

The Fermat cubic, elliptic functions, continued fractions, and a combinatorial excursion

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1 “ALGEBRAIC” CONTINUED FRACTIONS

S (Stieltjes)

$$\frac{1}{z} \tan z = \frac{1}{1 - \frac{z^2}{3 - \frac{z^2}{5 - \frac{z^2}{\ddots}}}}$$

J (Jacobi)

$$\sum_{n \geq 0} n! \cdot z^n = \frac{1}{1 - 1 \cdot z - \frac{1^2 \cdot z^2}{1 - 3 \cdot z - \frac{2^2 \cdot z^2}{\ddots}}}$$

CF: Iterate $X \mapsto 1/X$; $X = [X] + \{X\}$. Here: $f = f(0) + zf'(0) + z^2\{f\}$.

(Irrationality of π (Lambert) and summation of divergent series (Euler). Also related to orthogonal polynomials Padé approximants, moment problems, etc.)

Explicit CFs are *very rare*: From Perron, Wall, Chihara, etc, *perhaps less than 100 continued fractions are known for special functions*.

Theorem (Apéry 1978): $\zeta(3) = \sum 1/n^3$ is irrational.

$$\zeta(3) = \frac{6}{\varpi(0) - \frac{1^6}{\varpi(1) - \frac{2^6}{\varpi(2) - \frac{3^6}{\ddots}}}}, \quad \text{with } \varpi(n) := (2n + 1)(17n(n + 1) + 5).$$

(Stieltjes) $\sum_{n \geq 0} \frac{1}{(n + z)^3} = \frac{1}{\sigma(0) - \frac{1}{\sigma(1) - \frac{2^6}{\sigma(2) - \frac{3^6}{\ddots}}}},$

with $\sigma(n) = (2n + 1)(2z(z + 1) + n(n + 1) + 1)$. Cf Berndt/Ramanujan.

Theorem (Conrad 2002): For a certain function sm :

$$\int_0^\infty \text{sm}(u) e^{-u/x} du = \frac{x^2}{1 + b_0 x^3 - \frac{1 \cdot 2^2 \cdot 3^2 \cdot 4 x^6}{1 + b_1 x^3 - \frac{4 \cdot 5^2 \cdot 6^2 \cdot 7 x^6}{1 + b_2 x^3 - \frac{7 \cdot 8^2 \cdot 9^2 \cdot 10 x^6}{\ddots}}}},$$

where $b_n = 2(3n + 1)((3n + 1)^2 + 1)$, and

$$\text{sm}(z) = \text{Inv} \int_0^z \frac{dt}{(1 - t^3)^{2/3}} = \text{Inv} z \cdot {}_2F_1 \left[\frac{1}{3}, \frac{2}{3}, \frac{4}{3}; z^3 \right].$$

Plan: some cute combinatorics surrounding the functions

- The Fermat cubic $x^3 + y^3 = 1$ and Dixonian functions
- A first model related to Pólya urns and branching processes
- A second model of Dixonian function by permutations
 - ★ based on parity constraints (cf Viennot, F., Dumont)
- A third model of Dixonian function by weighted Dyck paths, related to continued fractions, and permutations
 - ★ based on patterns of order 3 (cf F.-Françon)

Side effects: An analytic-combinatorial approach to urn processes that are 2×2 balanced.

2 FERMAT CURVES: CIRCLE & CUBIC

The Fermat curve \mathbf{F}_m is the complex algebraic curve

$$x^m + y^m = 1.$$

Circle \mathbf{F}_2 : Consider $s' = c, \quad c' = -s$, with $s(0) = 0, \quad c(0) = 1$.

