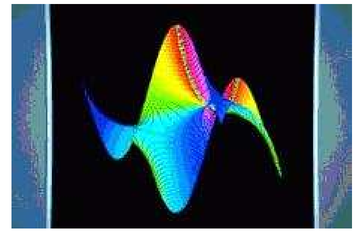


Probabilistic Counting:
from analysis to algorithms to programs

Philippe Flajolet, INRIA, Rocquencourt

`http://algo.inria.fr/flajolet`

Give a (large) sequence s over some (large) domain $\mathcal{D}$,

$$s = s_1\, s_2 \cdots s_\ell, \qquad s_j \in \mathcal{D},$$

View sequence s as a *multiset*

$$\mathcal{M} = m_1^{f_1}\, m_2^{f_2}\, \cdots\, m_n^{f_n}.$$

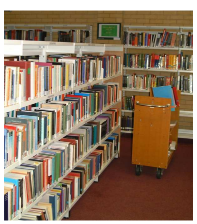
— **A.** **Length :=** ℓ;
— **B.** **Cardinality** := $\mathrm{card}\{s_j\} \equiv n$;
— **C.** **Mice** := # elements repeated 1,2,. . . ,10 times;
— **D.** **Icebergs** := # elem. with *relative* frequency $\frac{1}{\ell} f_v > \frac{1}{100}$;
— **E.** **Elephants** := # elem. with *absolute* frequency $f_v > 200$;
— **F.** **Frequency moments** := $\left(\sum f_v^r\right)^{1/r}$.

Alon, Matias, Szegedy; Bar-Yossef; Indyk; Motwani; RAP@Inria. . .
Fl-Martin (1985); Fl (1992); Louchard (1997); Durand-Fl (2003); FlFuGaMe $\rightsquigarrow$ AofA07, Prodinger, Fill-Janson-Mahmoud-Szpankowski . . .

$$s = s_1\, s_2\, \cdots s_\ell, \qquad s_j \in \mathcal{D}.$$

Length can be $\ell \gg 10^9$. Cardinality can be $n \propto 10^7$.


Routers in the range of Terabits/sec (10^{12}b/s).



Google indexes 6 billion pages & prepares to index 100 Petabytes of data (10^{17}B).

Can estimate a few key characteristics, QUICK and EASY

Length; Cardinality; Icebergs; Mice; Elephants; Freq. moments. . .

Rules of the game

- **Limited storage**: cannot store elements; use $\approx$ **one page** of print$\equiv$4kB..
- **Limited time**: proceed online = single pass, read once data.
- Allow to **estimate** rather than compute exactly.

Assume **hash function** $h : \mathcal{D} \mapsto [0, 1]$ scrambles data *uniformly*:

Angel-daemon scenario: n values, replicated and permuted at will, then made into random uniform $[0, 1]$.

What for?

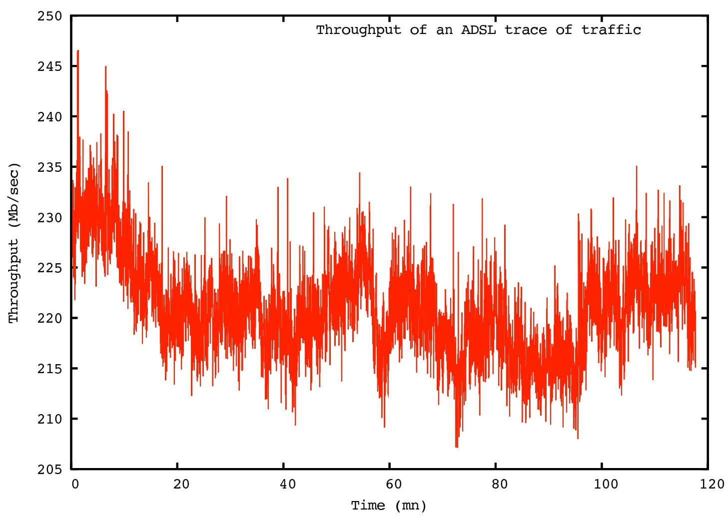
— Network management, worms and viruses, traffic monitoring

