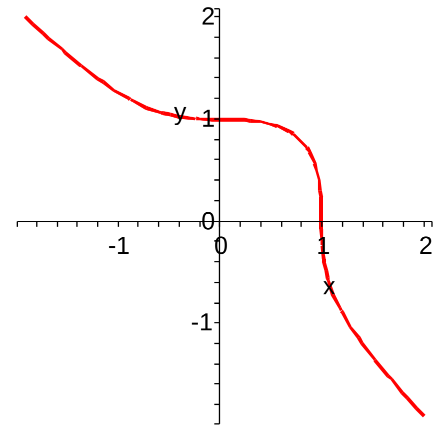

The Fermat cubic, elliptic functions, continued fractions, and a combinatorial excursion

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1 “ALGEBRAIC” CONTINUED FRACTIONS

S (Stieltjes)

$$\frac{1}{z}\tan z = \cfrac{1}{1 - \cfrac{z^2}{3 - \cfrac{z^2}{5 - \cfrac{z^2}{\ddots}}}},$$

J (Jacobi)

$$\sum_{n\geq 0} n!\cdot z^n = \cfrac{1}{1 - 1\cdot z - \cfrac{1^2\cdot z^2}{1 - 3\cdot z - \cfrac{2^2\cdot z^2}{\ddots}}}.$$

CF: Iterate $X \mapsto 1/X$; $X = \lfloor X \rfloor + \{X\}$. Here: $f = f(0) + zf'(0) + z^2\text{“}\{f\}\text{”}$.

(Irrationality of π (Lambert) and summation of divergent series (Euler). Also related to orthogonal polynomials Padé approximants, moment problems, etc.)

Explicit CFs are *very rare*: From Perron, Wall, Chihara, etc, *perhaps less than 100 continued fractions are known for special functions.*

Theorem (Apéry 1978): $\zeta(3) = \sum 1/n^3$ *is irrational.*

$$\zeta(3) = \cfrac{6}{\varpi(0) - \cfrac{1^6}{\varpi(1) - \cfrac{2^6}{\varpi(2) - \cfrac{3^6}{\ddots}}}},$$

with $\varpi(n) := (2n+1)(17n(n+1)+5)$.

(Stieltjes) $$\sum_{n\geq 0} \frac{1}{(n+z)^3} = \cfrac{1}{\sigma(0) - \cfrac{1^6}{\sigma(1) - \cfrac{2^6}{\sigma(2) - \cfrac{3^6}{\ddots}}}},$$

with $\sigma(n) = (2n+1)(2z(z+1)+n(n+1)+1)$. Cf Berndt/Ramanujan.

Theorem (Conrad 2002): For a certain function sm:

$$\int_0^\infty \mathrm{sm}(u) e^{-u/x}\, du = \cfrac{x^2}{1 + b_0 x^3 - \cfrac{1\cdot 2^2 \cdot 3^2 \cdot 4\, x^6}{1 + b_1 x^3 - \cfrac{4 \cdot 5^2 \cdot 6^2 \cdot 7\, x^6}{1 + b_2 x^3 - \cfrac{7 \cdot 8^2 \cdot 9^2 \cdot 10\, x^6}{\ddots}}}},$$

where $b_n = 2(3n+1)((3n+1)^2+1)$, and

$$\mathrm{sm}(z) = \mathrm{Inv} \int_0^z \frac{dt}{(1-t^3)^{2/3}} = \mathrm{Inv}\, z \cdot {}_2F_1\left[\frac{1}{3}, \frac{2}{3}, \frac{4}{3}; z^3\right].$$

Plan: some cute combinatorics surrounding the functions

— The Fermat cubic $x^3 + y^3 = 1$ and Dixonian functions

— A first model related to Pólya urns and branching processes

— A second model of Dixonian function by permutations

⋆ based on parity constraints (cf Viennot, F., Dumont)

— A third model of Dixonian function by weighted Dyck paths, related to continued fractions, and permutations

⋆ based on patterns of order 3 (cf F.-Françon)

Side effects: An analytic-combinatorial approach to *urn processes* that are 2×2 balanced.

