Singularity Analysis: A Perspective

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Analysis of Algorithms

\[ \downarrow \]

Average-Case, Probabilistic

\[ \downarrow \]

Properties of Random Structures?

- Counting and asymptotics
  \[ n! \sim n^n e^{-n} \sqrt{2\pi n} \]

- Asymptotic laws
  \[ \Omega_n \xrightarrow{D} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt. \]
  (e.g., Monkey and typewriter!)

- Probabilistic, stochastic
- Analytic Combinatorics: Generating Functions
1. Introduction

“Symbolic” Methods

Rota-Stanley; Foata-Schutzenberger; Joyal and UQAM group; Jackson-Goulden, &c; F.; ca 1980±. F-Salvy-Zimmermann 1991 \(\sim\) Computer Algebra.

Basic combinatorial constructions admit of direct translations as operators over generating functions (GF’s).
$C$ : class of comb. structures;

$C_n$ : # objects of size $n$

\[
\begin{align*}
C(z) & := \sum C_n z^n \\
\hat{C}(z) & := \sum C_n \frac{z^n}{n!}
\end{align*}
\]

\[
\begin{align*}
C(z, u) & := \sum C_{n,k} z^n u^k \\
\hat{C}(z, u) & := \sum C_{n,k} u^k \frac{z^n}{n!}
\end{align*}
\]

Ordinary GF’s for unlabelled structures. Exponential GF’s for labelled structures.
“Dictionaries”

= Constructions viewed as Operators over GF’s.

<table>
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<tr>
<th>Constr.</th>
<th>Operations</th>
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<td>Union</td>
<td>+</td>
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<td>Product</td>
<td>×</td>
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<tr>
<td>Sequence</td>
<td>$(1 - f)^{-1}$</td>
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<tr>
<td>MultiSet</td>
<td>Pólya Exp.</td>
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<tr>
<td>Cycle</td>
<td>Pólya Log.</td>
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<td></td>
<td>(unlab.)</td>
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|                        | (lab.)              |

Exp$(f) := \exp (f(z) + \frac{1}{2} f(z^2) + \cdots)$

Log$(f) := \log \frac{1}{1-f(z)} + \cdots$

Books: Goulden-Jackson, Bergeron-LL, Stanley, F-Sedgewick

⇒ How to extract coeff., especially, asymptotically??
“Complex–analytic Structures”

Interpret:
- Counting GF as analytic transformation of \( \mathbb{C} \);
- Comb. Construction as analytic functional.

Singularities are crucial to asymptotic prop’s!

(cf. analytic number theory, complex analysis, etc)

Asymptotic counting via Singularity Analysis (S.A.)
Asymptotic laws via Perturbation + S.A.
\[
\frac{1}{2i\pi} \int \frac{1}{1 - z - z^2} \frac{dz}{z^{n+1}} \quad \exists f(z), \quad f(z) = (1 - z - z^2)^{-1}.
\]


**Location** of singularity at $z = \rho$: coeff. $[z^n]f(z) = \rho^{-n} \cdot \text{coeff.} [z^n]f(\rho z)$

**Nature** of singularity at $z = 1$:

\[
\begin{align*}
\frac{1}{(1-z)^2} & \quad \rightarrow \quad n + 1 \quad \sim \quad n \\
\frac{1}{1-z} \log \frac{1}{1-z} & \quad \rightarrow \quad H_n \equiv \frac{1}{1} + \ldots + \frac{1}{n} \quad \sim \quad \log n \\
\frac{1}{1-z} & \quad \rightarrow \quad 1 \quad \sim \quad 1 \\
\frac{1}{\sqrt{1-z}} & \quad \rightarrow \quad 1 \frac{2^{2n}}{n} \quad \sim \quad \frac{1}{\sqrt{\pi n}}
\end{align*}
\]

\[
\{ \begin{align*}
\text{Location of sing's:} & \quad \text{Exponential factor} \quad \rho^{-n} \\
\text{Nature of sing's:} & \quad \text{“Polynomial” factor} \quad \vartheta(n)
\end{align*} \]

Generating Function $\rightsquigarrow$ Coefficients

Solving a “Tauberian” problem

<table>
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<tr>
<th>Real-Tauberian</th>
<th>Darboux-Pólya</th>
<th>Singularity An.</th>
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</table>

- (large $\Rightarrow$ large)
- (smooth $\Rightarrow$ small)
- (Full mappings)

