.A. applies automatically to classical classification.

Asymptotics of coeff is decidable
— general case: modulo oracle for connection problem;
— strictly positive case: unconditionally.

EXAMPLE **5**. *Quadtrees—Partial Match* [FGPR'92]

Divide-and-conquer recurrence with coeff. in $\mathbb{Q}(n)$
Fuchsian equation of order $d$ (dimension) for GF
$Q_n^{(d=2)} \approx n^{(\sqrt{17}-3)/2}$.

E.g., $d = 2$: Hypergeom ${}_2F_1$ with algebraic arguments. □



## • Functional Equations and Substitution.

• Early example of *balanced 2–3 trees* by Odlyzko, 1979.

$$T(z) = z + T(\tau(z)), \qquad \tau(z) := z^2 + z^3.$$

Infinitely many exponents with common real part implies periodicities: $T_n \sim \frac{\phi^n}{n}\Omega(\log n)$.

• Singular iteration for *height of trees* (binary and other simple varieties; F-Gao-Odlyzko-Richmond; cf Rényi-Szekeres):

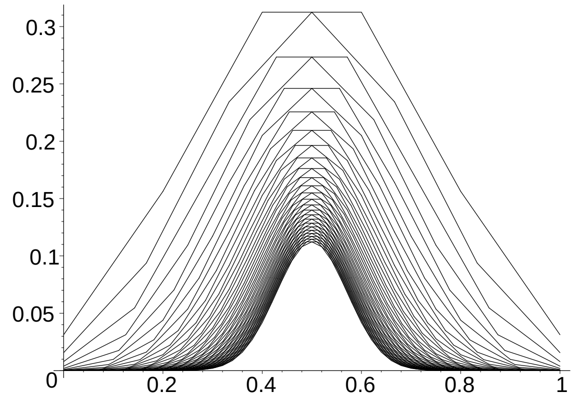
$$y_h = z + y_{h-1}^2, \qquad y_0 = z.$$

— Moments and convergence in law; Local limit law of $\vartheta$-type. (Applies to branching processes conditioned on total progeny.)

• *Digital search trees* via $q$–hypergeometrics: singularities accumulate geometrically $\leadsto$ periodicities [F-Richmond]:

$$\partial_z^k f(z) = t(z) + 2e^{z/2} f(\frac{z}{2}).$$

• *Order of binary trees* (Horton-Strahler, Register function; F-Prodinger) via Mellin tr. of GF and & singularities.




# 5. Limit Laws

Classical probability theory: sums of independent Random Variables $\leadsto$ powers of fixed function (probability GF, Fourier tr.) $\leadsto$ $\mathcal{N}$ormal Law.

For problems expressed by Bivariate GF (BGF): field founded by E. Bender et al. + developments by F, Soria, Hwang, ...

Idea: BGF $F(z,u)=\sum f_n(u)z^n$, where $f_n(u)$ describes parameter on objects of size $n$. If (for $u$ near 1)

$$f_n(u) \approx \omega(u)^{\kappa_n}, \qquad \kappa_n \to \infty,$$

then speak of Quasi-Powers approximation. Recycle continuity theorem, Berry-Esseen, Chernov, etc. $\Longrightarrow$ $\mathcal{N}$ormal law and many goodies...

(speed of convergence, large deviation fn, local limits)

Two important cases:

- Movable singularity:

$$F(z,u) \approx \left(1 - \frac{z}{\rho(u)}\right)^{-\alpha} \Longrightarrow \frac{f_n(u)}{f_n(1)} \approx \left(\frac{\rho(1)}{\rho(u)}\right)^{n}.$$

- Variable exponent:

$$F(z,u) \approx \left(1 - \frac{z}{\rho}\right)^{-\alpha(u)} \Longrightarrow \frac{f_n(u)}{f_n(1)} \approx \begin{cases} n^{\alpha(u)-\alpha(1)} \\ \left(e^{\alpha(u)-\alpha(1)}\right)^{\log n} \end{cases}.$$

Requires *uniformity* afforded by *Singularity Analysis* ($\neq$ Tauber or Darboux).