2 FERMAT CURVES: CIRCLE & CUBIC

The Fermat curve $\mathbf{F}_m$ is the complex algebraic curve

$$x^m + y^m = 1.$$

Circle $\mathbf{F}_2$: Consider $\boxed{s' = c, \quad c' = -s}$, with $s(0) = 0, \quad c(0) = 1.$

The transcendental functions s, c do parameterize the circle,

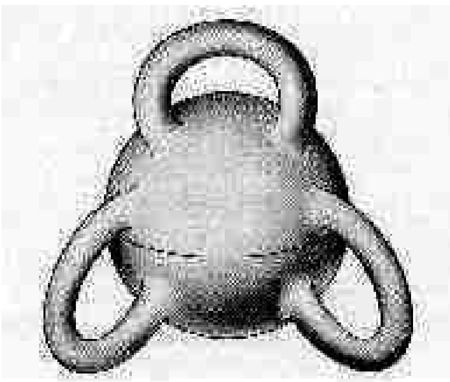
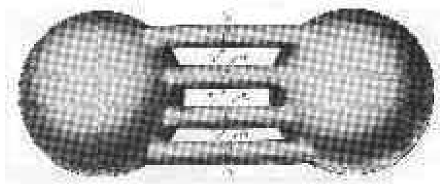
$$s(z)^2 + c(z)^2 = 1, \qquad \text{since} \qquad (s^2 + c^2)' = 2ss' + 2cc' = 2sc - 2cs = 0.$$

Also: inversion from *abelian integral* $\int R(z, y)\, dz$ on $\mathbf{F}_2$:

$$\int_0^{\sin z} \frac{dt}{(1 - t^2)^{1/2}} = z, \qquad \cos(z) = \sqrt{1 - \sin(z)^2}$$

For combinatorialists: $\tan z = \frac{\sin z}{\cos z}, \qquad \sec z = \frac{1}{\cos z}$ enumerate alternating (aka up-and-down, zig-zag) permutations (Désiré André, 1881).

The “complexity” of integral calculus over an algebraic curve depends on its (topological) genus.

Sphere with 3 holes, $g = 3$

For Fermat curve $\mathbf{F}_p$, genus is $\frac{1}{2}(p-1)(p-2)$.

- $\mathbf{F}_2 \Longrightarrow g = 0$;
- $\mathbf{F}_3 \Longrightarrow g = 1$; Normal forms of Weierstraß and Jacobi + Dixon;
- $\mathbf{F}_4 \Longrightarrow g = 3, \ldots$

A clever generalization of $\sin, \cos$: the *nonlinear* system

$$\boxed{s' = c^2, \qquad c' = -s^2} \qquad \text{with } s(0) = 0,\ c(0) = 1.$$

We have: $s(z)^3 + c(z)^3 = 1$: the pair $\langle s(z), c(z) \rangle$ parametrizes $\mathbf{F}_3$.
Follow **Dixon (1890)** and set: $\mathrm{sm}(z) \equiv s(z), \qquad \mathrm{cm}(z) \equiv c(z)$.
(See sn, cn by Jacobi, sl, cl for lemniscate.)

$$\begin{cases} \mathrm{sm}(z) & = & z - 4\dfrac{z^4}{4!} + 160\dfrac{z^7}{7!} - 20800\dfrac{z^{10}}{10!} + 6476800\dfrac{z^{13}}{13!} - \cdots \\ \mathrm{cm}(z) & = & 1 - 2\dfrac{z^3}{3!} + 40\dfrac{z^6}{6!} - 3680\dfrac{z^9}{9!} + 8880000\dfrac{z^{12}}{12!} - \cdots . \end{cases}$$

Alfred Cardew Dixon

Born: 22 May 1865 in Northallerton, Yorkshire, England

Died: 4 May 1936 in Northwood, Middlesex, England

Generally, ACD considers $X^3 + Y^3 - 3\alpha XY = 1$.

