On Buffon Machines & Numbers

Philippe Flajolet, Maryse Pelletier, Michèle Soria

AofA'09, Fréjus --- June 2009

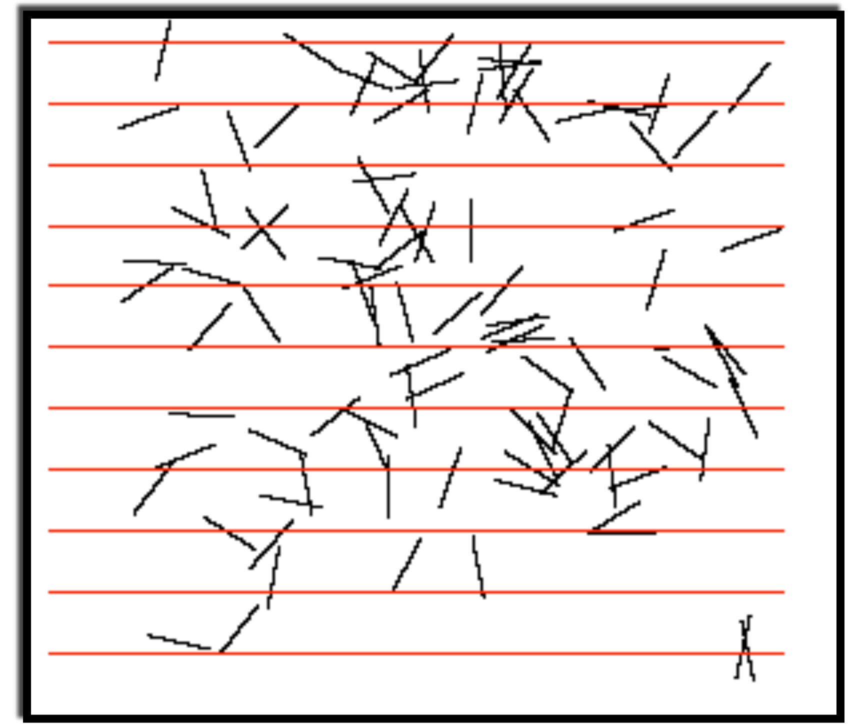
[INRIA-Rocquencourt & LIP6, Paris]



1733: Countess Buffon drops her knitting kit on the floor.

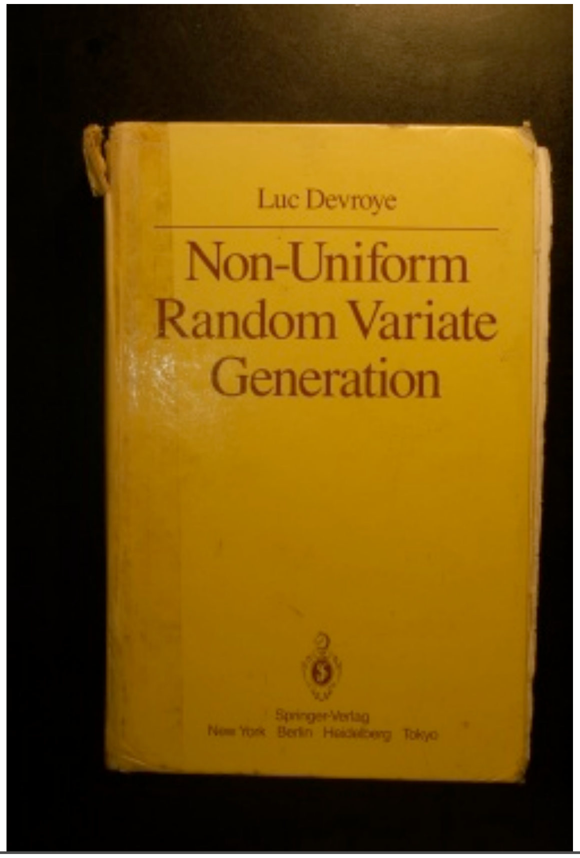
Count Buffon picks it up and notices that about 63% of the needles intersect a line on the floor.

Oh-Oh! 0.6366 is almost 2/pi (!)...



- A large body of literature on **real-number simulations**, starting with von Neumann, Ulam, Metropolis,...

- Luc Devroye's monumental synthesis, which is available on the web:
  @ http://cg.scs.carleton.ca/~luc/



What to do if you travel and don't want to carry floor planks and knitting needles?

Assume you have a coin!

Insist on PERFECT simulations!

- **Assume you have a coin.**
  + Insist on perfect simulations.