Combinatorial constructions $\rightsquigarrow$ Analytic Functionals

$\implies$ Analytic continuation prevails for comb. GF’s
2. Basic Singularity Analysis

**Theorem 1.** Basic scale translates:

\[ \sigma_{\alpha,\beta}(z) := (1 - z)^{-\alpha} \left( \frac{1}{z} \log \frac{1}{1-z} \right)^\beta \]

\[ \implies [z^n] \sigma_{\alpha,\beta} \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)} (\log n)^\beta. \]

**Proof.** Cauchy’s coefficient integral, \( f(z) = (1 - z)^{-\alpha} \)

\[ [z^n] f(z) = \frac{1}{2i\pi} \int_\gamma f(z) \frac{dz}{z^{n+1}} \]

\[ \downarrow \quad (z = 1 + \frac{t}{n}) \quad \downarrow \]

\[ \frac{1}{2i\pi} \int_\mathcal{H} \left( -\frac{t}{n} \right)^{-\alpha} e^{-t} \frac{dt}{n} \]

\[ n^{\alpha-1} \times \frac{1}{\Gamma(\alpha)}. \]
Theorem 2. $O$–transfers: Under continuation in a $\Delta$-domain,

$$f(z) = O(\sigma_{\alpha,\beta}(z)) \implies [z^n]f(z) = O([z^n]\sigma_{\alpha,\beta}(z)).$$

**Proof:**
\[
\begin{align*}
\begin{cases}
    f(z) &= \lambda \sigma(z) + \mu \tau(z) + \ldots + O(\omega(z)) \\
    f_n &= \lambda \sigma_n + \mu \tau_n + \ldots + O(\omega_n).
\end{cases}
\end{align*}
\]

Usage:

Similarly: \textit{o-transfer}.

- Dominant singularity at \( \rho \) gives factor \( \rho^{-n} \).
- Finitely many singularities work fine.
**Example 1.** 2-regular graphs [Comtet] (Originally by Darboux-Pólya.)

\[ G = \mathcal{M} \left( \frac{1}{2} e_{\geq 3}(Z) \right) \]

\[ \hat{G}(z) = \exp \left( \frac{1}{2} \log \frac{1}{1 - z} - \frac{z}{2} - \frac{z^2}{4} \right) \]

\[ \hat{G}(z) \underset{z \to 1}{\sim} \frac{e^{-3/4}}{\sqrt{1 - z}} \]

\[ \frac{G_n}{n!} \underset{n \to \infty}{\sim} \frac{e^{-3/4}}{\sqrt{\pi n}}. \]

\[ > \text{equivalent(exp(-z/2-z^2/4)/sqrt(1-z),z,n,4); # By SALVY} \]

\[ \begin{array}{ccc}
\frac{1}{2} & \text{exp(-3/4) (1/n)} & \frac{1}{2} \\
\frac{3}{2} & \text{exp(-3/4) (1/n)} & \frac{5}{2} \\
\frac{5}{8} & \text{exp(-3/4) (1/n)} & + \frac{1}{128} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\pi & \pi & \pi
\end{array} \]
Example 2. Richness index of trees [F-Sipala-Steyaert,90]
= Number of different terminal subtrees. Catalan case:

\[ K(z) = \frac{1}{2z} \sum_{k \geq 0} \frac{1}{k+1} \binom{2k}{k} \left( \sqrt{1 - 4z - 4z^{k+1}} - \sqrt{1 - 4z} \right) \]

\[ K(z) \approx \frac{1}{z \to 1/4} \frac{1}{\sqrt{Z \log Z}}, \quad Z := 1 - 4z \]

Mean index \( n \to \infty \quad C \frac{n}{\sqrt{\log n}}, \quad C \equiv \sqrt{\frac{8 \log 2}{\pi}}. \)

= Compact tree representations as DAGs = Common Subexpression Pf.
Extensions

- Slowly varying $\implies$ slowly varying: Log-log $\implies$ Log-Log, ...
- Full asymptotic expansions
- Uniformity of coefficient extraction $[z^n] \{ F_u(z) \}_{u \in \Omega} = \sim$ later!
- Some cases with natural boundary [Fl-Gourdon-Panario-Pouyanne]

**Example 3. Distinct Degree Factorization** [DDF] in Polynomial Fact $\sim$

Greene–Knuth:

$$[z^n] \prod_{k=1}^{\infty} \left( 1 + \frac{z^k}{k} \right).$$

Hybrid w/ Darboux: $e^{-\gamma} + \frac{e^{-\gamma}}{n} + \cdots + \star \frac{(-1)^n}{n^3} + \star \frac{\omega^n}{n^3} + \cdots$

Cf. Hardy-Ramanujan’s partition analysis “without contrast”.
3. Closure Properties

Function of S.A.-type = amenable to singularity analysis
- is continuable in a $\Delta$-domain,
- admits singular expansion in scale $\{\sigma_{\alpha,\beta}\}$.