2.1 A hypergeometric connection.

One can make $s \equiv \mathrm{sm}$ and $c \equiv \mathrm{cm}$ somehow "explicit".
Start from the defining system and differentiate

$$s' = c^2 \quad \stackrel{\partial}{\Longrightarrow} \quad s'' = 2cc' \quad \stackrel{E}{\Longrightarrow} \quad s'' = -2cs^2 \quad \stackrel{E}{\Longrightarrow} \quad s'' = -2c\sqrt{s'}.$$

Then "cleverly" multiply by $\sqrt{s'}$ to integrate ($\int$):

$$s''\sqrt{s'} = -2s^2 s' \quad \stackrel{\int}{\Longrightarrow} \quad \frac{2}{3}(s')^{3/2} = -\frac{2}{3}s^3 + K.$$

$$\int_0^{\mathrm{sm}(z)} \frac{dt}{(1-t^3)^{2/3}} = z, \qquad \mathrm{cm}(z) = \sqrt[3]{1 - \mathrm{sm}(z)^3}$$

= Abelian integral over $\mathbf{F}_3$ + incomplete Beta integral + hypergeometric

Classical hypergeometric function:

$$ {}_2F_1[\alpha, \beta, \gamma; z] := 1 + \frac{\alpha \cdot \beta}{\gamma} \frac{z}{1!} + \frac{\alpha(\alpha+1) \cdot \beta(\beta+1)}{\gamma(\gamma+1)} \frac{z^2}{2!} + \cdots . $$

$\mathrm{Inv}(f)$ is the inverse of f w.r.t. composition: $\mathrm{Inv}(f) = g$ if $f \circ g = g \circ f = Id$.

Proposition: *Function* sm *is defined by inversion,*

$$ \mathrm{sm}(z) = \mathrm{Inv} \int_0^z \frac{dt}{(1-t^3)^{2/3}} = \mathrm{Inv}\; z \cdot {}_2F_1\left[\frac{1}{3}, \frac{2}{3}, \frac{4}{3}; z^3\right]. $$

The function cm *is then defined near* 0 *by* $\mathrm{cm}(z) = \sqrt[3]{1 - \mathrm{sm}^3(z)}$.

3 A STARTLING FRACTION.

From Eric van Fossen Conrad, PhD Columbus, OH, 2002.

$$\int_0^\infty \mathrm{sm}(u) e^{-u/x}\, du = \cfrac{x^2}{1 + b_0 x^3 - \cfrac{1\cdot 2^2 \cdot 3^2 \cdot 4\, x^6}{1 + b_1 x^3 - \cfrac{4 \cdot 5^2 \cdot 6^2 \cdot 7\, x^6}{1 + b_2 x^3 - \cfrac{7 \cdot 8^2 \cdot 9^2 \cdot 10\, x^6}{\ddots}}}},$$

where $\quad b_n = 2(3n+1)((3n+1)^2+1).$

Proof: Follow Stieltjes and Rogers. Cleverly introduce

$$S_n := \int_0^\infty \mathrm{sm}^n(u) e^{-u/x}\, du.$$

Then integration by parts shows that

$$\frac{S_n}{S_{n-3}} = \frac{n(n-1)(n-2)x^3}{1 + 2n(n^2+1)x^3 - n(n+1)(n+2)x^3 \dfrac{S_{n+3}}{S_n}}.$$

$\Longrightarrow$ "Pump" out the continued fraction.

Six J-fractions: $\mathrm{sm}, \mathrm{sm}^2, \mathrm{sm}^3, \mathrm{cm}, \mathrm{cm}\cdot\mathrm{sm}, \mathrm{cm}\cdot\mathrm{sm}^2$:

$$1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, 7, 7, 8, 8, \ldots$$

+ **Three** S-fractions: $\mathrm{sm}, \mathrm{cm}, \mathrm{sm}\cdot\mathrm{cm}$.