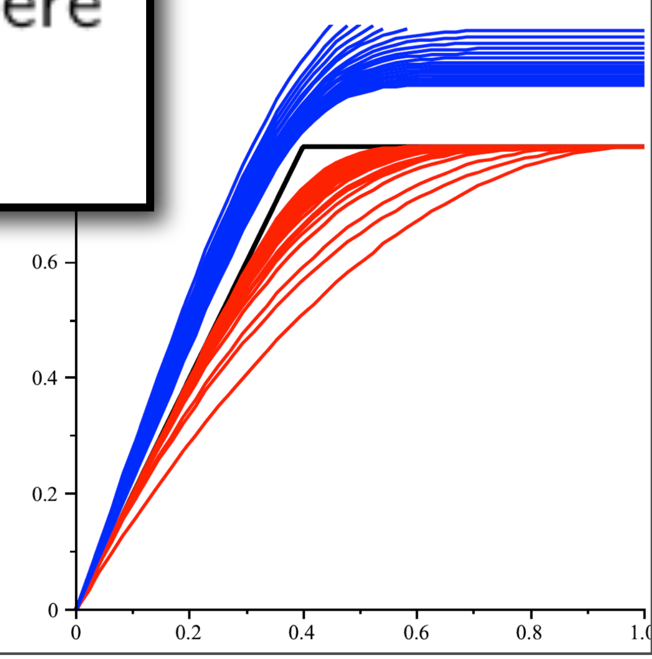
- The problem is trivial!!!!!!
  Everything that is computable can be simulated.

- Numbers & functions:

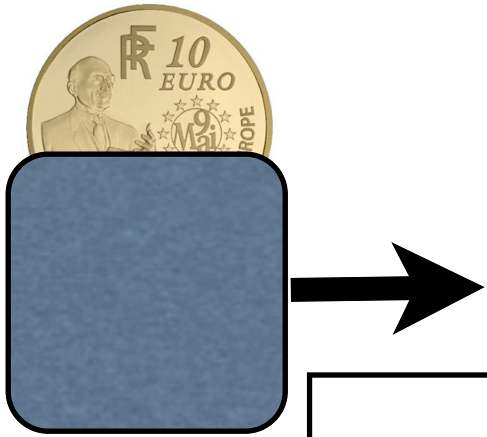
approximate α with $u_n < \alpha < v_n$, where $u_n, v_n \in \mathbb{Q}$.
approximate $\alpha(x)$ with $u_n(x) < \alpha(x) < v_n(x)$, where $u_n(x), v_n(x) \in \mathbb{Q}[x]$.

1/π

$$\frac{1}{3}, \frac{7}{22}, \frac{106}{333}, \frac{113}{355}, \frac{33102}{103993}, \frac{33215}{104348}$$



1. The framework

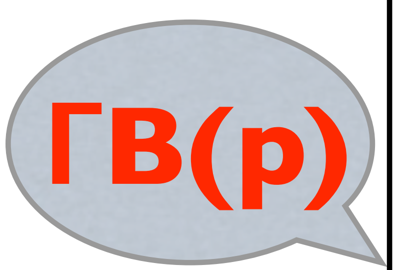


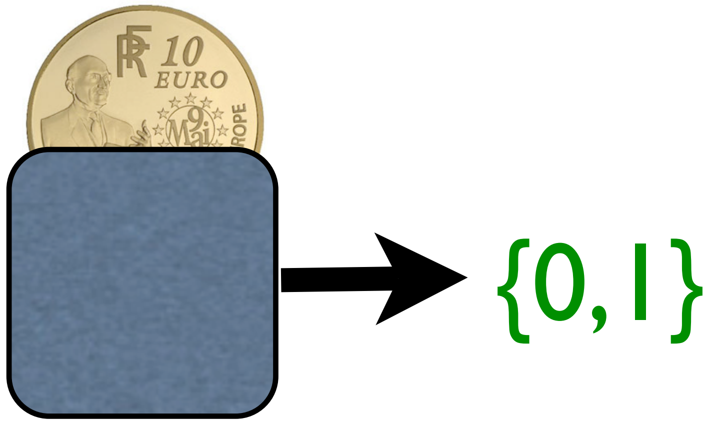
{0,1}

- A *Buffon machine* is a machine or program that has access to a pure source of perfect coin flips and outputs {0,1}-values, or, in some cases, integers.

- It **may not involve multi-precision arithmetics**, only basic probabilistic processes, be simple(!) and efficient(!).