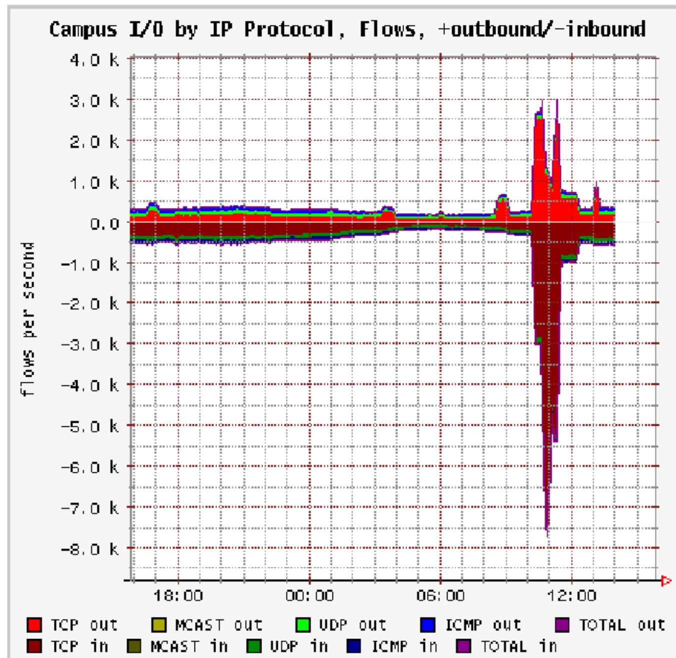
— Databases: Query optimization = size estimation; also "sketches".

— Document classification (Broder), cf Google, citeseer, ...

— Data mining of web graph, internet graph, etc

Traces of attacks: Number of active connections in time slices.





(Raw ADSL traffic) *(Attack)*

Incoming/Outgoing flows at 40Gbits/second. Code Red Worm: 0.5GBytes of compressed data per hour (2001). CISCO: in 11 minutes, a worm infected 500,000,000 machines.

Left: ADSL FT@Lyon 1.5×10^8 packets (21h–23h). Right: (Estan-Varghese-Fisk) different incoming/outgoing connections

Claims:

— High Tech algorithms based on probabilities.
— Efficient programs: Produce **short algorithms & programs** with $\mathcal{O}(10)$ instructions. Gains by factors in the range 100-1000 (!)
— No maths, no algorithms!

AofA: Symbolic methods and generating functions, complex asymptotics (singularities, saddle-point), limit laws and quasipowers, transforms (Mellin), analytic depoissonization. . .

Constants play a crucial rôle.

1 APPROXIMATE COUNTING

In streaming framework: given $s_1 s_2 \cdots s_\ell$, get length ℓ.
Means: maintain an efficient counter of events.

The oldest algorithm (Morris CACM:1977): Counting a large number of events in small memory.

First analysis (F. 1985). Prodinger (1992–4).

Approximate Counting

• Information Theory: need $\log_2 N$ bits to count till N.

• Approximate counting: use $\log_2 \log N + O(1)$ for ε–approximation, in relative terms and **in probability**.

How to find an unbounded integer while posing few questions?

— Ask if in $[1—2]$, $[2—4]$, $[4—8]$, $[8–16]$, etc?

— Conclude by binary search (cost is $2\log_2 n$).

= A general paradigm for unbounded search:

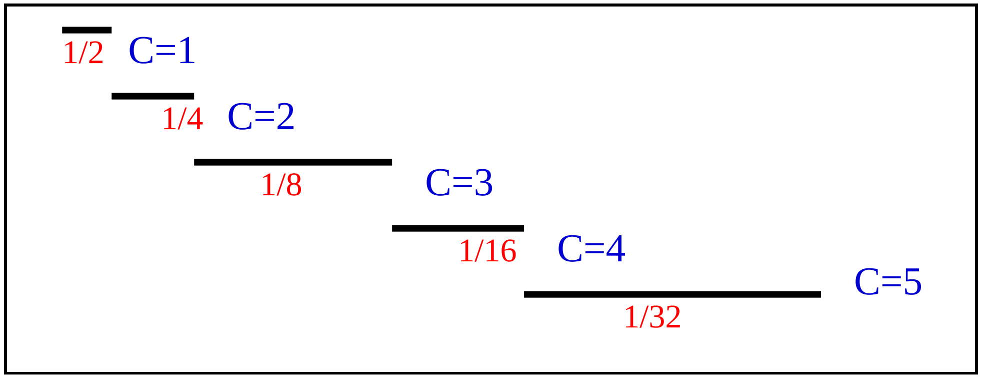
• Ethernet proceeds by period doubling + randomization.

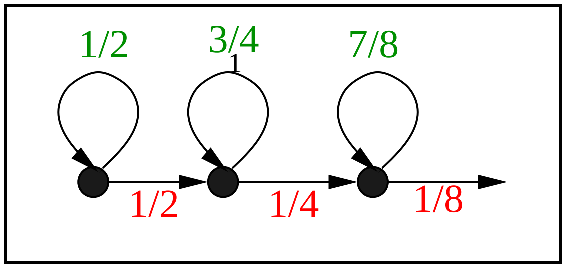
• Wake up procedures for mobile communication (Lavault$^+$)

• Adaptive data structures: e.g., extendible hashing tables.

♡ Approximate Counting

Emulate a counter subject to $X := X + 1$.



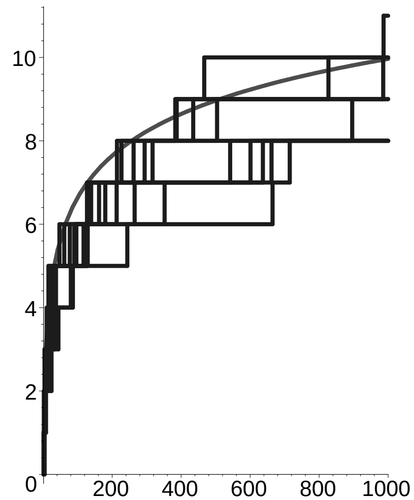


Algorithm: Approximate Couting /* binary base */
— Initialize: $C := 1$;
— Increment: do $C := C + 1$ with probability 2^{-C};
— Output: $2^{C} - 2$.

Alternate base $q \to 1$ controls cost/accuracy tradeoff.

Expect C near $\log_2 n$ after n steps, then use only $\log_2 \log n$ bits.

10 runs of of APCO: value of C ($n = 10^3$)



Theorem: • Basic binary algorithm is *unbiased*: $\mathbb{E}_n(2^C - 2) = n$.

• *Accuracy*, .i.e., *standard error*$\equiv \frac{\text{std-dev.}}{n}$ is $\sim \frac{1}{\sqrt{2}}$.

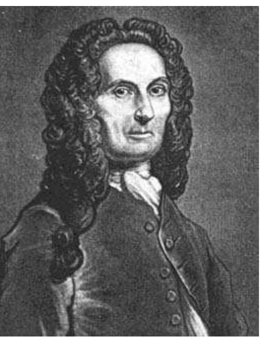
• Asymptotics of *distribution* is (binary case):

$$\mathbb{P}(C = \ell) \sim \Phi\left(\frac{n}{2^\ell}\right), \qquad \Phi(x) := \frac{1}{Q_\infty} \sum_{k \geq 0} (-1)^k q^{k(k-1)/2} \frac{e^{-xq^{-k}}}{Q_k},$$

where $Q_k := (1-q)(1-q^2)\cdots(1-q^k)$ and $q = \frac{1}{2}$ for binary case.

Count till N using $\log_2 \log N + \delta$ bits, with accuracy $\sim 0.59 \cdot 2^{-\delta/2}$.
Beats information theory: 8 bits for counts $\leq 2^{16}$ w/ accuracy $\approx 15\%$.