4 BALLS GAMES

Cf. *Théorie analytique des probabilités* Laplace (1812).

> **Pólya urn model.** *An urn contains black and white balls. At each epoch, a ball in the urn is chosen at random.*

Described by a placement matrix. Here:

$$\mathcal{M}_{12} = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}, \qquad \begin{array}{ccc} \bigcirc & \longrightarrow & \bullet\bullet \\ \bullet & \longrightarrow & \bigcirc\bigcirc \end{array}$$

A *history* of length n (Françon78) is any description of a legal sequence of n moves of the Pólya urn. For instance ($n = 5$):

$$\underline{\mathsf{x}} \longrightarrow \mathsf{y}\underline{\mathsf{y}} \longrightarrow \mathsf{y}\underline{\mathsf{x}}\mathsf{x} \longrightarrow \underline{\mathsf{y}}\mathsf{yyx} \longrightarrow \mathsf{x}\underline{\mathsf{x}}\mathsf{yyx} \longrightarrow \mathsf{xyyyyx},$$

What are the "history numbers"? The sequence for $(1,0) \mapsto (0,\star)$ starts as $0, 1, 0, 0, 4, 0, 0, 160$. Cf sm?

4.1 Urns and Dixonian functions.

Take the (autonomous, nonlinear) ordinary differential system

$$\Sigma : \qquad \frac{dx}{dt} = y^2, \quad \frac{dy}{dt} = x^2, \quad \text{with} \quad x(0) = x_0, \quad y(0) = y_0,$$

$\langle x(t), y(t) \rangle$ parameterizes the "Fermat hyperbola": $y^3 - x^3 = 1$.

For $x_0 = 0, y_0 = 1$, get trivial variants: $\text{smh}(z) = -\text{sm}(-z), \text{cmh}(z) = \text{cm}(-z)$.

Define a linear transformation δ acting on polynomials $\mathbb{C}[x, y]$:

$$\delta[x] = y^2, \qquad \delta[y] = x^2, \qquad \delta[u \cdot v] = \delta[u] \cdot v + u \cdot \delta[v],$$

(Cf the elegant presentation of Chen grammars by (Dumont96) and the "combinatorial integral calculus" of Leroux–Viennot.)

(i) **Combinatorially**, the nth iterate $\delta^n[x^a y^b]$ is such that

$$\#\ \text{histories from } (a_0, b_0) \text{ to } (k, \ell) = \text{coeff}[x^k y^\ell]\, \delta^n[x^{a_0} y^{b_0}],$$

(ii) **Algebraically**, the operator δ describes the "logical consequences" of the differential system Σ = $\{\dot{x} = y^2, \dot{y} = x^2\}$:

$$\delta^n[x^a y^b] = \frac{d^n}{dt^n} x(t)^a y(t)^b \quad \text{expressed in} \quad x(t), y(t),$$

♡ **Taylor** $\Longrightarrow H(x(t), y(t); z) = x(t+z)^{a_0} y(t+z)^{b_0}$; set $t = 0 \ldots$

♡♡♡ **Combinatorial Interpretation I** ♡♡♡

Proposition: The EGFs of histories of the urn $\mathcal{M}_{12}$ starting with one ball: and ending with balls . . .

All of the other colour: $\dfrac{\operatorname{sm}(z)}{\operatorname{cm}(z)} = -\operatorname{sm}(-z)$.

All of the original colour: $\dfrac{1}{\operatorname{cm}(z)} = \operatorname{cm}(-z)$.

Homogeneous monomial **differential systems** $\iff$ $k \times k$ **balanced urns**.