The transcendental functions s, c do parameterize the circle,

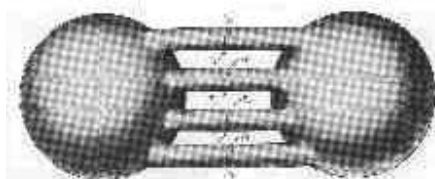
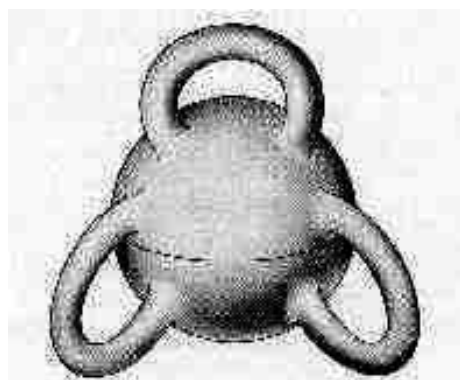
$$s(z)^2 + c(z)^2 = 1, \quad \text{since} \quad (s^2 + c^2)' = 2ss' + 2cc' = 2sc - 2cs = 0.$$

Also: inversion from *abelian integral* $\int R(z, y) dz$ on \mathbf{F}_2 :

$$\int_0^{\sin z} \frac{dt}{(1-t^2)^{1/2}} = z, \quad \cos(z) = \sqrt{1 - \sin(z)^2}$$

For combinatorialists: $\tan z = \frac{\sin z}{\cos z}, \quad \sec z = \frac{1}{\cos z}$ enumerate alternating (aka up-and-down, zig-zag) permutations (Désiré André, 1881).

The “complexity” of integral calculus over an algebraic curve depends on its (topological) **genus**.



Sphere with 3 holes, $g = 3$

For Fermat curve \mathbf{F}_p , genus is $\frac{1}{2}(p-1)(p-2)$.

- $\mathbf{F}_2 \implies g = 0$;
- $\mathbf{F}_3 \implies g = 1$; Normal forms of Weierstraß and Jacobi + Dixon;
- $\mathbf{F}_4 \implies g = 3, \dots$

A clever generalization of \sin, \cos : the *nonlinear* system

$$\boxed{s' = c^2, \quad c' = -s^2} \quad \text{with } s(0) = 0, c(0) = 1.$$

We have: $s(z)^3 + c(z)^3 = 1$: the pair $\langle s(z), c(z) \rangle$ parametrizes F_3 .

Follow **Dixon (1890)** and set: $\text{sm}(z) \equiv s(z), \quad \text{cm}(z) \equiv c(z)$.

(See sn, cn by Jacobi, sl, cl for lemniscate.)

$$\begin{cases} \text{sm}(z) &= z - 4\frac{z^4}{4!} + 160\frac{z^7}{7!} - 20800\frac{z^{10}}{10!} + 6476800\frac{z^{13}}{13!} - \dots \\ \text{cm}(z) &= 1 - 2\frac{z^3}{3!} + 40\frac{z^6}{6!} - 3680\frac{z^9}{9!} + 8880000\frac{z^{12}}{12!} - \dots \end{cases}$$

Alfred Cardew Dixon

Born: 22 May 1865 in Northallerton, Yorkshire, England

Died: 4 May 1936 in Northwood, Middlesex, England



Generally, ACD considers $X^3 + Y^3 - 3\alpha XY = 1$.

2.1 A hypergeometric connection.

One can make $s \equiv \text{sm}$ and $c \equiv \text{cm}$ somehow “explicit”.
Start from the defining system and differentiate

$$s' = c^2 \xrightarrow{\partial} s'' = 2cc' \xrightarrow{E} s'' = -2cs^2 \xrightarrow{E} s'' = -2c\sqrt{s'}.$$

Then “cleverly” multiply by $\sqrt{s'}$ to integrate (\int):

$$s''\sqrt{s'} = -2s^2s' \xrightarrow{\int} \frac{2}{3}(s')^{3/2} = -\frac{2}{3}s^3 + K.$$

$$\int_0^{\text{sm}(z)} \frac{dt}{(1-t^3)^{2/3}} = z, \quad \text{cm}(z) = \sqrt[3]{1 - \text{sm}(z)^3}$$

= Abelian integral over \mathbf{F}_3 + incomplete Beta integral + **hypergeometric**

Classical hypergeometric function:

$${}_2F_1[\alpha, \beta, \gamma; z] := 1 + \frac{\alpha \cdot \beta}{\gamma} \frac{z}{1!} + \frac{\alpha(\alpha + 1) \cdot \beta(\beta + 1)}{\gamma(\gamma + 1)} \frac{z^2}{2!} + \dots$$

$\text{Inv}(f)$ is the inverse of f w.r.t. composition: $\text{Inv}(f) = g$ if $f \circ g = g \circ f = \text{Id}$.