Recurrences: $P_{n+1,\ell} = (1-q^{\ell})P_{n,\ell} + q^{\ell-1}P_{n,\ell-1}$.
$\mathbb{E}_n(2^C) = n+2$, $\mathbb{V}(2^C) = n(n+1)/2$ (Morris1977).



Symbolic methodology:

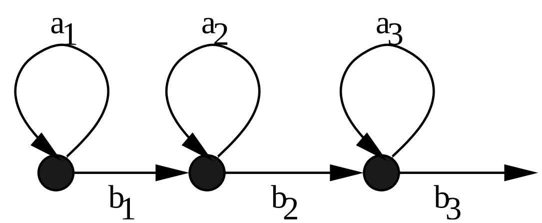
(i) Describe events; (ii) translate to *generating functions (GFs).*

An alphabet $\mathcal{A}$ with weights for Bernoulli trials. For a language describing an event $\mathcal{E}$, the GF is

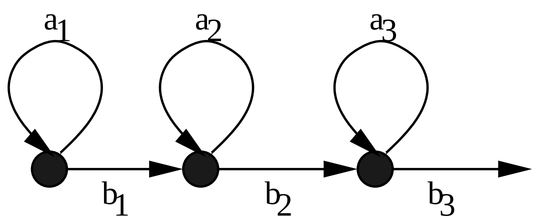
$$\boxed{E(z) \equiv \sum_n E_n z^n = \sum_n \mathbb{P}_n(\mathcal{E}) z^n}$$

$a \in \mathcal{A}$	$\mapsto$	αz
$E \uplus F$	$\mapsto$	$E(z) + F(z)$
$E \odot F$	$\mapsto$	$E(z) \times F(z)$
$E^\star$	$\mapsto$	$(1 - E(z))^{-1}$

$\frac{1}{1-f} = 1 + f + f^2 + \cdots \simeq (f)^\star$



$a_1^\star \cdot b_1 \cdot a_2^\star \cdot b_2 \cdot a_3^\star \cdot b_3$



$$\frac{(a_1)^\star b_1 (a_2)^\star b_2 (a_3)^\star}{\frac{1}{1-a_1} b_1 \frac{1}{1-a_2} b_2 \frac{1}{1-a_3}}$$

- Perform probabilistic valuation $a_j \mapsto q^j$; $b_j \mapsto 1-q^j$:

$$H_3(z) = \frac{q^{1+2} z^2}{(1-(1-q)z)(1-(1-q^2)z)(1-(1-q^3 z))}.$$

- Do partial fraction expansion to get exact probabilities.
- Do $(1-a)^n \approx e^{-na}$ to get main approximation:

$$\mathbb{P}(C=\ell) \sim \Phi\left(\frac{n}{2^\ell}\right), \qquad \Phi(x) := \frac{1}{Q_\infty} \sum_{k \geq 0} (-1)^k q^{k(k-1)/2} \frac{e^{-xq^k}}{Q_k},$$

where $Q_k := (1-q)(1-q^2)\cdots(1-q^k)$, and $q = \frac{1}{2}$ for binary case.

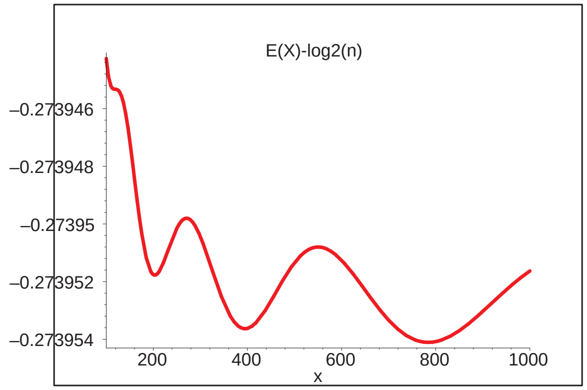
cf F.+Sedgewick, *Analytic Combinatorics* C.U.P., 2007.

♣ Dyadic superpositions of models: $\mathbb{P}_n(C = \ell) \sim \Phi\left(n/2^{\ell}\right)$.

