

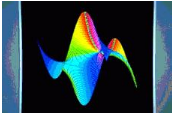

Analytic Combinatorics of the Mabinogion and OK-Corral Urns

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Based on joint work with *Thierry Huillet* and *Vincent Puyhaubert*

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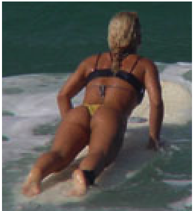


Mabinogion

MABINOGION

Population of N sheep that bleat either $A[aah]$ or $B[eeh]$.
At times $t = 0, 1, 2, \ldots$, a randomly chosen sheep bleats and convinces one sheep of the other kind to change its opinion.

- *Time to reach unanimity?*
- *Probability that one minority group wins?*

E.g.: French election campaign (2007): $N = 60,000,000$.
Probability of reversing of majority of 51%?
In the "fair" case $(N/2, N/2)$, time to reach unanimity?

OK Corral

OK CORRAL



Population of N gangsters of gang either A or B.
At times $t = 0, 1, 2, \ldots$, a randomly chosen gangster kills a member of the other group.

- *Time to win?*
- *Probability that one minority group survives?*

E.g.: The OK Corral fight at Tombstone.
(Wyatt Earp and Doc Holliday)

Urn models (1)

- An **urn** contains **balls** of 2 possible colours

- A **fixed set of rules** governs the urn evolution:

Draw: \ *Place*	Red (A)	Blue (B)
Red (A)	α	β
Blue (B)	γ	δ

Balanced urns: $\boxed{\alpha+\beta=\gamma+\delta=:\sigma}$

Classically: $\beta,\gamma \geq 0$ and $\sigma > 0$.

Convention: The ball "drawn" is *not* withdrawn (not taken out)!

Urn models (2)

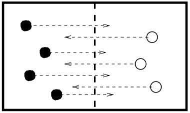
All (classical) balanced 2×2 models are "integrable"!

- $=$ An urn $\boxed{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}}$;
- $=$ A partial differential operator $\boxed{\mathfrak{D} = x^{\alpha+1} y^{\beta} \partial_x + x^{\gamma} y^{\delta+1} \partial_y}$;
- $=$ An ordinary nonlinear system $\boxed{\{\dot{X} = X^{\alpha+1} Y^{\beta}, \quad \dot{Y} = X^{\gamma} Y^{\delta+1}\}}$.

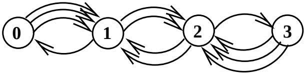
Refs: [Fl-Ga-Pe'05]; [Fl-Dumas-Puyhaubert'06] [Conrad-Fl'06] [Hwang-Kuba-Panholzer'07+]; [Mahmoud$\star$].
Cf also: Janson$\star^{\heartsuit\heartsuit\heartsuit}$

§1. Ehrenfest & Mabinogion

Ehrenfest

-  Ehrenfest's two chambers $\mathcal{E} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$;
- Formally: $\mathfrak{D} = x\partial_y + y\partial_x; \quad \{\dot{X} = Y; \quad \dot{Y} = Y\};$
- Combinatorics of set partitions: *histories* from $(N, 0)$ to $(k, N-k)$ are partitions with $N-k$ even classes and k odd classes:

$$\mathbb{P}\left[(N,0) \to (k, N-k),\ N \text{ steps}\right] = \frac{n!}{N^n} \cdot \binom{N}{k} \cdot [z^n] \sinh^k(z) \cosh^{N-k}(z)$$

Also: *path in a special graph* 

Also: *special walks* on the interval $\boxed{\mathbf{k-1}} \xleftarrow{k/N} \boxed{\mathbf{k}} \xrightarrow{(N-k)/N} \boxed{\mathbf{k+1}}$

Mabinogion (1)

Ehrenfest: $\boxed{\mathbf{k-1}} \xleftarrow{k/N} \boxed{\mathbf{k}} \xrightarrow{(N-k)/N} \boxed{\mathbf{k+1}}$

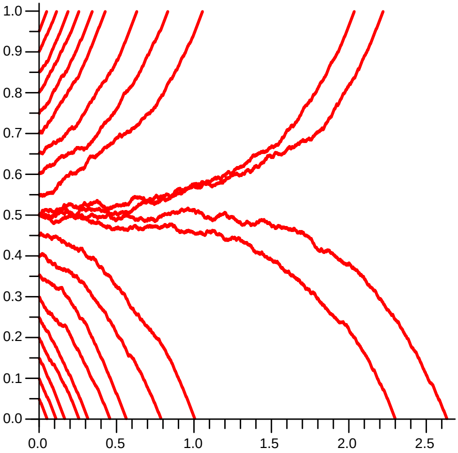
Mabinogion: $\boxed{\mathbf{k-1}} \xleftarrow{(N-k)/N} \boxed{\mathbf{k}} \xrightarrow{k/N} \boxed{\mathbf{k+1}}$ + **absorption**

