Analytic Combinatorics of the Mabinogion and OK-Corral Urns

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Population of $N$ sheep that bleat either $A[aah]$ or $B[eeh]$. At times $t = 0, 1, 2, \ldots$, a randomly chosen sheep bleats and convinces one sheep of the other kind to change its opinion.

- Time to reach unanimity?
- Probability that one minority group wins?

E.g.: French election campaign (2007): $N = 60,000,000$. Probability of reversing of majority of 51%? In the “fair” case ($N/2, N/2$), time to reach unanimity?
Population of $N$ gangsters of gang either $A$ or $B$. At times $t = 0, 1, 2, \ldots$, a randomly chosen gangster kills a member of the other group.

- **Time to win?**
- **Probability that one minority group survives?**

E.g.: The OK Corral fight at Tombstone. (Wyatt Earp and Doc Holliday)
An urn contains balls of 2 possible colours.

A fixed set of rules governs the urn evolution:

Draw: \[
\begin{cases}
\text{Red (A)} & \alpha \\
\text{Blue (B)} & \beta \\
\gamma & \delta
\end{cases}
\]

Balanced urns: \[\alpha + \beta = \gamma + \delta =: \sigma\]
Classically: \[\beta, \gamma \geq 0 \text{ and } \sigma > 0\].
Convention: The ball “drawn” is not withdrawn (not taken out)!
Urn models (2)

All (classical) balanced $2 \times 2$ models are “integrable”!

- An urn
  $$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix};$$

- A partial differential operator
  $$\mathcal{D} = x^{\alpha+1}y^\beta \frac{\partial}{\partial x} + x^\gamma y^{\delta+1} \frac{\partial}{\partial y};$$

- An ordinary nonlinear system
  $$\{ \dot{X} = X^{\alpha+1}Y^\beta, \quad \dot{Y} = X^\gamma Y^{\delta+1} \}.$$

Refs: [Fl-Ga-Pe’05]; [Fl-Dumas-Puyhaubert’06] [Conrad-Fl’06] [Hwang-Kuba-Panholzer’07+]; [Mahmoud*].
Cf also: Janson* ♥♥♥
§1. Ehrenfest & Mabinogion
Ehrenfest

Ehrenfest’s two chambers $\mathcal{E} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$;

Formally: $\mathcal{D} = x \partial_y + y \partial_x$; \{ $\dot{X} = Y$; $\dot{Y} = Y$ \};

Combinatorics of set partitions: histories from $(N,0)$ to $(k, N-k)$ are partitions with $N-k$ even classes and $k$ odd classes:

$$\mathbb{P}[(N,0) \rightarrow (k, N-k), \text{ $N$ steps}] = \frac{n!}{N^n} \cdot \binom{N}{k} \cdot [z^n] \sinh^k(z) \cosh^{N-k}(z)$$

Also: path in a special graph

Also: special walks on the interval

$\begin{array}{c}
k - 1 \\
k \\
(N-k)/N \\
k + 1
\end{array}$

$\begin{array}{c}
k/N \\
\rightarrow
\end{array}$

$\begin{array}{c}
\leftarrow
\end{array}$
Mabinogion (1)

Ehrenfest: \[ k - 1 \xleftarrow{k/N} k \xrightarrow{(N-k)/N} k + 1 \]

Mabinogion: \[ k - 1 \xleftarrow{(N-k)/N} k \xrightarrow{k/N} k + 1 \]

Absorption

Time-reversal relates \( \mathcal{M}[N] \) and \( \mathcal{E}[N + 2] \), with fudge factors

\[ P(T = n + 1) = \frac{N - 1}{N^{n+1}} \binom{N-2}{k-1} n! [z^n](\sinh z)^{k-1}(\cosh z)^{N-k-1}. \]
**Theorem M2.** Probability $\Omega_{N,k}$ of majority reversal: $k = xN$ is initial # of A’s, with $x > \frac{1}{2}$; A’s become extinct

$$- \lim_{N \to \infty} \frac{1}{N} \log \Omega_{N,k} = \log 2 + x \log x + (1 - x) \log(1 - x).$$

**Proof.** Laplace transform + Laplace method (peak at end-point).

$$\Omega_{N,k} = \cdots \int_0^\infty e^{-z}(\sinh z)^{k-1}(\cosh z)^{N-k-1} \, dz = \cdots \int_0^1 (1 - y)^{k-1}(1 + y)^{N-k-1} \, dy \sim 2^{-N+1} \binom{N - 2}{k - 2} \frac{1}{2x - 1}.$$

$N = 60,000,000$: 51% → $10^{-5.215}$; 50.1% → $10^{-54}$. 
**Theorem M3.** Time $T$ till absorption, when $k = xN$ is initial number of A's, with $x < \frac{1}{2}$:

$$
\mathbb{P}\left( \frac{T - N\tau}{\sigma \sqrt{N}} \right) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-w^2/2} \, dw.
$$

$$
\tau(x) = \frac{1}{2} \log \frac{1}{1 - 2x}; \quad \sigma(x)^2 = \frac{x(1-x)}{(1-2x)^2} + \frac{1}{2} \log(1 - 2x).
$$

Proof. Laplace transform

$$
\mathbb{E} \left[ e^{uT} \right] = \cdots \int_{0}^{\infty} e^{-z} \left( \sinh \frac{uz}{N} \right)^{k-1} \left( \cosh \frac{uz}{N} \right)^{N-k-1} \, dz
$$

Characteristic functions $u = e^{it/\sqrt{N}}$. Laplace method (with peak inside the interval) and perturbation.
Experiments

Distribution of absorption time

UNFAIR URN

FAIR URN

Normal

UNFAIR URN vs FAIR URN

Ehrenfest and Mabinogion
Friedman and OK-Corral
Theorem M4. FAIR URN $N = \nu + \nu$: time $\widehat{T}$ till absorption

$$P(\widehat{T} = n) \sim \frac{2}{\nu} Ce^{-t} e^{-e^{-2t}}; \quad n = \frac{1}{2} \nu \log \nu + t\nu.$$ 

(i) Exact distribution $\propto \sum_{j=1}^{\nu/2} \text{Geom} \left( \left( \frac{2j-1}{\nu} \right)^2 \right)$.