Approximate Counting

Mean$(X) - \log_2 n$

$\longrightarrow$



$$\mathbb{E}_n(C) \sim \sum_{\ell} \ell \Phi\left(\frac{n}{2^{\ell}}\right)$$

Real analysis is possible: Knuth 1965, Guibas 1977+, Fill-Mahmoud-Szpankowski-Janson, Robert-Mohamed, . . .

• **Complex asymptotic methodology:** **Mellin transform** (FlDuGo95, FlSe*)

$$f^\star(s) := \int_0^\infty f(x) x^{s-1}\, dx.$$

Need **singularities** in **complex plane**.

Mellin: *Probabilistic counting, loglog counting* + Lempel-Ziv compression (Jacquet-Szpa) + dynamic hashing + tree protocols (Jacquet+) + Quadtries &c.

Mellin transform $f^*(s) = \int_0^\infty f(x)\,dx$, from real to *complex*.

♡ **Maps** asymptotics of f at 0 and $+\infty$ **to singularities of** $f^\star$ in $\mathbb{C}$:

$$C \cdot x^\alpha \quad \overset{\mathcal{M}}{\longleftrightarrow} \quad \pm \frac{C}{s+\alpha}.$$

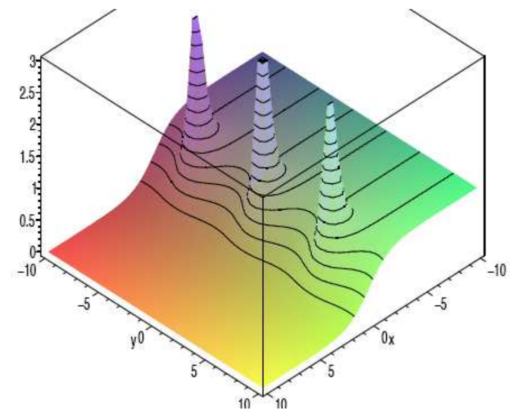
Reason: Inversion theorem $\frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} f^\star(s) x^{-s}\,ds$ + Residues.

♡ **Factorizes** *harmonic sums:*

$$\sum \lambda \cdot f(\mu x) \quad \overset{\mathcal{M}}{\longrightarrow} \quad f^\star(s) \cdot \sum \frac{\lambda}{\mu^s}.$$

For dyadic sums: $\sum f(x 2^{-k}) \quad \overset{\mathcal{M}}{\longrightarrow} \quad \frac{f^\star(s)}{1-2^s}$

$\alpha = 2ik\pi / \log 2 \Longrightarrow x^{-\alpha} = e^{-2ik\pi \log_2 x}$



Cultural flashes

— **Complexity:** Morris (1977): Counting a large number of events in small memory. The power of probabilistic machines & approximation (Freivalds 1977).

— **Special functions:** Mellin analysis involves partition identities for Dirichlet series. Prodinger has connections with q-hypergeometric functions.

$$\sum_{n\geq 0} q^{n(n+1)/2} \frac{x^n w^n}{(1+xq)\cdots(1+xq^{n+1})} = \sum_{n\geq 0} (-qx)^n \left[(1-w)\cdots(1-q^{n-1}w)\right] .$$

— **Probability theory:** Exponentials of Poisson processes (Yor et al).

$\sum_i E_i\, q^i$, where $E_i \in \mathrm{Exp}(1)$.

— **Communication:** The TCP protocol = Additive Increase Multiplicative Decrease (AIMD) leads to similar functions (Robert et al, 2001).
Ethernet: Get waiting time for a packet subject to k collisions (Robert).
Ethernet is unstable (Aldous 1986) but tree protocols *are* stable (Jacquet+).

2 CARDINALITY ESTIMATORS

Given stream (read-once sequence), estimate number of distinct elements.