**Theorem 3.** Generalized polylogarithms

$$\text{Li}_{\alpha,k} := \sum (\log n)^k n^{-\alpha} z^n$$

are of S.A.-type.

**Proof.** Cauchy-Lindelöf representations

$$\sum \varphi(n)(-z)^n = -\frac{1}{2i\pi} \int_{1/2-i\infty}^{1/2+i\infty} \varphi(s) z^s \frac{\pi}{\sin \pi s} \, ds.$$ 

+ Mellin transform techniques (Ford, Wong, F.).
**Example 4.** Entropy of Bernoulli distribution

\[ H_n := - \sum_{k} \pi_{n,k} \log \pi_{n,k}, \quad \pi_{n,k} \equiv \binom{n}{k} p^k (1 - p)^{n-k} \]

involves

\[ \sum \log(k!) z^k = (1 - z)^{-1} \text{Li}_{0,1}(z) \]

\[ \frac{1}{2} \log n + \frac{1}{2} + \log \sqrt{2\pi p(1-p)} + \cdots. \]

Redundancy, coding, information th.; Jacquet-Szpankowski via Analytic dePoissonization.

- Elements like \( \log n, \sqrt{n} \) in combinatorial sums
**Theorem 4.** Functions of S.A.-type are closed under integration and differentiation.

**Proof.** Adapt from Olver, Henrici, etc.

**Theorem 5.** Functions of S.A.-type are closed under Hadamard product

\[ f(z) \circ g(z) := \sum_n (f_n g_n) z^n. \]

**Proof.** Start from Hadamard’s formula

\[ f(z) \circ g(z) = \frac{1}{2i\pi} \int_\gamma f(t) g \left( \frac{w}{t} \right) \frac{dt}{t}. \]

+ adapt Hankel contours [H., Jungen, R. Wilson \sim Fill-F-Kapur]
Example 5. Divide-and-conquer recurrences

\[ f_n = t_n + \sum \pi_{n,k} (f_k + f_{n-k}) \]

\[ \text{Sing}(f(z)) = \Phi(\text{Sing}(t(z))) \]

\[ \text{Asympt}[f_n] = \Psi(\text{Sing}(t)). \]

E.g., Catalan statistics: need \( \sum \binom{2n}{n} \log n \cdot z^n \).

4. Functional Equations

- **Rational functions.** Linear system $\mathbb{Q}_0[z]$ implies polar singularities:

$$[z^n]f(z) \approx \sum \omega^n n^k, \quad \omega \in \overline{\mathbb{Q}}, \quad k \in \mathbb{Z}_{\geq 0}.$$  

+ irreducibility: Perron-Frobenius $\implies$ *simple dom. pole.*

- **Word problems** from regular language models;
- **Transfer matrices** [Bender-Richmond]: dimer in strip, knights, etc.
- $\sim$ Vallée’s generalization to *dynamical sources* via transfer operators.

- **Algebraic functions,** by Puiseux expansions $(Z^{p/q}) \ll \text{S.A. or Darboux!}$

$$[z^n]f(z) \approx \sum \sum \omega^n n^{p/q}, \quad \omega \in \overline{\mathbb{Q}}, \quad p/q \in \mathbb{Q},$$

*Asymptotics of coeff. is decidable* [Chabaud-F-Salvy].

- **Word problems** from context-free models;
- **Trees:** Geom. configurations (non-crossing graphs, polygonal triangs.);
- **Planar Maps** [Tutte...]; **Walks** [Banderier Bousquet-M., Schaeffer], ...