(ii) Saddle point for $[z^n](\sinh z)^N$ when $n \approx N \log N$, with suitable perturbations.

$$\widehat{T}_\infty \overset{d}{=} \sum_{\ell \geq 1} \frac{\varepsilon\ell - 1}{\ell - 1/2}, \quad \varepsilon\ell \in \text{Exp}(1).$$

[Simatos-Robert-Guillemin’08] [Biane-Pitman-Yor*]
§2. Friedman & OK-Corral
Friedman

Adverse campaign model: \( \mathcal{F} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \)

= A counterbalance of influences [Friedman 1949]

- System: \( \dot{X} = XY, \quad \dot{Y} = XY \) is exactly solvable

Distribution is expressed in terms of Eulerian numbers:

- rises in perms;
- leaves in increasing Cayley trees;
- other urn models [Mahmoud*] [FlDuPu06].

\[
A(z, u) = \sum_{n,k} A_{n,k} u^k \frac{z^n}{n!} = \frac{1 - u}{1 - ue^z(1-u)}
\]

\[
A_{n,k} = \sum_{0 \leq j \leq k} (-1)^j \binom{n+1}{j} (k-j)^n
\]
OK Corral

Two gangs of $m$ and $n$ gunwomen
At any time, one shooter shoots

- Survival probabilities?
- Time till one gang wins?

[Williams & McIlroy 1998]
[Kingman 1999]
[Kingman & Volkov 2003]
OK Corral (1)

Friedman $\mathcal{F} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; OK Corral $\mathcal{O} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$; i.e., $\mathcal{O} = -\mathcal{F}$

Time reversal: $P_O(m, n \searrow s) = \frac{s}{m+n} P_F(s, 0 \rightarrow m, n)$

**Theorem O1.** Probability of $s$ survivors of type A, from $(m, n)$

$$P_O(m, n, s) = \frac{s!}{(m+n)!} \sum_{k=1}^{m} (-1)^{m-k} \binom{k-1}{s-1} \binom{m+n}{n+k} k^{m+n-s}$$

Involves generalized Eulerian numbers and expansions + identities.

**Theorem O2.** The probability that first group survives is

$$P_O S(m, n) = \frac{1}{(m+n)!} \sum_{k=1}^{m} A_{m+n-k} = \frac{1}{(m+n)!} \sum_{k=1}^{n} (-1)^{m-k} \binom{m+n}{n+k} k^{m+n}$$
Theorem 03. NEARLY FAIR FIGHTS: If \( \frac{m-n}{\sqrt{m+n}} = \theta \), then

\[
P_{\text{OS}}(m, n) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\theta \sqrt{3}} e^{-t^2/2} dt + O\left(\frac{1}{n}\right).
\]

Theorem 04. UNFAIR FIGHTS: Probability of survival with \( m = \alpha n \) and \( \alpha < 1 \) is exponentially small.

It is related to the Large Deviation rate for Eulerian statistics.

Make use of explicit GFs and usual Large Deviation techniques [Quasi-Powers, shifting of the mean]
Theorem 05. [Kingman] Number of survivors:

- If \((m - n) \gg \sqrt{m + n}\) then in probability \(S \to \sqrt{m^2 - n^2}\);
- If \((m - n) \ll \sqrt{m + n}\) then
\[
P(S \leq \lambda n^{3/4}) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\lambda^2 \sqrt{3/8}} e^{-t^2/2} \, dt.
\]

Only dominant asymptotics, indirectly from Kingman et al. Get:

Theorem 06. MOMENTS

\[
\mathbb{E}[S^\ell] = \frac{1}{(m+n)!} \sum_{k=1}^{m} (-1)^{m-k} \binom{m+n}{n-k} k^{m+n-1} [f_\ell(k)Q(k) + g_\ell(k)],
\]

where \(f_\ell, g_\ell\) are polynomials and \(Q(k) = \frac{k}{k} + \frac{k(k-1)}{k^2} + \cdots\) is Ramanujan’s = birthday paradox function.

\(\mapsto\) Complex asymptotics à la Lindelöf–Rice.
Conclusions?

- **Analytic combinatorics** has something to say about balanced urn models, including some nonstandard ones ≠ probabilistic approaches [Mahmoud–Smythe–Janson].

- Analysis of imbalanced models??? Higher dimensions?

“Rather surprisingly, relatively sizable classes of nonlinear systems are found to have an extra property, **integrability**, which changes the picture completely. Integrable systems [...] form an **archipelago of solvable models in a sea of unknown**, and can be used as stepping stones to investigate properties of ‘nearby’ non-integrable systems.”

Eilbeck, Mikhailov, Santini, and Zakharov (2001)