— Adaptive sampling
— Probabilistic Counting
— LogLog Counting

2.1 Adaptive Sampling

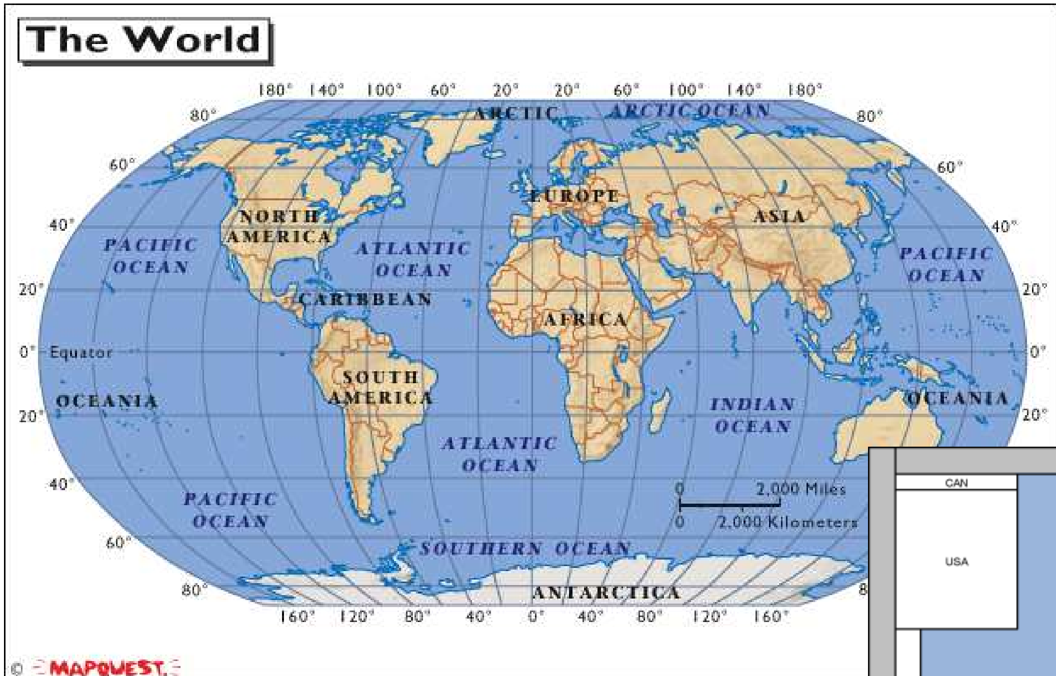
• An algorithm of M. Wegman (1980$^+$) that does *cardinality estimation* for $s = s_1 \ldots s_\ell$ and *more*:

> Samples *uniformly over domains (sets)* of multisets = of independent interest for data bases.

• $\neq$ straight sampling (by positions). Cf Vitter (TOMS 1985), Devroye 1986, ...

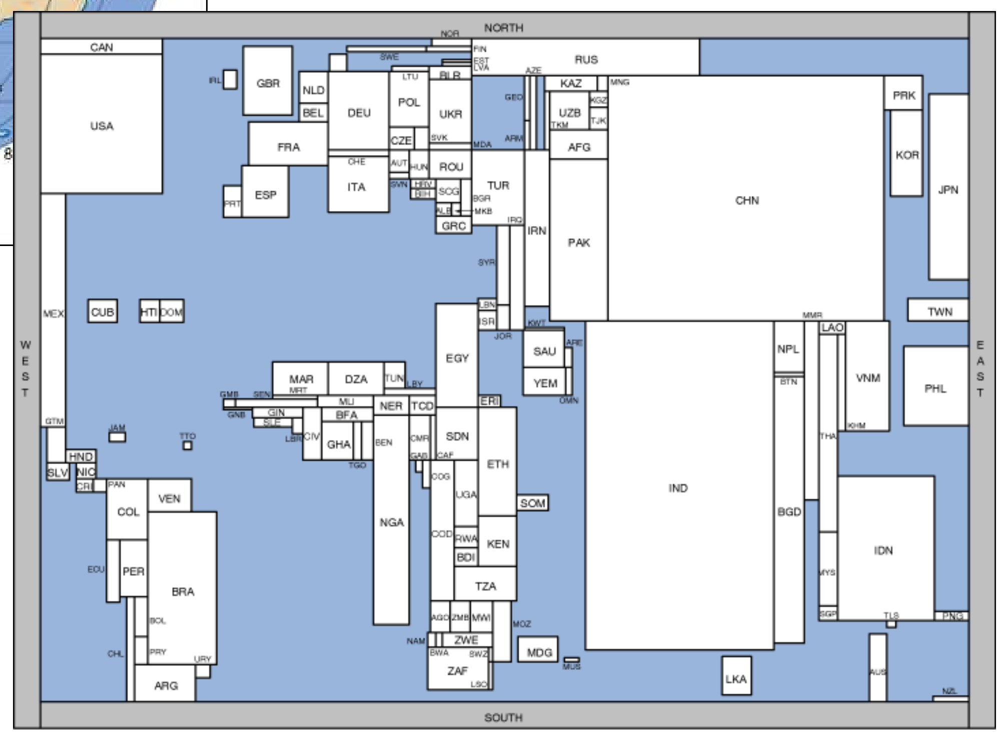
First analysis (F. 1992). Louchard (2000).