Square-root singularity is “universal” for many recursive classes = controlled “failure” of Implicit Function Theorem $Z \propto Y^2$
Entails coeff. asymptotic $\approx \omega^n n^{-3/2}$ with critical exponent $-3/2$.
E.g., unbalanced 2–3 trees (Meir-Moon): $f = z\phi(f)$, $\phi(u) = 1 + u^2 + u^3$.
Pólya’s combinatorial chemistry programme:

$$f(z) = z \exp(f(z)) \equiv ze^{f(z)} + \frac{1}{2}f(z)^2 + \frac{1}{3}f(z)^3 + \ldots$$

Starting with Pólya 1937; Otter 1949; Harary-Robinson et al. 1970’s;
• **“Holonomic” functions.** Defined as solutions of linear ODE’s with coeff in \( \mathbb{C}(z) \) [Zeilberger] \( \equiv \mathcal{D} \)-finite.

\[
\mathcal{L}[f(z)] = 0, \quad \mathcal{L} \in \mathbb{C}(z)[\partial_z].
\]

• Stanley, Zeilberger, Gessel: Young tableaux and permutation statistics; regular graphs, constrained matrices, etc.

**Fuchsian case** (or “regular” singularity) \((Z^\beta \log^k Z)\):

\[
[z^n]f(z) \approx \sum \omega^n n^\beta (\log n)^k, \quad \omega, \beta \in \mathbb{Q}, \quad k \in \mathbb{Z}_{\geq 0}.
\]

S.A. applies automatically to classical classification.

Asymptotics of coeff is decidable
— general case: modulo oracle for connection problem;
— strictly positive case: “usually” OKay.
**Example 6.** *Quadtrees—Partial Match* [FGPR’92]

Divide-and-conquer recurrence with coeff. in \( \mathbb{Q}(n) \)

Fuchsian equation of order \( d \) (dimension) for GF

\[
Q_{n}^{(d=2)} \propto n^{(\sqrt{17}-3)/2}.
\]

E.g., \( d = 2 \): Hypergeom \( _2F_1 \) with algebraic arguments.

Extended by Hwang et al. Cf also Hwang’s *Cauchy ODE* cases.
Panholzer-Prodinger+Martinez, …
• **Functional Equations and Substitution.**

• Early example of *balanced 2–3 trees* by Odlyzko, 1979.

\[ T(z) = z + T(\tau(z)), \quad \tau(z) := z^2 + z^3. \]

Infinitely many exponents with common real part implies periodicities:

\[ T_n \sim \frac{\phi^n}{n} \Omega(\log n). \]
• Singular iteration for *height of trees* (binary and other simple varieties; F-Gao-Odlyzko-Richmond; cf Renyi-Szekeres):

\[ y_h = z + y_{h-1}^2, \quad y_0 = z. \]

— Moments and convergence in law; Local limit law of \( \vartheta \)-type. Applies to branching processes conditioned on total progeny.

Cf Chassaing-Marckert for // probabilistic approaches \( \rightsquigarrow \textbf{width} \)

• *Digital search trees* via \( q \)-hypergeometrics: singularities accumulate geometrically \( \rightsquigarrow \textbf{periodicities} [\text{F-Richmond}] \):

\[ \partial_z^k f(z) = t(z) + 2e^{z/2} f(\frac{z}{2}). \]

• *Order of binary trees* (Horton-Strahler, Register function; F-Prodinger) via **Mellin** tr. of GF and \& singularities.
5. Limit Laws

- Moment pumping from bivariate GF

Early theories by Kirschenhofer-Prodinger-Tichy (1987)

Factorial moment of order $k$: $[z^n] \left( \frac{\partial^k}{\partial z^k} F(z, u) \right)_{u=1}$

**Example 7.** Airy distribution of areas shows up in area below paths, path length in trees, Linear Probing Hashing, inversions in increasing trees, connectivity of graphs.

\[
\frac{\partial}{\partial z} F(z, q) = F(z, q) \cdot \frac{F(z, q) - qF(qz, q)}{1 - q}
\]

Louchard-Takács[Darboux]; Knuth; F-Poblete-Viola // Chassaing-Marckert
Classical probability theory: sums of Random Variables $\sim$ powers of fixed function (PGF, Fourier tr.) $\sim$ Normal Law.

For problems expressed by Bivariate GF (BGF): field founded by E. Bender et al. + developments by F, Soria, Hwang, …

Idea: BGF $F(z, u) = \sum f_n(u)z^n$, where $f_n(u)$ describes parameter on objects of size $n$. If (for $u$ near 1)

$$f_n(u) \approx \omega(u)^{\kappa_n}, \quad \kappa_n \to \infty,$$

then speak of Quasi-Powers approximation. Recycle continuity theorem, Berry-Esseen, Chernov, etc. $\implies$ Normal law and many goodies...