- Buffon machines have no permanent memory => *they can only produce i.i.d random variables; typically, Bernoulli.*



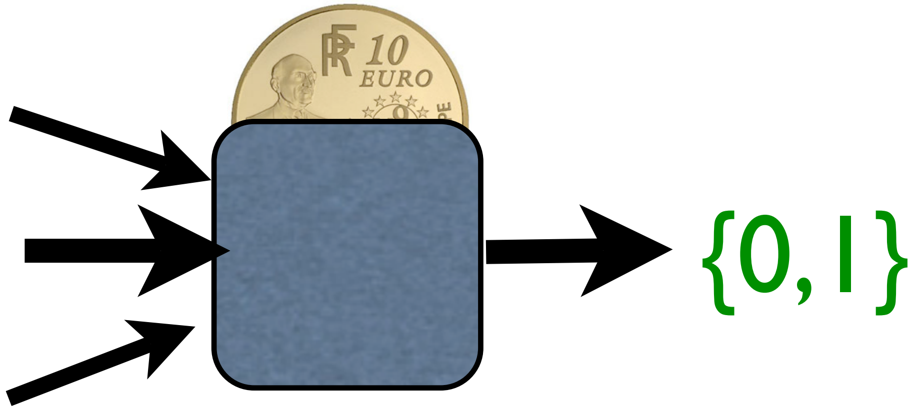


- Can you do such numbers as

$$1/\sqrt{2}, \quad e^{-1}, \quad \log 2, \quad \frac{1}{\pi}, \quad \pi - 3, \quad \frac{1}{e-1}, \quad ???$$

with only basic coin flips and no arithmetics.

- Simulation: expected # flips is finite.
Strong simulation: + has exponential tails.



- A *Buffon machine* may *also* *call black boxes* sampling from Bernoulli distributions of unknown parameters.

- A machine computes φ(p), if given a machine ΓB(p) for Bern(p) [p unknown!] as subroutine, its output is a Bern(φ(p)).

- In this way Buffon machines can be composed from simpler ones...

- **Meta-thorem**: *You can do, constructively, simply and efficiently:*

  - *All rational numbers and functions in (0,1)*
  - *All positive algebraic functions (context-free)*
  - *Closure under half-sum, product, composition*
  - *Exponentials, logarithms; polylogs; trig functions*
  - *Closure under integration; inverse trigs*
  - *Hypergeometrics of "binomial type"*
  - *+ Poisson and logarithmic-series generators*

- We shall see nine ways to get π, some with 5 coin flips on average, with typically about a dozen lines of code...

- Builds on ideas of
  von Neumann, Knuth-Yao

- Encapsulates constructions by
  Wästlund, Nacu, Peres, Mossel

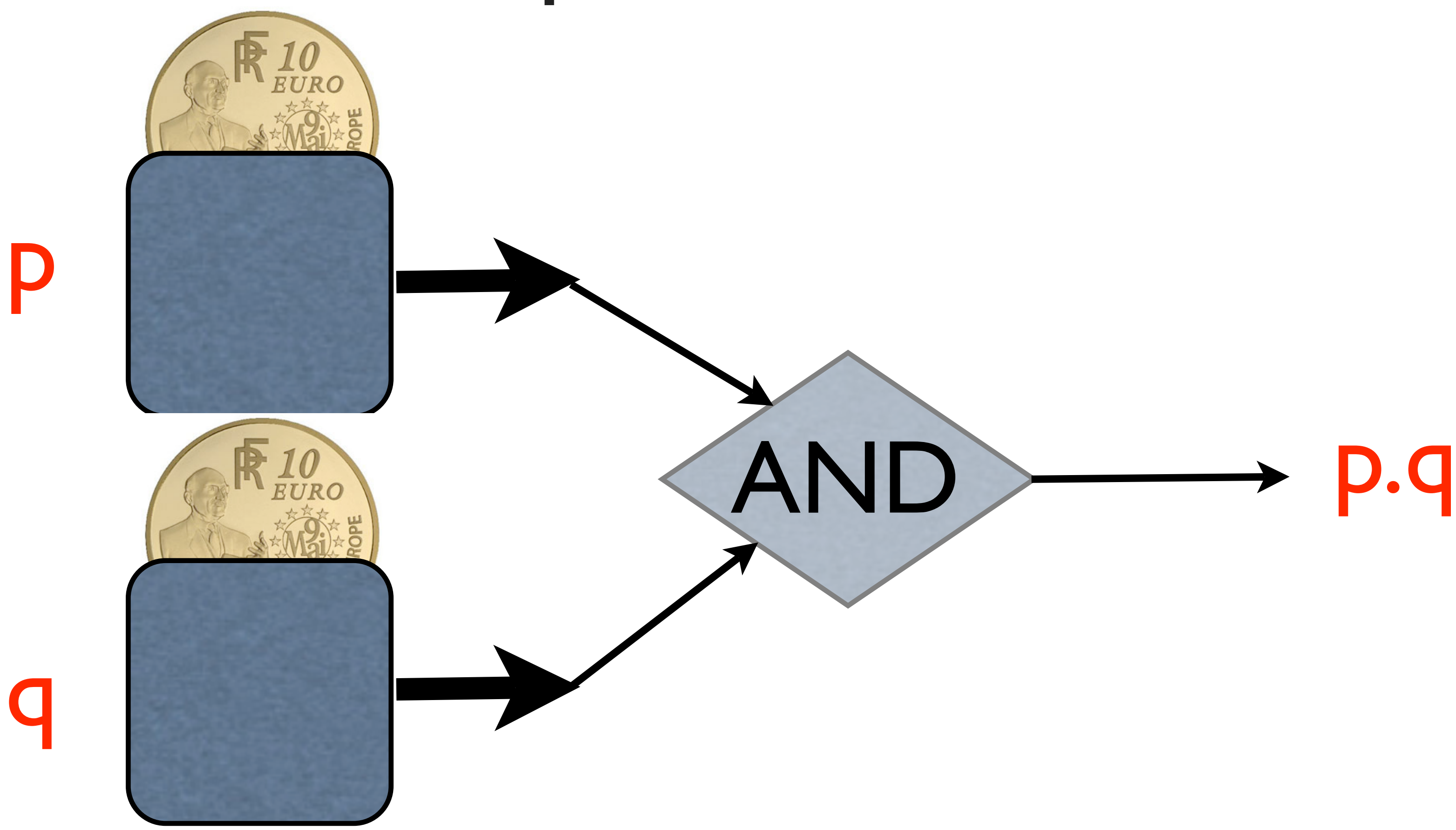
- Develops new constructions:
  VN-generator, integration; Poisson & logarithmic distributions.