Proposition: *Function sm is defined by inversion,*

$$\text{sm}(z) = \text{Inv} \int_0^z \frac{dt}{(1 - t^3)^{2/3}} = \text{Inv} z \cdot {}_2F_1 \left[\frac{1}{3}, \frac{2}{3}, \frac{4}{3}; z^3 \right].$$

The function cm is then defined near 0 by $\text{cm}(z) = \sqrt[3]{1 - \text{sm}^3(z)}$.

3 A STARTLING FRACTION.

From Eric van Fossen CONRAD, PhD Columbus, OH, 2002.



$$\int_0^{\infty} \text{sm}(u) e^{-u/x} du = \frac{x^2}{1 + b_0 x^3 - \frac{1 \cdot 2^2 \cdot 3^2 \cdot 4 x^6}{1 + b_1 x^3 - \frac{4 \cdot 5^2 \cdot 6^2 \cdot 7 x^6}{1 + b_2 x^3 - \frac{7 \cdot 8^2 \cdot 9^2 \cdot 10 x^6}{\ddots}}}},$$

where $b_n = 2(3n + 1)((3n + 1)^2 + 1)$.

Proof: Follow Stieltjes and Rogers. Cleverly introduce

$$S_n := \int_0^\infty sm^n(u) e^{-u/x} du.$$

Then integration by parts shows that

$$\frac{S_n}{S_{n-3}} = \frac{n(n-1)(n-2)x^3}{1 + 2n(n^2 + 1)x^3 - n(n+1)(n+2)x^3 \frac{S_{n+3}}{S_n}}.$$

\implies “Pump” out the continued fraction.

Six J -fractions: $sm, sm^2, sm^3, cm, cm \cdot sm, cm \cdot sm^2$:

$$1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, 7, 7, 8, 8, \dots$$

+ **Three** S -fractions: $sm, cm, sm \cdot cm$.

4 **BALLS GAMES**

Cf. *Théorie analytique des probabilités* Laplace (1812).

Pólya urn model. *An urn contains black and white balls. At each epoch, a ball in the urn is chosen at random.*

Described by a placement matrix. Here:

$$\mathcal{M}_{12} = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}, \quad \begin{array}{l} \bigcirc \longrightarrow \bullet \bullet \\ \bullet \longrightarrow \bigcirc \bigcirc \end{array}$$

A *history* of length n (Françon78) is any description of a legal sequence of n moves of the Pólya urn. For instance ($n = 5$):

$$\underline{x} \longrightarrow y\underline{y} \longrightarrow y\underline{x}x \longrightarrow \underline{y}yyx \longrightarrow x\underline{x}yyx \longrightarrow xy\underline{y}yyx,$$

What are the “history numbers”? The sequence for $(1, 0) \mapsto (0, \star)$ starts as 0, 1, 0, 0, 4, 0, 0, 160. Cf *sm*?

4.1 Urns and Dixonian functions.

Take the (autonomous, nonlinear) ordinary differential system

$$\Sigma : \quad \frac{dx}{dt} = y^2, \quad \frac{dy}{dt} = x^2, \quad \text{with } x(0) = x_0, \quad y(0) = y_0,$$

$\langle x(t), y(t) \rangle$ parameterizes the “Fermat hyperbola”: $y^3 - x^3 = 1$.

For $x_0 = 0, y_0 = 1$, get trivial variants: $\text{smh}(z) = -\text{sm}(-z), \text{cmh}(z) = \text{cm}(-z)$.