DataBases: Given ⟨`persons, towns`⟩, get geography from demography?



⟵ **Adaptive Sampling**

Sampling ⟶



(©Bettina Speckmann, TU Eindhoven)

Sample *values* (i.e., without multiplicity)?

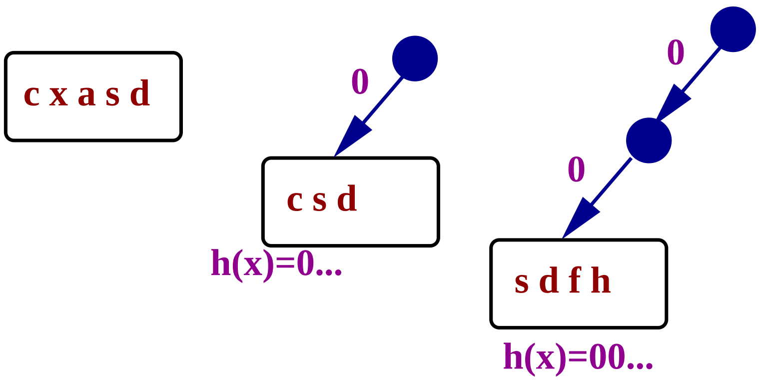
Algorithm: Adaptive Sampling (*without multiplicities*)

/* Get a sample of size $\leq m$ according to **distinct values.** */

— **On overflow:** Increase sampling depth and decrease sampling rate = *use farther bits to filter.*

Sample of size $\leq m$:

depth $d = 0, 1, 2, \ldots$



Analysis makes use of digital trees, generating functions and Mellin transforms.

First Counting Algorithm for *cardinalities*.

Let d := sampling *depth*; ξ :=sample size.

Theorem: $X := 2^d \xi$ estimates the cardinality of S using m words:

- It is unbiased: $\mathbb{E}_n(X) = n$.
- Standard error is $\sim \frac{1}{\sqrt{(m-1)\log 2}} \doteq \frac{1.20}{\sqrt{m}}$.
- Distribution is a Louchard compound (Louchard00).

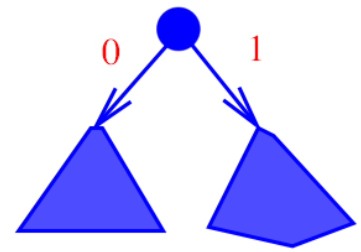
With $m = 1,000$W, get 4% accuracy.