Note: Get full composition (=Gaussian), large deviations, etc.

$$\mathbb{P}(X_n = 0) \sim c\rho^{-n}, \qquad \rho = \frac{\sqrt{3}}{6\pi}\Gamma\left(\frac{1}{3}\right)^3, \quad n \equiv 1 \pmod 3.$$

Note: The knight's moves of Bousquet-Melou & Petkovšek.

```
    o----.             o----.
         |                  |
         |                  |
         .                  . multiplicity p
         |                  |
         |                  |
P=(p,q)  o----.----.      P o----.----.
                   |                  | multiplicity q
                   |                  |
                   o                  o
```

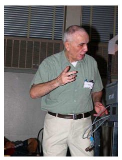
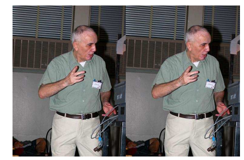
The OGF of walks that start at $(1,0)$ and end on the horizontal axis is

$$G(x) = \sum_{i \geq 0} (-1)^i \left(\xi^{\langle i \rangle}(x) \xi^{\langle i+1 \rangle}(x) \right)^2,$$

where ξ, a branch of the (genus 0) cubic $x\xi - x^3 - \xi^3 = 0$ is $\xi(x) = x^2 \sum_{m \geq 0} \binom{3m}{m} \frac{x^{3m}}{2m+1}$.

4.2 Continuous-time branching = Yule process.

Foatons and Viennons live an exponential time and disintegrate. . .

 $\mapsto$ ;  $\mapsto$ 

Proposition. *Consider the Yule process with two types of particles. The probabilities that particles are all of the second type at time t are*

$$X(t) = e^{-t}\,\mathrm{smh}(1 - e^{-t}), \qquad Y(t) = e^{-t}\,\mathrm{cmh}(1 - e^{-t}),$$

depending on whether the system at time 0 is initialized with one particle of the first type (X) or of the second type (Y).

Remarks on urn processes. For urn

$$\begin{pmatrix} -\alpha & \beta \\ \gamma & -\delta \end{pmatrix}, \qquad -\alpha+\beta=\gamma-\delta,$$

associate a partial differential operator:

$$\Gamma = x^{1-\alpha}y^{\beta}\frac{\partial}{\partial x} + x^{\gamma}y^{1-\delta}\frac{\partial}{\partial y}.$$

$\leadsto$ Develop a general theory of *Pólya Urn Processes* (FlDuPu06).

$\leadsto$ Can characterize all six matrices such that $e^{z\Gamma}$ is expressible by elliptic functions (FlGaPe05). One such model $\in \{\mathrm{sm}, \mathrm{cm}\}$.

$$A = \begin{pmatrix} -2 & 3 \\ 4 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 2 \\ 3 & -2 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix},$$
$$D = \begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} -1 & 3 \\ 5 & -3 \end{pmatrix}, \quad F = \begin{pmatrix} -1 & 4 \\ 5 & -2 \end{pmatrix}.$$

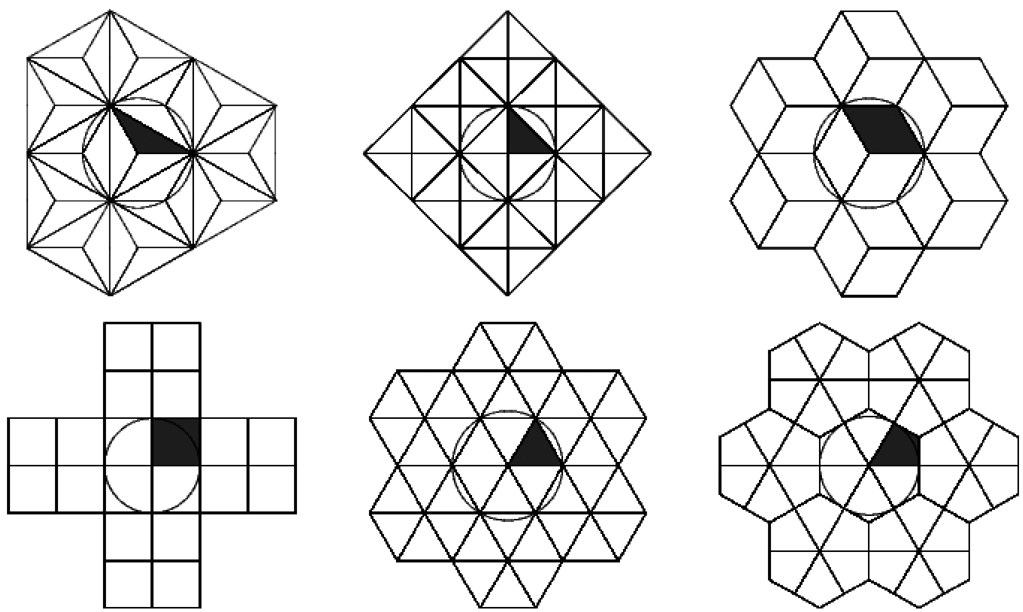
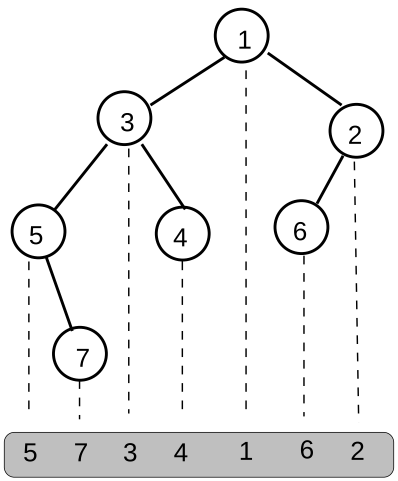