Singularity Perturbation analysis (smoothness)

$\downarrow$

Uniform Quasi-Powers for coeffs

$\downarrow$

$\mathcal{N}$ormal limit law

EXAMPLE **6**. *Polynomials over finite fields.*

```
> Factor(x^7+x+1) mod 29;
   3    2                2                 2
 (x  + x  + 3 x + 15) (x  + 25 x + 25) (x  + 3 x + 14)
```

- $\mathcal{P}$olynomial is a $\mathfrak{S}$*equence* of coeffs: $\mathcal{P}$ has Polar singularity.

- By unique factorization, $\mathcal{P}$ is also $\mathfrak{M}$*ultiset of* $\mathcal{I}$*rreducibles*:

  $\mathcal{I}$ has log singulariy.

$\Longrightarrow$ Prime Number Theorem for Polynomials $I_n \sim \dfrac{q^n}{n}$.

- Marking number of $\mathcal{I}$–factors is approx $u$th power:

$$P(z,u) \approx \left(e^{I(z)}\right)^u.$$

*Variable Exponent* $\Longrightarrow$ $\mathcal{N}$ormality of # of irred. factors.
(cf Erdős-Kac for integers.) □

(Analysis of algorithms, [F-Gourdon-Panario])

ACCGATCATTAGCAGATTATCATTTACTGAGAGTACTTAACATGCCA

**Example 7**. *Patterns in Random Strings*

= Perturbation of linear system of eqns.
(& many problems with finite automata, paths in graphs)

Linear system $X = X_0 + \mathbf{T}X$ w/ Perron-Frobenius.
Auxiliary mark $u$ induces smooth singularity displacement.
For "natural" problems: $\mathcal{N}$ormal limit law.

Also sets of patterns; similarly for patterns in increasing labelled trees, in permutations, etc. □

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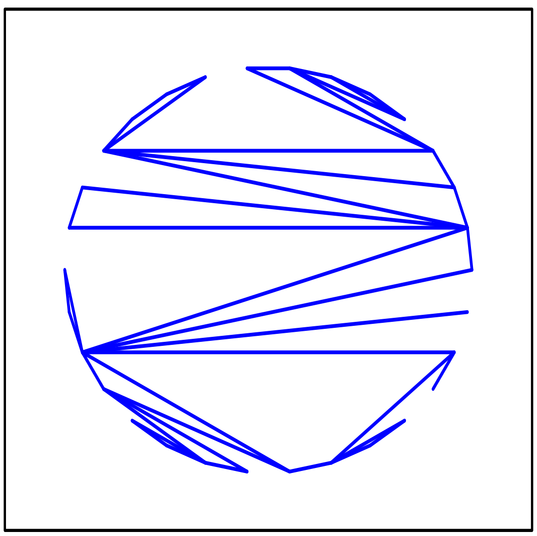
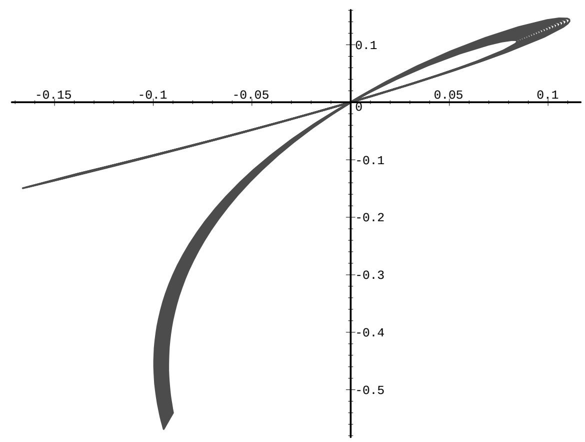