2. Basic construction rules

- **Decision trees and loopless programs**

  Do Bernoulli of param. 3/8,5/8; *dyadic rationals*

  “Compute” *Boolean combinations*



p.q
1-p
(p+q)/2
p^2

Name	*realization*	*function*
Conjunction ($P \wedge Q$)	if $P() = 1$ then return($Q()$) else return(0)	$p \wedge q = p \cdot q$
Disjunction ($P \vee Q$)	if $P() = 0$ then return($Q()$) else return(1)	$p \vee q = p + q - pq$
Complementation ($\neg P$)	if $P() = 0$ then return(1) else return(0)	$1 - p$
Squaring	$(P \wedge P)$	p^2
Conditional ($P \rightarrow Q \vert R$)	if $R() = 1$ then return($P()$) else return($Q()$)	$rp + (1 - r)q.$

• Finite graphs and Markov chains

- To do a ΓB(3/7), flip three times; in 3 cases, return(1); in 4 cases return(0); otherwise repeat.
  Can do all rational p

- From a ΓB(p); repeatedly try till 1 is observed. If number of trials is even, then return(1).
  Computes 1/(1+p) = $(1-p)[1+p^2+p^4+ \ldots]$

- Mossel, Nacu, Peres, Wästlund:

Theorem 1 ([21, 22, 27]). *(i) Any polynomial $f(x)$ with rational coefficients that maps $(0,1)$ into $(0,1)$ is strongly realizable by a finite graph. (ii) Any rational function $f(x)$ with rational coefficients that maps $(0,1)$ into $(0,1)$ is strongly realizable by a finite graph.*

Also: *do a geometric ΓG(p) from a Bernoulli ΓB(p)*

3. The von Neumann schema

- Choose a class of permutations with P_n the number of those of size n.

$\Gamma\mathrm{VN}[\mathcal{P}](\lambda) :=$ { **do** {
  $N := \Gamma G(\lambda)$; ← geometric
  let $\mathbf{U} := (U_1, \ldots, U_N)$ be a vector of $[0,1]$–uniform variables.
  set $\tau := \mathrm{trie}(\mathbf{U})$; **let** $\sigma := \mathrm{type}(\mathbf{U})$;
  if $\sigma \in \mathcal{P}_N$ **then** return(N) } }.

- Probability of success with N=n is

$$\frac{(1-\lambda)P_n\lambda^n/n!}{(1-\lambda)\sum_n P_n\lambda^n/n!} = \frac{1}{P(\lambda)}\frac{P_n\lambda_n}{n!},$$

- Thus, get *Poisson and logarithmic distributions*

permutations ($\mathcal{P}$):	all ($\mathcal{Q}$)	sorted ($\mathcal{R}$)	cyclic ($\mathcal{S}$)
distribution:	$(1-\lambda)\lambda^n$ geometric	$e^{-\lambda}\frac{\lambda^n}{n!}$ Poisson	$\frac{1}{L}\frac{\lambda^n}{n}$, $L := \log(1-\lambda)^{-1}$ logarithmic.

- For **VN schema**, *path-length of tries* determines # coin flips.