Define a linear transformation δ acting on polynomials $\mathbb{C}[x, y]$:

$$\delta[x] = y^2, \quad \delta[y] = x^2, \quad \delta[u \cdot v] = \delta[u] \cdot v + u \cdot \delta[v],$$

(Cf the elegant presentation of Chen grammars by (Dumont96) and the “combinatorial integral calculus” of Leroux–Viennot.)

(i) **Combinatorially**, the n th iterate $\delta^n[x^a y^b]$ is such that

$$\# \text{ histories from } (a_0, b_0) \text{ to } (k, \ell) = \text{coeff}[x^k y^\ell] \delta^n[x^{a_0} y^{b_0}],$$

(ii) **Algebraically**, the operator δ describes the “logical consequences” of the differential system $\Sigma = \{\dot{x} = y^2, \dot{y} = x^2\}$:

$$\delta^n[x^a y^b] = \frac{d^n}{dt^n} x(t)^a y(t)^b \quad \text{expressed in } x(t), y(t),$$

♡ **Taylor** $\implies H(x(t), y(t); z) = x(t+z)^{a_0} y(t+z)^{b_0}$; set $t = 0 \dots$



Combinatorial Interpretation I



Proposition: The EGFs of histories of the urn \mathcal{M}_{12} starting with one ball: and ending with balls ...

All of the other colour: $\frac{\text{sm}(z)}{\text{cm}(z)} = -\text{sm}(-z)$.

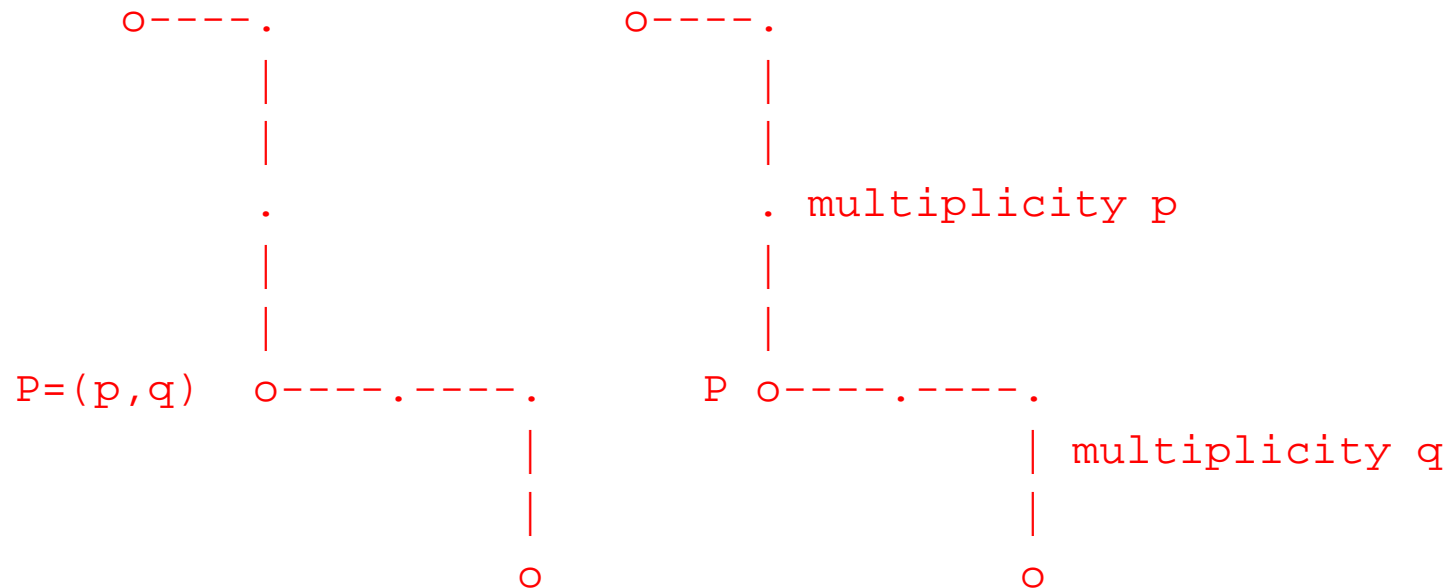
All of the original colour: $\frac{1}{\text{cm}(z)} = \text{cm}(-z)$.