(speed of convergence, large deviation fn, local limits)
Two important cases:

- **Movable singularity**:

  \[ F(z, u) \approx \left(1 - \frac{z}{\rho(u)}\right)^{-\alpha} \quad \Rightarrow \quad \frac{f_n(u)}{f_n(1)} \approx \left(\frac{\rho(1)}{\rho(u)}\right)^n. \]

- **Variable exponent**:

  \[ F(z, u) \approx \left(1 - \frac{z}{\rho}\right)^{-\alpha(u)} \quad \Rightarrow \quad \frac{f_n(u)}{f_n(1)} \approx \begin{cases} \frac{n^{\alpha(u)-\alpha(1)}}{\left(e^{\alpha(u)-\alpha(1)}\right)^{\log n}}. \end{cases} \]

Requires **uniformity** afforded by **Singularity Analysis**

(≠ Tauber or Darboux).

**Singularity Perturbation** analysis (smoothness)

\[ \Downarrow \]

Uniform Quasi-Powers for coeffs

\[ \Downarrow \]

Normal limit law
Example 8. Polynomials over finite fields.

\[ \text{Factor}(x^7+x+1) \mod 29; \]
\[ (x^3 + x^2 + 3x + 15) (x^2 + 25x + 25) (x^2 + 3x + 14) \]

- Polynomial is a sequence of coeffs: \( \mathcal{P} \) has Polar singularity.
- By unique factorization, \( \mathcal{P} \) is also multiset of irreducibles: \( \mathcal{I} \) has log singularity.

\[ \implies \text{Prime Number Theorem for Polynomials} \quad I_n \sim \frac{q^n}{n}. \]

- Marking number of \( \mathcal{I} \)-factors is approx\( u \)th power:
  \[ P(z, u) \approx \left(e^{I(z)}\right)^u. \]

Variable Exponent \( \implies \) Normality of \# of irreducible factors.
(cf Erdős-Kac for integers.)

(Analysis of polynomial fact. algorithms, [F-Gourdon-Panario])
Example 9. Patterns in Random Strings = Perturbation of linear system of eqns. (& many problems with finite automata, paths in graphs)

Linear system $X = X_0 + TX$ w/ Perron-Frobenius. Auxiliary mark $u$ induces smooth singularity displacement. For “natural” problems: Normal limit law. cf [Régnier & Szpankowski], … □

Also sets of patterns; similarly for patterns in increasing labelled trees, in permutations, in binary search trees [F-Gourdon-Martinez]. Generalized patterns and/or sources by Szpankowski, Vallée, …
**Example 10.** *Non crossing graphs.* [F-Noy]

= Perturbation of algebraic equation.

\[
G^3 + (2z^2 - 3z - 2)G^2 + (3z + 1)G' = 0
\]

\[
G^3 + (2u^3 z^2 - 3u^2 z + u - 3)G^2 + (3u^2 - 2u + 3)G + u - 1 = 0
\]

Movable singularity scheme applies: *Normality.*

+ Patterns in context-free languages, in combinatorial tree models, in functional graphs: Drmota’s version of Drmota-Lalley-Woods.  

□
Example 11. Profile of Quadtrees.

\[ F(z, u) = 1 + 2^3 u \int_0^z \frac{dx_1}{x_1(1 - x_1)} \int_0^{x_1} \frac{dx_2}{1 - x_2} \int_0^{x_2} F(x_3, u) \frac{dx_3}{1 - x_3}. \]

Solution is of the form \((1 - z)^{-\alpha(u)}\) for algebraic branch \(\alpha(u)\);
Variable Exponent \(\implies\) Normality of search costs.

Applies to many linear differential models that behave like cycles-in-perms.
**Example 12. Urn models.** $2 \times 2$–balanced.

$$(u^5 z - u) \frac{\partial G}{\partial z} + (1 - u^6) \frac{\partial G}{\partial u} + u^5 G = 0$$

[FGP’03] ↔ Janson, Mahmoud, Puyhaubert, Panholzer-Prodinger, ...

- **Coalescence of singularities and/or exponents:** e.g. Maps = Airy Law ≡ Stable$(\frac{3}{2})$ [BFSS’01]. Cf Pemantle, Wilson, Lladser, ....
Conclusions

For combinatorial counting and limit laws:
Modest technical apparatus & generic technology.
High-level for applications, esp., analysis of algorithms.
Plug-in on Symbolic Combinatorics & Symbolic Computation.
Discussion of Schemas & “Universality” in metric aspects of random discrete structures.
E.g. Borges’ theorem for words, trees, labelled trees, mappings, permutations, increasing trees, maps, etc.

Thank you! ♥♥♥