Time-reversal relates $\mathcal{M}[N]$ and $\mathcal{E}[N+2]$, with fudge factors

Theorem M1. *Absorption time T of the Mabinogion urn:*

$$\mathbb{P}(T = n+1) = \frac{N-1}{N^{n+1}} \binom{N-2}{k-1} n!\, [z^n](\sinh z)^{k-1}(\cosh z)^{N-k-1}.$$

Trajectories



Mabinogion (2)

Theorem M2. *Probability $\Omega_{N,k}$ of majority reversal: $k = xN$ is initial # of A's, with $x > \frac{1}{2}$; A's become extinct*

$$-\lim_{N\to\infty} \frac{1}{N} \log \Omega_{N,k} = \log 2 + x\log x + (1-x)\log(1-x).$$

Proof. Laplace transform + Laplace method (peak at end-point).

$$\begin{aligned}
\Omega_{N,k} &= \cdots \int_0^\infty e^{-z}(\sinh z)^{k-1}(\cosh z)^{N-k-1}\,dz \\
&= \cdots \int_0^1 (1-y)^{k-1}(1+y)^{N-k-1}\,dy \\
&\sim 2^{-N+1}\binom{N-2}{k-1}\frac{1}{2x-1}.
\end{aligned}$$

$N = 60,000,000$: $51\% \to 10^{-5,215}$; $50.1\% \to 10^{-54}$.

Mabinogion (3)

Theorem M3. *Time T till absorption , when $k = xN$ is initial number of A's, with $x < \frac{1}{2}$:*

$$\mathbb{P}\left(\frac{T - N\tau}{\sigma\sqrt{N}}\right) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-w^2/2}\, dw.$$

$$\tau(x) = \frac{1}{2} \log \frac{1}{1-2x};\ \sigma(x)^2 = \frac{x(1-x)}{(1-2x)^2} + \frac{1}{2}\log(1-2x).$$

Proof. Laplace transform

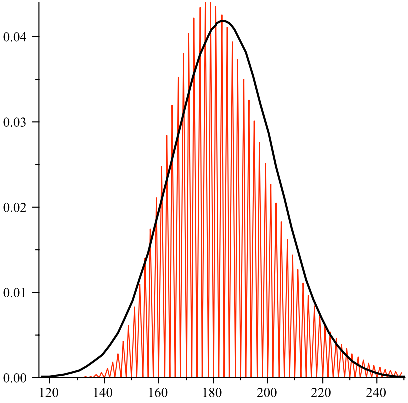
$$\mathbb{E}\left[e^{uT}\right] = \cdots \int_0^\infty e^{-z} \left(\sinh \frac{uz}{N}\right)^{k-1} \left(\cosh \frac{uz}{N}\right)^{N-k-1} dz$$

Characteristic functions $u = e^{it/\sqrt{N}}$. Laplace method (with peak inside the interval) and perturbation.

Experiments

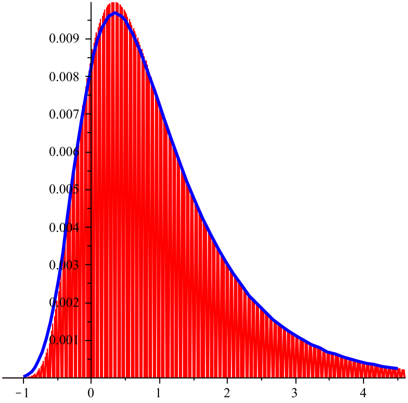
Distribution of absorption time

UNFAIR URN



Normal

FAIR URN



$\rightsquigarrow$

Mabinogion (4)

Theorem M4. *FAIR URN $N = \nu + \nu$: time $\widehat{T}$ till absorption*

$$\mathbb{P}(\widehat{T} = n) \sim \frac{2}{\nu} C e^{-t} e^{-e^{-2t}}; \qquad n = \frac{1}{2}\nu \log \nu + t\nu.$$

(i) Exact distribution $\propto \sum_{j=1}^{\nu/2} \mathbf{Geom}\left(\left(\frac{2j-1}{\nu}\right)^2\right)$.