PGF:

$$h_n(q) = \frac{1}{1 - q^n 2^{1-n}} \sum_{k=1}^{n-1} \frac{1}{2^n} \binom{n}{k} h_k(q) h_{n-k}(q).$$

Proposition 1. (*i*) *Given a class $\mathcal{P}$ of permutations and a parameter $\lambda \in (0,1)$, the von Neumann schema $\Gamma\mathrm{VN}[\mathcal{P}](\lambda)$ produces* exactly *a discrete random variable with probability distribution*

$$\mathbb{P}(N = n) = \frac{1}{P(\lambda)} \frac{P_n \lambda_n}{n!}.$$

(*ii*) *The number K of iterations has expectation $1/s$, where $s = (1-\lambda)P(\lambda)$, and its distribution is $1 + \mathrm{Geo}(s)$.*

(*iii*) *The number C of flips consumed by the algorithm (not counting[2] the ones in $\Gamma\mathrm{G}(\lambda)$) is a random variable with probability generating function*

$$\mathbb{E}(q^C) = \frac{H^+(\lambda, q)}{1 - H^-(\lambda, q)}. \tag{10}$$

where H^+, H^- are determined by (9):

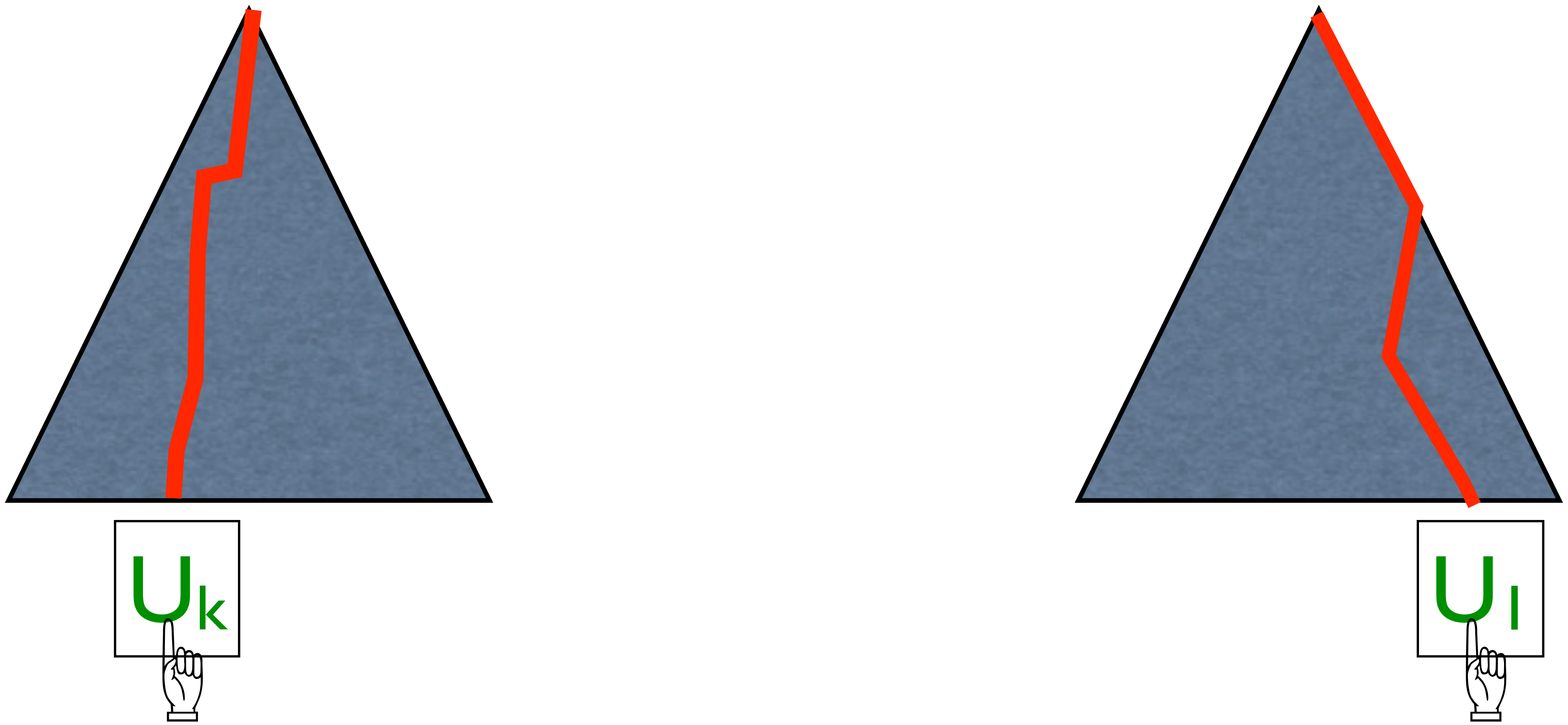
$$H^+(z,q) = (1-z) \sum_{n=0}^{\infty} \frac{P_n}{n!} h_n(q) z^n, \qquad H^-(z,q) = (1-z) \sum_{n=0}^{\infty} \left(1 - \frac{P_n}{n!}\right) h_n(q) z^n.$$

The distribution has exponential tails.