Homogeneous monomial **differential systems** $\iff k \times k$ **balanced urns**.

Note: Get full composition (=Gaussian), large deviations, etc.

$$\mathbb{P}(X_n = 0) \sim c\rho^{-n}, \quad \rho = \frac{\sqrt{3}}{6\pi} \Gamma\left(\frac{1}{3}\right)^3, \quad n \equiv 1 \pmod{3}.$$

Note: The knight's moves of Bousquet-Melou & Petkovšek.



The OGF of walks that start at $(1, 0)$ and end on the horizontal axis is

$$G(x) = \sum_{i \geq 0} (-1)^i \left(\xi^{(i)}(x) \xi^{(i+1)}(x) \right)^2,$$

where ξ , a branch of the (genus 0) cubic $x\xi - x^3 - \xi^3 = 0$ is $\xi(x) = x^2 \sum_{m \geq 0} \binom{3m}{m} \frac{x^{3m}}{2m+1}$.

4.2 Continuous-time branching = Yule process.

Foatons and Viennons live an exponential time and disintegrate. . .



Proposition. Consider the *Yule process with two types of particles*. The probabilities that *particles are all of the second type at time t* are

$$X(t) = e^{-t} \operatorname{smh}(1 - e^{-t}), \quad Y(t) = e^{-t} \operatorname{cmh}(1 - e^{-t}),$$

depending on whether the system *at time 0 is initialized with one particle of the first type (X) or of the second type (Y)*.

Remarks on urn processes. For urn

$$\begin{pmatrix} -\alpha & \beta \\ \gamma & -\delta \end{pmatrix}, \quad -\alpha + \beta = \gamma - \delta,$$

associate a **partial differential operator**:

$$\Gamma = x^{1-\alpha} y^\beta \frac{\partial}{\partial x} + x^\gamma y^{1-\delta} \frac{\partial}{\partial y}.$$

~> Develop a general theory of *Pólya Urn Processes* (FIDuPu06).

~> Can characterize all **six** matrices such that $e^{z\Gamma}$ is expressible by **elliptic functions** (FIGaPe05). One such model $\in \{\mathbf{sm}, \mathbf{cm}\}$.

$$\begin{aligned}
 A &= \begin{pmatrix} -2 & 3 \\ 4 & -3 \end{pmatrix}, & B &= \begin{pmatrix} -1 & 2 \\ 3 & -2 \end{pmatrix}, & C &= \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}, \\
 D &= \begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix}, & E &= \begin{pmatrix} -1 & 3 \\ 5 & -3 \end{pmatrix}, & F &= \begin{pmatrix} -1 & 4 \\ 5 & -2 \end{pmatrix}.
 \end{aligned}$$