FIGURE 7. The six elliptic cases in order A, B, C, D, E, F: The diagrams formed by the fundamental polygon together with its rotated images. (The elementary kite is darkened.)

5 FIRST PERMUTATION MODEL

A permutation can always be represented as a tree, which is binary, rooted, and increasing.



$$\text{Tree}(w) = \langle \xi, \text{Tree}(w'), \text{Tree}(w'') \rangle$$

Level of node $\equiv$ distance to root. **Type of node** $\leadsto$ Peak, Valley, db-rise, db-fall.

Peaks	Valleys	Double rises	Double falls
$\sigma_{j-1} < \sigma_j > \sigma_{j+1}$	$\sigma_{j-1} > \sigma_j < \sigma_{j+1}$	$\sigma_{j-1} < \sigma_j < \sigma_{j+1}$	$\sigma_{j-1} > \sigma_j > \sigma_{j+1}$

♡♡♡ **Combinatorial Interpretation II** ♡♡♡

Proposition: *Consider the class $\mathcal{X}$ (resp. $\mathcal{Y}$) of permutations such that elements at any odd (resp. even) level are valleys only. Then the exponential generating functions are*

$$X(z) = \mathrm{smh}(z) = -\,\mathrm{sm}(-z), \qquad Y(z) = \mathrm{cmh}(z) = \mathrm{cm}(-z).$$

(Follows from standard combinatorics, reading off $X' = Y^2, Y' = X^2$.)

Other interpretations based on **parity**:

- Viennot, a *first* in 1980: Jacobi permutations, alternate reverse. . .
- Flajolet: alternating permutations, parity of peaks.
- Dumont on Schett, based on cycle structure. . .

6 THE SECOND PERMUTATION MODEL

Inspired by Fl-Françon (1989) = a model for Jacobi sn, cn when $r = 2$.

Definition: An r–*repeated* permutation of size rn is a permutation such that for each j, the (existing) elements $jr+1, jr+2, \ldots, jr+r-1$ are all of the same ordinal type (P, V, DR, DF).

Proposition: *Ordinary generating function for r–repeated is:*

$$\sum_{n\geq 0} R_{rn} z^n = \cfrac{1}{1 - 2\cdot 1^r\, z - \cfrac{1\cdot 2^2 \cdots r^2 \cdot (r+1)\cdot z^2}{1 - 2\cdot (r+1)^r\, z - \cfrac{(r+1)\cdot(r+2)^2\cdots(2r)^2\cdot(2r+1)\cdot z^2}{\ddots}}},$$

Numerators of degree $2r$; denominators of degree r.