(ii) Saddle point for $[z^n](\sinh z)^N$ when $n \approx N \log N$, with suitable perturbations.

$$\widehat{T}_\infty \overset{d}{\equiv} \sum_{\ell \geq 1} \frac{\varepsilon_\ell - 1}{\ell - 1/2}, \qquad \varepsilon_\ell \in \mathrm{Exp}(1).$$

[Simatos-Robert-Guillemin'08] [Biane-Pitman-Yor*]

§2. Friedman & OK-Corral

Friedman

Adverse campaign model: $\mathcal{F} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

= A counterbalance of influences [Friedman 1949]

- System: $\{\dot{X} = XY, \quad \dot{Y} = XY\}$ is exactly solvable

Distribution is expressed in terms of Eulerian numbers:

- rises in perms;
- leaves in increasing Cayley trees;
- other urn models [Mahmoud*] [FlDuPu06].

$$\boxed{A(z,u) = \sum_{n,k} A_{n,k} u^k \frac{z^n}{n!} = \frac{1-u}{1-ue^{z(1-u)}}}$$

$$A_{n,k} = \sum_{0 \le j \le k} (-1)^j \binom{n+1}{j} (k-j)^n$$

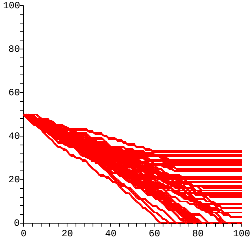
OK Corral

Calamity Jane
©Morris & Goscinny

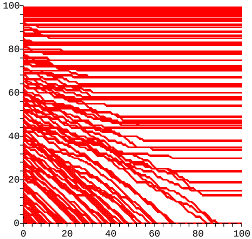
Two gangs of m and n gunwomen
At any time, one shooter shoots

- Survival probabilities?
- Time till one gang wins?

[Williams & McIlroy 1998]
[Kingman 1999]
[Kingman & Volkov 2003]



$(m, n) = (50, 50)$



$m + n = 100$

OK Corral (1)

Friedman $\mathcal{F} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; OK Corral $\mathcal{O} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$; i.e., $\mathcal{O} = -\mathcal{F}$

Time reversal: $\mathbb{P}_O(m, n \searrow s) = \dfrac{s}{m+n} P_{\mathcal{F}}(s, 0 \to m, n)$

Theorem O1. *Probability of s survivors of type A, from (m, n)*

$$\mathbb{P}_{\mathcal{O}}(m, n, s) = \frac{s!}{(m+n)!} \sum_{k=1}^{m} (-1)^{m-k} \binom{k-1}{s-1} \binom{m+n}{n+k} k^{m+n-s}$$

Involves generalized Eulerian numbers and expansions + identities.

Theorem O2. *The probability that first group survives is*

$$\mathbb{P}_{\mathcal{O}S}(m, n) = \frac{1}{(m+n)!} \sum_{k=1}^{m} A_{m+n-k} = \frac{1}{(m+n)!} \sum_{k=1}^{n} (-1)^{m-k} \binom{m+n}{n+k} k^{m+n}$$

OK Corral (2)

Theorem O3. *NEARLY FAIR FIGHTS: If* $\frac{m-n}{\sqrt{m+n}} = \theta$, *then*

$$\mathbb{P}_{\mathcal{O}S}(m,n) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\theta\sqrt{3}} e^{-t^2/2}\, dt + O\left(\frac{1}{n}\right).$$

Theorem O4. *UNFAIR FIGHTS: Probability of survival with* $m = \alpha n$ *and* $\alpha < 1$ *is* **exponentially small**.
It is related to the Large Deviation rate for Eulerian statistics.

Make use of explicit GFs and usual Large Deviation techniques [Quasi-Powers, shifting of the mean]

Theorem O5. *[Kingman] Number of survivors:*

- *If* $(m-n) \gg \sqrt{m+n}$ *then in probability* $S \to \sqrt{m^2-n^2}$*;*
- *If* $(m-n) \ll \sqrt{m+n}$ *then*

$$\mathbb{P}(S \leq \lambda n^{3/4}) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\lambda^2\sqrt{3/8}} e^{-t^2/2}\, dt.$$

Only dominant asymptotics, indirectly from Kingman *et al.* Get:

Theorem O6. MOMENTS

$$\mathbb{E}[S^\ell] = \frac{1}{(m+n)!} \sum_{k=1}^{m} (-1)^{m-k} \binom{m+n}{n-k} k^{m+n-1} \left[f_\ell(k) Q(k) + g_\ell(k) \right],$$

where f_ℓ, g_ℓ are polynomials and $Q(k) = \frac{k}{k} + \frac{k(k-1)}{k^2} + \cdots$ is Ramanujan's $=$ birthday paradox function.

$\rightsquigarrow$ Complex asymptotics à la Lindelöf–Rice.

Conclusions?

- Analytic combinatorics has something to say about balanced urn models, including some nonstandard ones $\neq$ probabilistic approaches [Mahmoud–Smythe–Janson].
- Analysis of imbalanced models??? Higher dimensions?

"Rather surprisingly, relatively sizable classes of nonlinear systems are found to have an extra property, integrability, which changes the picture completely.
Integrable systems [...] form an archipelago of solvable models in a sea of unknown, and can be used as stepping stones to investigate properties of 'nearby' non-integrable systems."

Eilbeck, Mikhailov, Santini, and Zakharov (2001)