- Using a *digital tree* (aka *trie*), we only need a single string register to recognize perm classes for Poisson and logarithmic distribs!

- Poisson = sorted perms: $U_1 < U_2 < U_3$

- Logarithmic = max-first perms: $U_1 > U_2 , U_3$



cf **Leader election**: Prodinger; Fill, Mahmoud, Szpankowski, Janson,...

Theorem 2. *The Poisson and logarithmic distributions of parameter $\lambda \in (0,1)$ have a strong simulation by a Buffon machine that only uses a single string register.*

☛ • ***Poisson***: Declare success (1) if N=0; failure o.w. Get exp(-λ), etc.

☛ • **Check P:** Do only one run; return(1) if success. E.g, for Poisson, gives (1-λ)exp(λ)

☛ • Use alternating (zigzag) perms & get trigs!

Theorem 3. *The following functions admit a strong simulation:*

$$e^{-x},\ e^{x-1},\ (1-x)e^{x},\ xe^{1-x},$$
$$\frac{x}{\log(1-x)^{-1}},\ \frac{1-x}{\log(1/x)},\ (1-x)\log\frac{1}{1-x},\ x\log(1/x),$$
$$\frac{1}{\cos(x)},\ x\cot(x),\ (1-x)\cos(x), (1-x)\tan(x).$$

- Polylogarithms, Bessel,...: do *r* experiments

$$\mathrm{Li}_r(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^r},$$

$$\mathrm{Li}_2(1/2) = \frac{\pi^2}{12} - \frac{1}{2}\log^2 2, \qquad \mathrm{Li}_3(1/2) = \frac{1}{6}\log^3 2 - \frac{\pi^2}{12}\log 2 + \frac{7}{8}\zeta(3).$$

Get log(2), then $\pi^2/24$, in less than 10 flips on average

4. Square roots, algebraic & hypergeometric functions

- Generate N∈Geo(λ) and succeed if we get a balanced score from 2N flips.

- The probability of success:

$$s(\lambda) := \sum_{n=0}^{\infty} (1-\lambda)\lambda^n \varpi_n = \sqrt{1-\lambda}$$

$$\varpi_n = \frac{1}{2^{2n}}\binom{2n}{n}.$$

Theorem 4. *The square-root construction of Equation (11) provides an exact Bernoulli generator of parameter $\sqrt{1-\lambda}$, given a $\Gamma B(\lambda)$. The mean number of coin flips required, not counting the ones involved in the calls to $\Gamma B(\lambda)$, is $\frac{2\lambda}{1-\lambda}$. Hence the function $\sqrt{1-x}$ is strongly realizable.*

Theorem 5 ([21]). *To each bistoch grammar G and non-terminal S, there corresponds a construction (Figure 3), which can be implemented by a deterministic pushdown automaton and calls to a $\Gamma B(\lambda)$ and is of type $\Gamma B(\lambda) \longrightarrow \Gamma B\left(S\left(\frac{\lambda}{2}\right)\right)$, where $S(z)$ is the algebraic function canonically associated with the grammar G and non-terminal S.*

- Get hypergeometrics of binomial type.

Ramanujan:

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \binom{2n}{n}^3 \frac{6n+1}{2^{8n+4}},$$

procedure Rama(); $\{$*returns the value 1 with probability* $1/\pi\}$

S1. let $S := X_1 + X_2$, where $X_1, X_2 \in \mathrm{Geom}(\frac{1}{4})$;

S2. with probability $\frac{5}{9}$ do $S := S + 1$;

S3. for $j = 1, 2, 3$ do

S4. draw a sequence of $2S$ coin flippings;

if (# Heads − # Tails) $\neq 0$ then return(0);

S5. return(1).