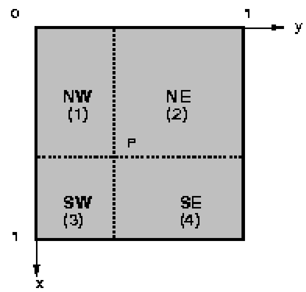


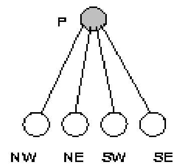

**Example 8**. *Non crossing graphs.* [F-Noy]
= Perturbation of algebraic equation.

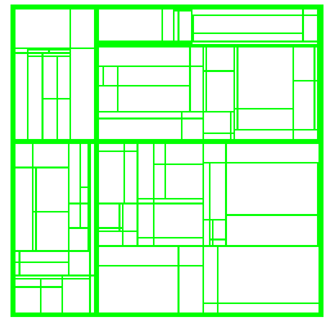
$$G^3 + (2z^2 - 3z - 2)G^2 + (3z + 1)G = 0$$
$$G^3 + (2u^3z^2 - 3u^2z + u - 3)G^2 + (3u^2 - 2u + 3)G + u - 1 = 0$$

Movable singularity scheme applies: $\mathcal{N}$ormality.

+ Patterns in context-free languages, in combinatorial tree models, in functional graphs [Cf. Drmota]. □







EXAMPLE **9**. *Profile of Quadtrees.*

$$F(z,u) = 1+2^3u\int_0^z \frac{dx_1}{x_1(1-x_1)}\int_0^{x_1}\frac{dx_2}{1-x_2}\int_0^{x_2} F(x_3,u)\frac{dx_3}{1-x_3}.$$

Solution is of the form $(1-z)^{-\alpha(u)}$ for algebraic branch $\alpha(u)$;
Variable Exponent $\Longrightarrow$ $\mathcal{N}$ormality of search costs. □

Applies to many linear differential models that then behave like *cycles-in-perms*.

- Coalescence of singularities and/or exponents: e.g. Airy Law $\equiv$ Stable($\frac{3}{2}$) [BFSS'01; cf also Pemantle].

# Conclusion

For combinatorial counting and limit laws:

Modest technical apparatus & *generic technology*.

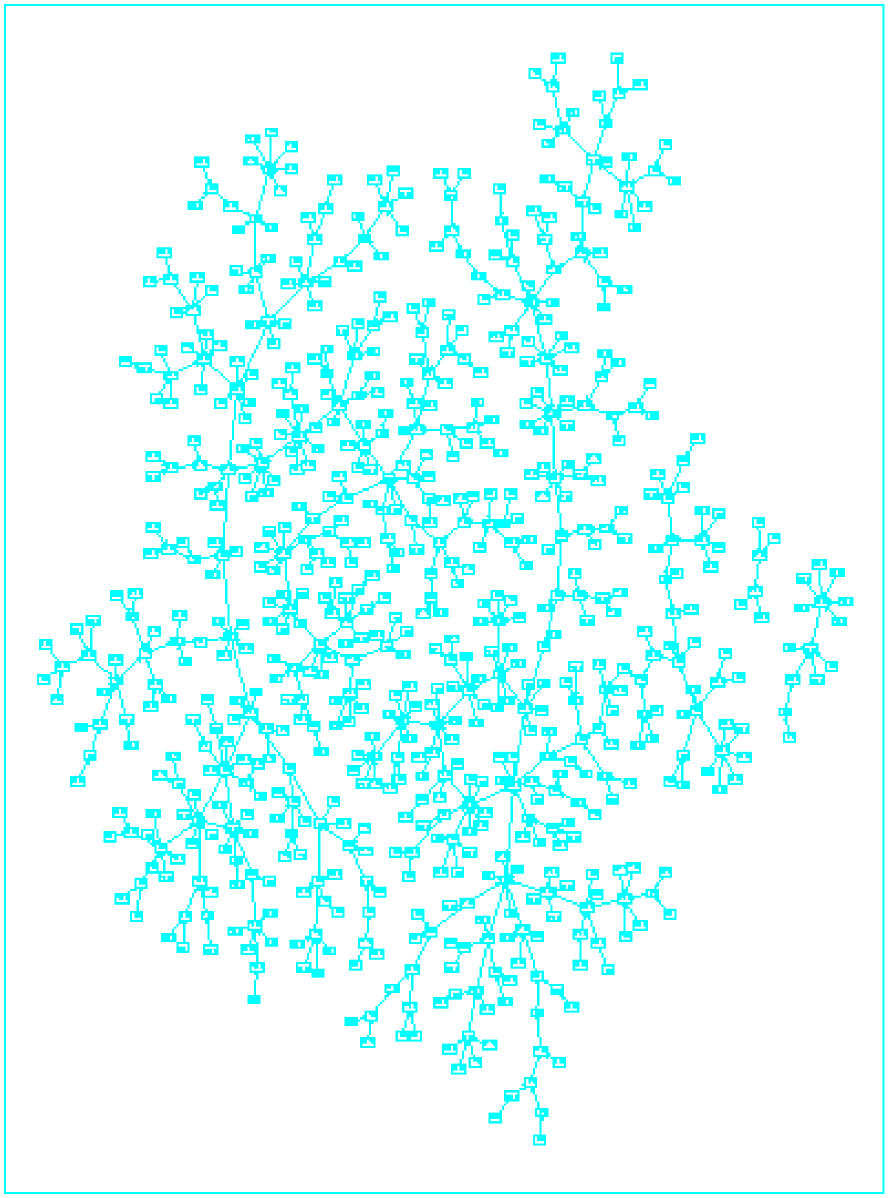
High-level for applications, esp., *analysis of algorithms*.