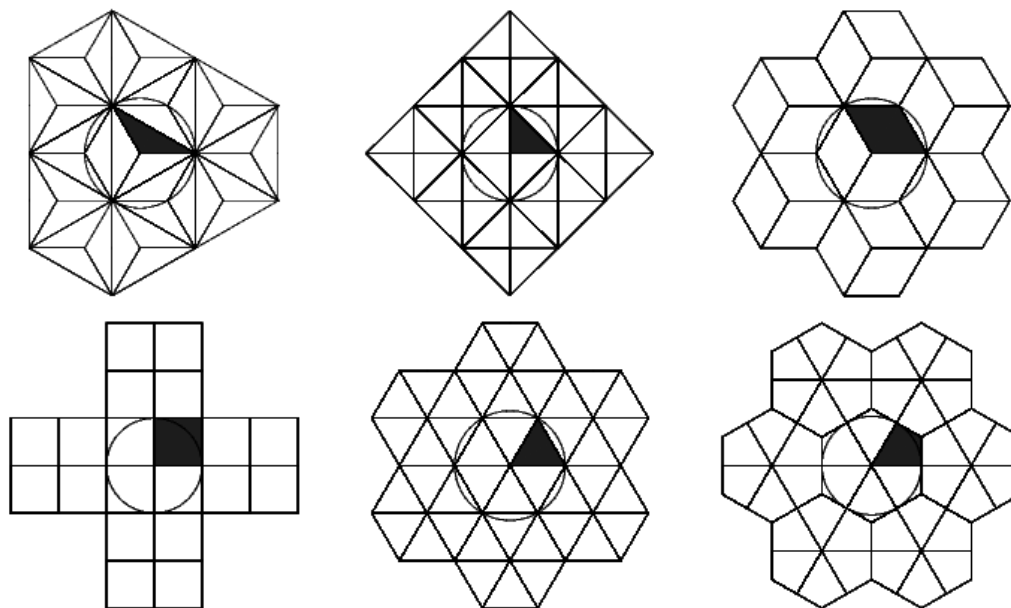
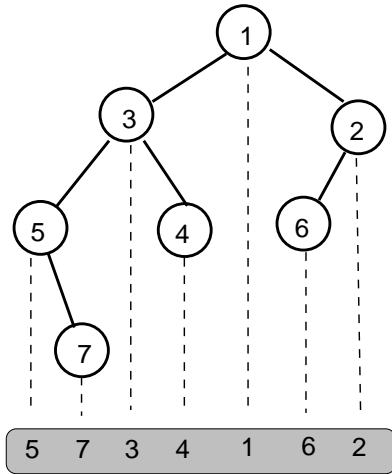


FIGURE 7. The six elliptic cases in order A, B, C, D, E, F : The diagrams formed by the fundamental polygon together with its rotated images. (The elementary kite is darkened.)

5 FIRST PERMUTATION MODEL

A permutation can always be represented as a tree, which is binary, rooted, and increasing.



$$\text{Tree}(w) = \langle \xi, \text{Tree}(w'), \text{Tree}(w'') \rangle$$

Level of node \equiv distance to root. **Type of node** \rightsquigarrow Peak, Valley, db-rise, db-fall.

Peaks	Valleys	Double rises	Double falls
$\sigma_{j-1} < \sigma_j > \sigma_{j+1}$	$\sigma_{j-1} > \sigma_j < \sigma_{j+1}$	$\sigma_{j-1} < \sigma_j < \sigma_{j+1}$	$\sigma_{j-1} > \sigma_j > \sigma_{j+1}$



Combinatorial Interpretation II



Proposition: Consider the class \mathcal{X} (resp. \mathcal{Y}) of permutations such that elements at any odd (resp. even) level are valleys only. Then the exponential generating functions are

$$X(z) = \text{smh}(z) = -\text{sm}(-z), \quad Y(z) = \text{cmh}(z) = \text{cm}(-z).$$

(Follows from standard combinatorics, reading off $X' = Y^2, Y' = X^2$.)

Other interpretations based on **parity**:

- **Viennot**, a *first* in 1980: Jacobi permutations, alternate reverse. . .
- Flajolet: alternating permutations, parity of peaks.
- Dumont on Schett, based on cycle structure. . .

6 THE SECOND PERMUTATION MODEL

Inspired by Fl-Françon (1989) = a model for Jacobi sn, cn when $r = 2$.

Definition: An r -repeated permutation of size rn is a permutation such that for each j , the (existing) elements $jr + 1, jr + 2, \dots, jr + r - 1$ are all of the same ordinal type (P, V, DR, DF).

Proposition: Ordinary generating function for r -repeated is:

$$\sum_{n \geq 0} R_{rn} z^n = \frac{1}{1 - 2 \cdot 1^r z - \frac{1 \cdot 2^2 \cdots r^2 \cdot (r+1) \cdot z^2}{1 - 2 \cdot (r+1)^r z - \frac{(r+1) \cdot (r+2)^2 \cdots (2r)^2 \cdot (2r+1) \cdot z^2}{\ddots}}},$$

Numerators of degree $2r$; denominators of degree r .