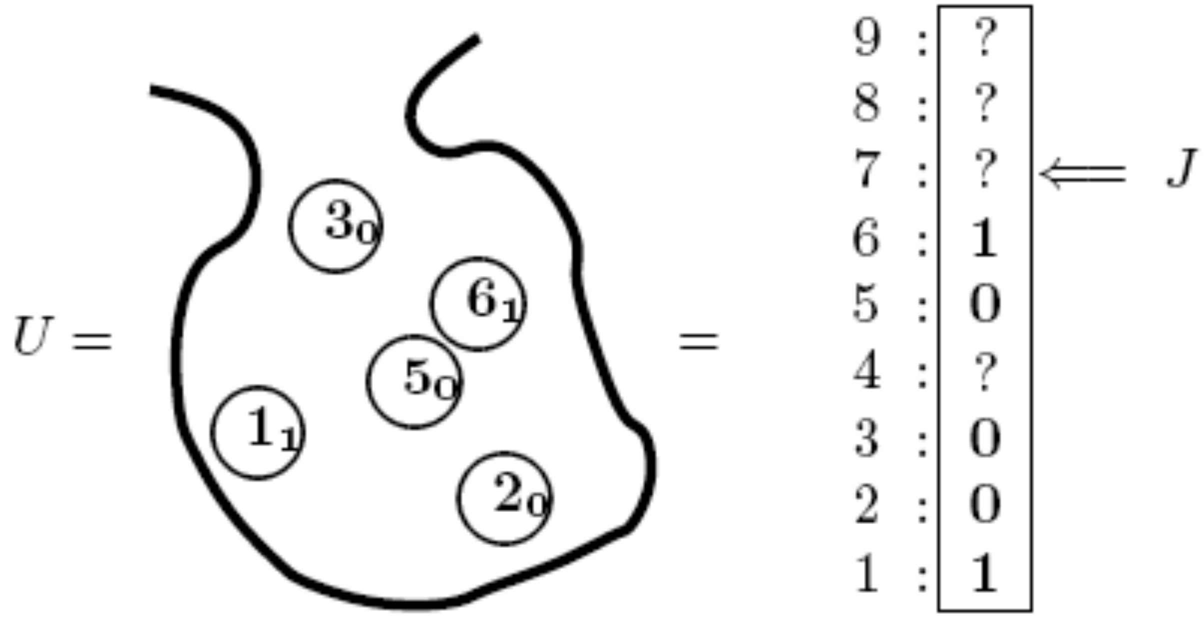
<11 coin flips on average

5. A Buffon integrator

- In a construction of a ΓB(φ(λ)) from a ΓB(λ), we substitute a ΓB(Uλ), with U uniform. Get an *integrator*:

$$\Phi(\lambda) = \frac{1}{\lambda}\int_0^\lambda \phi(w)\, dw.$$

- We can do *a product ΓB(Uλ)=ΓB(U).ΓB(λ) by an AND* (∧) as well as by *emulating a uniform U with a "bag"*:




```
Ghalf:=proc() local K;
# a geometric RV of param. 1/2
      K:=-1; do K:=K+1; if flip()=0
            then return(K) fi; od;

bag:=proc(U)  local J;
         J:=1+Ghalf();
         if type(U[J],name)
            then U[J]:=flip(); fi;
         return(U[J]); end;
```

Theorem 6. *Any construction* **C** *that produces a* $\Gamma B(\phi(\lambda))$ *from a* $\Gamma B(\lambda)$ *can be transformed into a construction of a* $\Gamma B(\Phi(\lambda))$, *where* $\Phi(\lambda) = \frac{1}{\lambda}\int_0^\lambda \phi(w)\,dw$, *by addition of a geometric bag. In particular, if* $\phi(\lambda)$ *is realizable, then its integral taken starting from* 0 *is also realizable. If in addition* $\phi(\lambda)$ *is analytic at 0, then its integral is strongly realizable.*

- Chain: p $\rightarrow$ p^2 $\rightarrow$ $1/(1+p^2)$ $\xrightarrow{\int}$ arctan(x)