6.1 Combinatorial aspects of continued fractions.

A *lattice path* aka *Motzkin path* is a sequence $s = (s_0, s_1, \ldots, s_n)$:

$$s_0 = s_n = 0, \quad s_j \in \mathbb{Z}_{\geq 0}, \qquad |s_{j+1} - s_j| \in \{-1, 0, +1\}.$$

Let $P(\mathbf{a}, \mathbf{b}, \mathbf{c})$ be the infinite-variable generating function of lattice paths with ascent $\leftrightarrow a_k$, descent $\leftrightarrow b_k$, level $\leftrightarrow c_k$.

Theorem (what Foata calls "the shallow Flajolet Theorem"):

$$P(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \cfrac{1}{1 - c_0 - \cfrac{a_0 b_1}{1 - c_1 - \cfrac{a_1 b_2}{1 - c_2 - \cfrac{a_2 b_3}{\ddots}}}}.$$

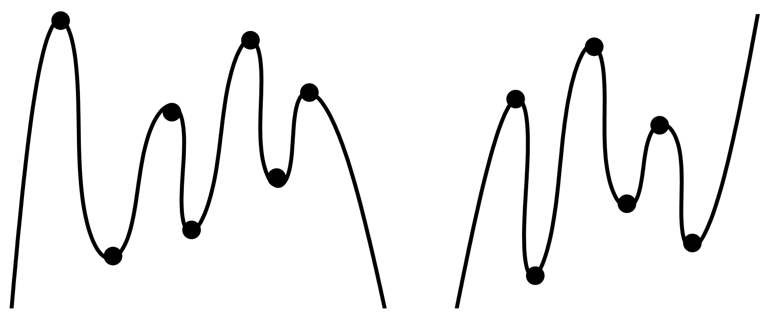
$\varpi =$

$$\begin{aligned} \mathfrak{m}(\varpi) &= a_0 a_1 c_2 b_2 a_1 a_2 b_3 b_2 b_1 a_0 b_1 \\ &= a_0^2 a_1^2 a_2 b_1^2 b_2^2 b_3 c_1. \end{aligned}$$

6.2 Lattice paths and permutations.

- A bijection due to Françon-Viennot (1979);
- What V.I. Arnold (2000) calls *snakes*

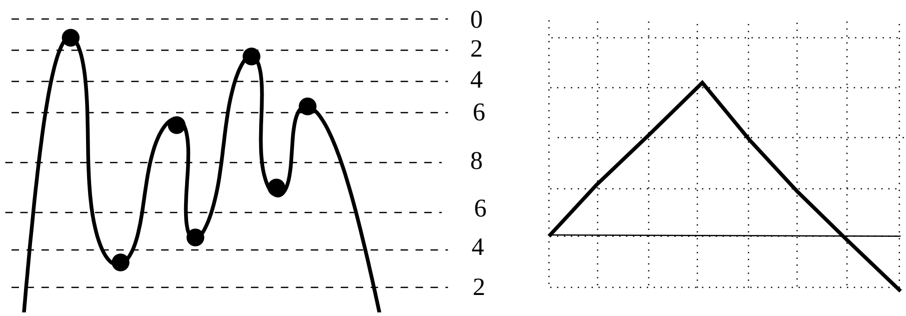
Consider piecewise monotonic smooth functions from $\mathbb{R}$ to $\mathbb{R}$, such that all the critical values are different, and take the equivalence classes up to orientation preserving maps of $\mathbb{R}^2$.