6.1 Combinatorial aspects of continued fractions.

A lattice path aka Motzkin path is a sequence $s = (s_0, s_1, \dots, s_n)$:

$$s_0 = s_n = 0, \quad s_j \in \mathbb{Z}_{\geq 0}, \quad |s_{j+1} - s_j| \in \{-1, 0, +1\}.$$

Let $P(\mathbf{a}, \mathbf{b}, \mathbf{c})$ be the infinite-variable generating function of lattice paths with ascent $\leftrightarrow a_k$, descent $\leftrightarrow b_k$, level $\leftrightarrow c_k$.

Theorem (what Foata calls “the shallow Flajolet Theorem”):

$$P(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \frac{1}{1 - c_0 - \frac{a_0 b_1}{1 - c_1 - \frac{a_1 b_2}{1 - c_2 - \frac{a_2 b_3}{\ddots}}}}.$$

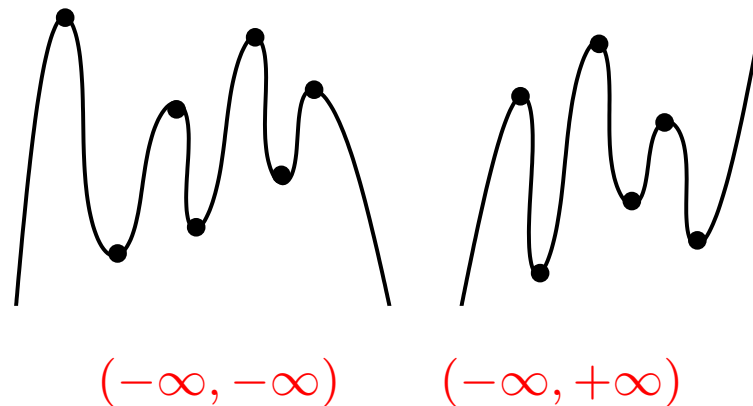


$$\begin{aligned} m(\varpi) &= a_0 a_1 c_2 b_2 a_1 a_2 b_3 b_2 b_1 a_0 b_1 \\ &= a_0^2 a_1^2 a_2 b_1^2 b_2^2 b_3 c_1. \end{aligned}$$

6.2 Lattice paths and permutations.

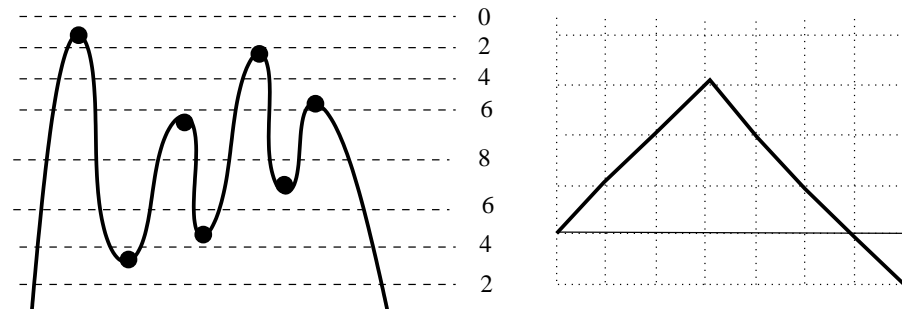
- A **bijection** due to Françon-Viennot (1979);
- What V.I. Arnold (2000) calls *snakes*

Consider piecewise monotonic smooth functions from \mathbb{R} to \mathbb{R} , such that all the critical values are different, and take the equivalence classes up to orientation preserving maps of \mathbb{R}^2 .