Theorem 7. *The following functions are strongly realizable* $(0 < x < 1)$*:*

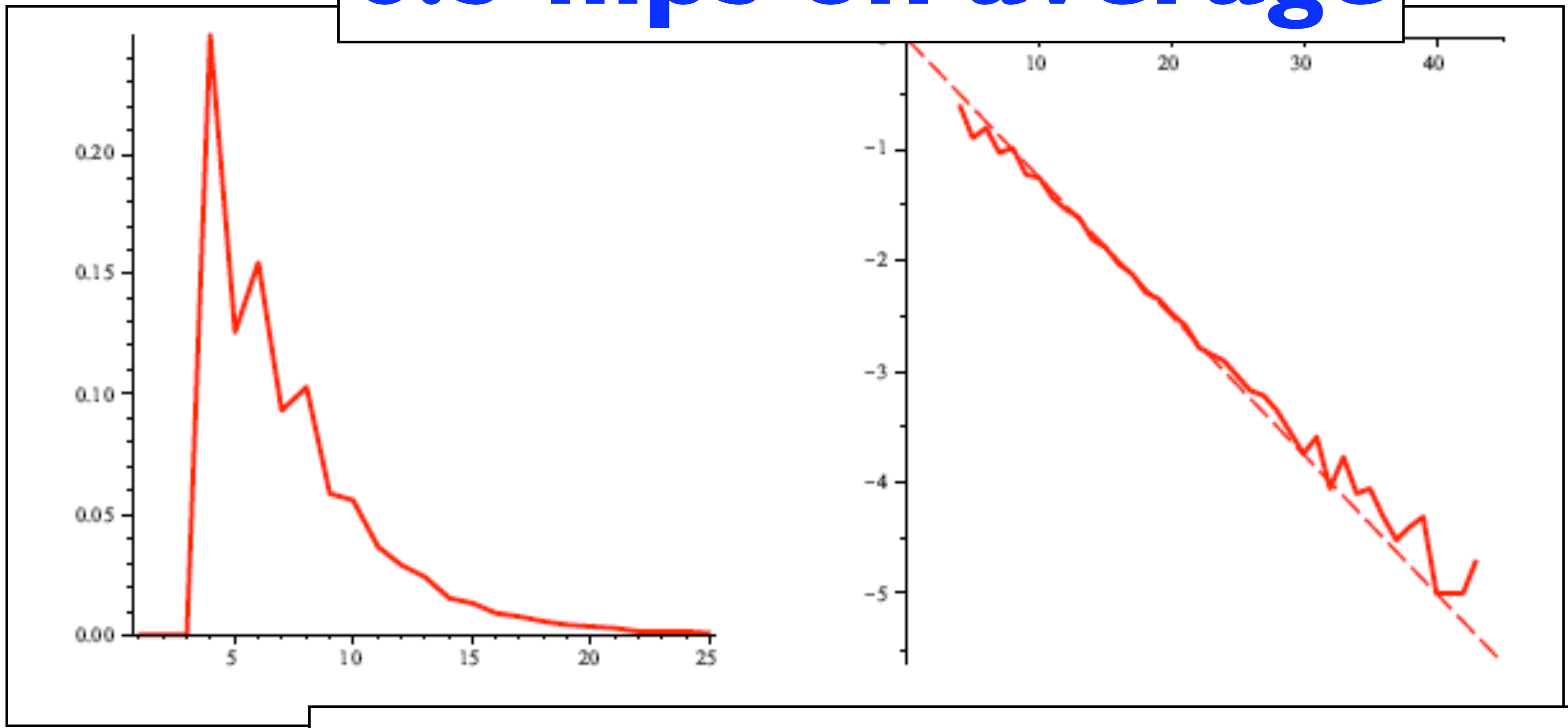
$$\log(1+x),\ \arctan(x),\ \frac{1}{2}\arcsin(x),\ \int_0^x e^{-w^2/2}\,dw.$$

- **Madhava-Gregory-Leibniz:** arctan(1)=π/4

```
MGL:=proc()    do
        if bag(U)=0 then return(1) fi;  if bag(U)=0 then return(1) fi;
        if bag(U)=0 then return(0) fi;  if bag(U)=0 then return(0) fi; od;  end.
```

- **Machin machine:** arctan(1/2)+arctan(1/3)=π/4.

6.5 flips on average



Distribution of costs (plain & log.)

6. Experiments

MAPLE:
an interpreter
~ 60 lines

```
> Z4:=expn(compl(ave(flip,ave(int1(int1(int1(even(prod(z, prod
  (x, y))), x, ONE), y, ONE), z, ONE),compl(sqrt0(int0(ave(logp
  (flip), sqrt0(prod(flip, int0(prod(Y, even(int0(even(prod(X,
  X)), X, expn(prod(flip,Z))))), Y, expn(prod(flip,flip)))))),
  Z, flip)))))))):
> test(Z4,10000);
mean_number_of_flips = 103.1645000
```

$$0.6313000000$$

```
> val(Z4);
```

$$e^{-\frac{1}{2}+\frac{3}{16}\zeta(3)-\frac{1}{4}\sqrt{2}\sqrt{\int_0^{\frac{1}{2}}\left(\frac{1}{2}\ln(2)+\frac{1}{4}\sqrt{\frac{e^{-\frac{1}{4}}}{1+\frac{\arctan\left(e^{-\frac{1}{2}Z}\right)}{e^{-\frac{1}{2}Z}}}}\right)dZ}}$$

```
> evalf(val(Z4));
```

$$0.6356033009$$

- Implements all earlier constructions: *it works*!

- Results for π-related constants:

$\mathrm{Li}_2(1/2)$	Rama	arcsin $[1; \frac{1}{\sqrt{2}}; \frac{1}{2}]$			arctan $[1/2+1/3; 1]$		$\zeta(4)$	$\zeta(2)$
$\frac{\pi^2}{24}$	$\frac{1}{\pi}$	$\frac{\pi}{4}$	$\frac{\pi}{8}$	$\frac{\pi}{12}$	$\frac{\pi}{4}$	$\frac{\pi}{8}$	$\frac{7\pi^4}{720}$	$\frac{\pi^2}{12}$
7.9	10.8	76.5 (∞)	16.2	4.9	4.5	26.7 (∞)	6.2	7.2.

Method; constant; mean # flips