$(-\infty, -\infty)$ $(-\infty, +\infty)$

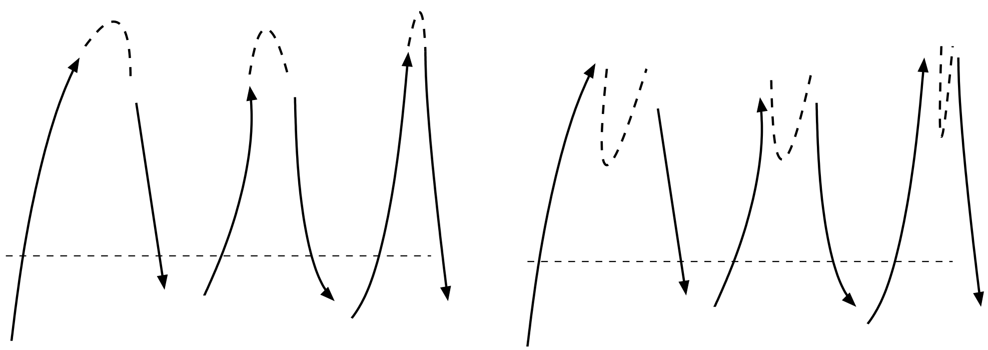
Clearly an equivalence class is an alternating permutation, and by André's theorem the EGFs are $\tan(z) = \frac{\sin(z)}{\cos(z)}$, $\sec(z) = \frac{1}{\cos(z)}$.

The sweepline algorithm: a snake and its associated Dyck path.



An encoding is obtained by the system of possibilities:

$$\Pi^{\text{odd}}: \qquad \alpha_j = (j+1), \quad \beta_j = (j+1), \quad \gamma_j = 0.$$



$$\int_0^\infty \tan(zt)e^{-t}\,dt = \cfrac{z}{1-\cfrac{1\cdot 2\,z^2}{1-\cfrac{2\cdot 3\,z^2}{\ddots}}}, \qquad \int_0^\infty \sec(zt)e^{-t}\,dt = \cfrac{1}{1-\cfrac{1^2\,z^2}{1-\cfrac{2^2\,z^2}{\ddots}}}.$$

All perms: encode double rises and double falls by level steps.

$$\sum_{n=1}^{\infty} n!z^n = \cfrac{z}{1-2z-\cfrac{1\cdot 2\,z^2}{1-4z-\cfrac{2\cdot 3\,z^2}{\ddots}}}, \qquad \sum_{n=0}^{\infty} n!z^n = \cfrac{1}{1-z-\cfrac{1^2\,z^2}{1-3z-\cfrac{2^2\,z^2}{\ddots}}}.$$

Stieltjes +Euler

6.3 The model of 3–repeated permutations.

♡♡♡ **Combinatorial Interpretation III** ♡♡♡

Proposition: The exponential generating function of *3–repeated polarized permutations* bordered by $(-\infty, -\infty)$ is

$$\mathrm{smh}(z).$$

Notes: By Fl-Françon, 2–repeated + recording rises (cf Eulerian #'s) gives Jacobian sn, cn, dn.

Corollary: *Combinatorial proofs of Conrad's fractions.*

Also: $\wp(z-\zeta_0, 0, -1)$ *expanded near its real zero,* $\zeta_0 = \frac{1}{3\pi}\Gamma\left(\frac{1}{3}\right)^3$*, has CF expansion with cubic denominators and sextic numerators.*
$\wp(z-\zeta_0; 0, -1) \equiv \mathrm{smh}(z)\cdot\mathrm{cmh}(z) = \mathrm{Inv}\, Y \cdot {}_2F_1\left[\frac{1}{3}, \frac{1}{2}, \frac{4}{3}; -4Y^2\right].$

7 PERSPECTIVES & QUESTIONS

Worth looking at nonlinear differential systems associated to algebraic curves and their Abelian integrals?

• **Q.** Three types of balls? (cf Schett-Dumont for a special elliptic case)
$A \to BC, B \to CA, C \to AB$ is elliptic (sn, cn); hyperelliptic generalizations.

• **Q.** What about numerators like k^6 and such in CF? Combinatorics?

• **Q.** Anything to say about orthogonal polynomials (cf Carlitz for sn, cn)?
Cf Galiano Valent *et al.* — *very* intriguing!

• **Q.** Any possibility of enumerating directly r–repeated perms for $r \geq 4$?

• **Q.** Anything (combinatorially) interesting regarding higher order systems associated to $\mathbf{F}_p$ for $p > 3$?
At least consequences for urn models (FlDuPu06).