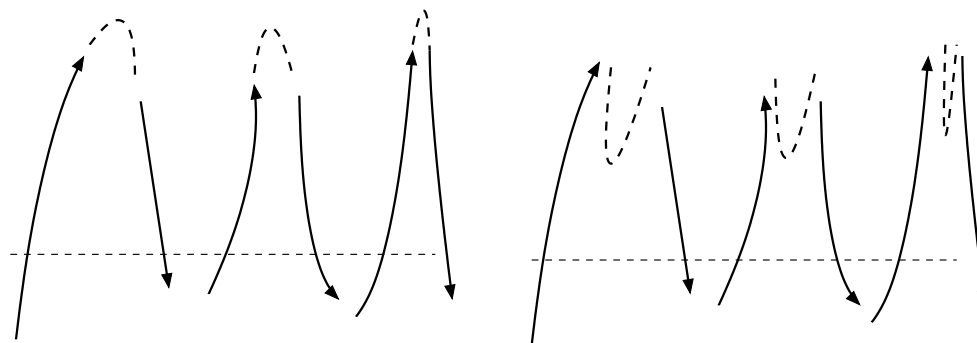
Clearly an equivalence class is an alternating permutation, and by André's theorem the EGFs are $\tan(z) = \frac{\sin(z)}{\cos(z)}$, $\sec(z) = \frac{1}{\cos(z)}$.

The swepline algorithm: a snake and its associated Dyck path.



An encoding is obtained by the system of possibilities:

$$\Pi^{\text{odd}} : \quad \alpha_j = (j + 1), \quad \beta_j = (j + 1), \quad \gamma_j = 0.$$



$$\int_0^\infty \tan(zt)e^{-t} dt = \frac{z}{1 - \frac{1 \cdot 2 z^2}{1 - \frac{2 \cdot 3 z^2}{\ddots}}}, \quad \int_0^\infty \sec(zt)e^{-t} dt = \frac{1}{1 - \frac{1^2 z^2}{1 - \frac{2^2 z^2}{\ddots}}}.$$

All perms: encode double rises and double falls by level steps.

$$\sum_{n=1}^\infty n!z^n = \frac{z}{1 - 2z - \frac{1 \cdot 2 z^2}{1 - 4z - \frac{2 \cdot 3 z^2}{\ddots}}}, \quad \sum_{n=0}^\infty n!z^n = \frac{1}{1 - z - \frac{1^2 z^2}{1 - 3z - \frac{2^2 z^2}{\ddots}}}.$$

Stieltjes +Euler

6.3 The model of 3-repeated permutations.



Combinatorial Interpretation III



Proposition: The exponential generating function of *3-repeated polarized permutations* bordered by $(-\infty, -\infty)$ is

$$\text{smh}(z).$$

Notes: By Fl-Françon, *2-repeated* + recording rises (cf Eulerian #'s) gives Jacobian sn, cn, dn .

Corollary: *Combinatorial proofs of Conrad's fractions.*

Also: $\wp(z - \zeta_0, 0, -1)$ expanded near its real zero, $\zeta_0 = \frac{1}{3\pi}\Gamma\left(\frac{1}{3}\right)^3$, has CF expansion with cubic denominators and sextic numerators.

$$\wp(z - \zeta_0; 0, -1) \equiv \text{smh}(z) \cdot \text{cmh}(z) = \text{Inv } Y \cdot {}_2F_1\left[\frac{1}{3}, \frac{1}{2}, \frac{4}{3}; -4Y^2\right].$$

7 PERSPECTIVES & QUESTIONS

Worth looking at nonlinear differential systems associated to algebraic curves and their Abelian integrals?

- **Q.** Three types of balls? (cf Schett-Dumont for a special elliptic case) $A \rightarrow BC, B \rightarrow CA, C \rightarrow AB$ is elliptic (sn, cn); hyperelliptic generalizations.
- **Q.** What about **numerators** like k^6 and such in CF? Combinatorics?
- **Q.** Anything to say about **orthogonal polynomials** (cf Carlitz for sn, cn)? Cf Galiano Valent *et al.* — very intriguing!
- **Q.** Any possibility of **enumerating directly r -repeated perms** for $r \geq 4$?
- **Q.** Anything (combinatorially) interesting regarding **higher order systems** associated to \mathbf{F}_p for $p > 3$?
At least consequences for urn models (FIDuPu06).