Plug-in on *Symbolic Combinatorics* & Symbolic Computation.

Discussion of *Schemas & Universality"* in metric aspects of random discrete structures.

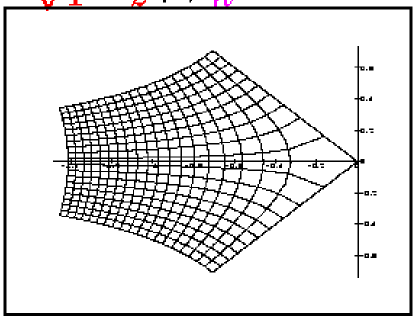
E.g. Borges' theorem for words, trees, labelled trees, mappings, permutations, increasing trees, etc.

THANK YOU!
♡♡♡

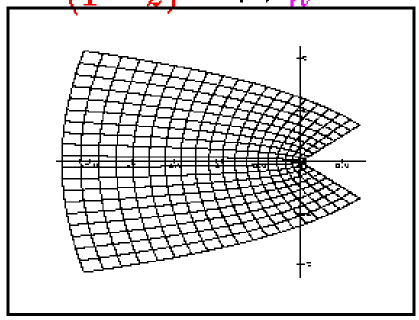


A random functional graph (mapping)

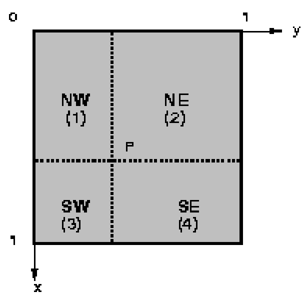
$-\sqrt{1-z} \mapsto n^{-3/2}$

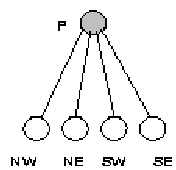


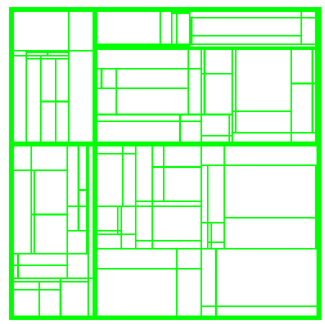
$-(1-z)^{3/2} \mapsto n^{-5/2}$



QTrees:



$$f^{\star}(s) := \int_0^{\infty} f^{\star}(x) x^{s-1}\, dx.$$