• Related to folk algorithm for leader election on channel: *"Talk, flip coin if noisy; sleep if Tails; repeat!*

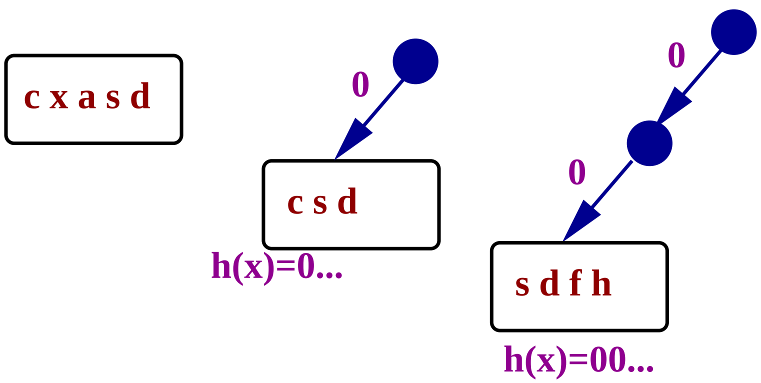
• Related to "tree protocols with counting" $\gg$ Ethernet. Cf (Greenberg-F-Ladner JACM 1987).

Analysis: Digital tree aka trie, paged version:

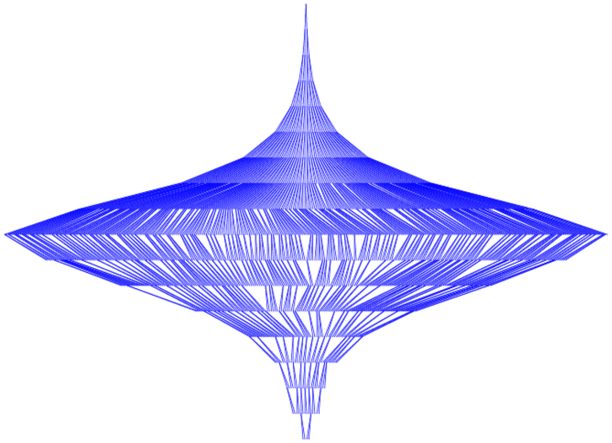
$$\left[\begin{array}{l} \mathrm{Trie}(\omega) \equiv \omega \text{ if } \mathrm{card}(\omega) \leq b \\ \\ \mathrm{Trie}(\omega) = \overbrace{\mathrm{Trie}(\omega \setminus 0) \qquad \mathrm{Trie}(\omega \setminus 1)}^{\bullet} \quad \text{if } \mathrm{card}(\omega) > b \end{array}\right.$$



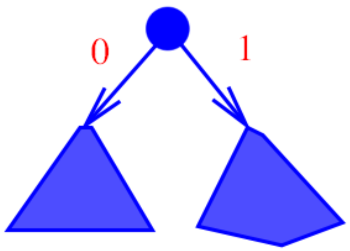
Depth in Adaptive Sampling is length of leftmost branch;
Bucket size is # of elements in leftmost page.



Refs: (Knuth Vol 3), (Sedgewick, Algs), (Mahmoud), (Szpankowski**). B. Vallée's dynamical sources; Bentley-Sedge trees + (Clément-F-Vallée), (Devroye*), etc.



Trie recurrences



For recursively defined parameters: $\alpha[\omega] = \tau[\omega] + \beta[\omega \setminus 0] \cdot \gamma[\omega \setminus 1]$:

$$\mathbb{E}_n(\alpha) := \mathrm{E}_n(\tau) + \sum_{k=0}^{n} \frac{1}{2^n} \binom{n}{k} \mathbb{E}_k(\beta) \cdot \mathbb{E}_{n-k}(\gamma).$$

Exponential generating functions (EGF): $A(z) := \sum_n \mathbb{E}_n(\alpha) \frac{z^n}{n!}$ *&c*

.

$$A(z) = T(z) + B(z) \cdot C(z).$$

For (left) recursive parameter ϕ: $\boxed{\Phi(z) = e^{z/2}\Phi\left(\frac{z}{2}\right) + \mathrm{Toll}(z)}$

Solve by iteration, extract coefficients; Mellin-ize!

More in AofA talks by Szpankowski & Devroye!

Hamlet

• **Straight Sampling** (13 elements):

and, and, be, both, i, in, is, leaue, my, no, ophe, state, the

$\boxed{\texttt{Google}}$ (leaue$\mapsto$leave, ophe$\mapsto \emptyset$) = $\boxed{\texttt{38,700,000}}$.

• **Adaptive Sampling** (10 elements):

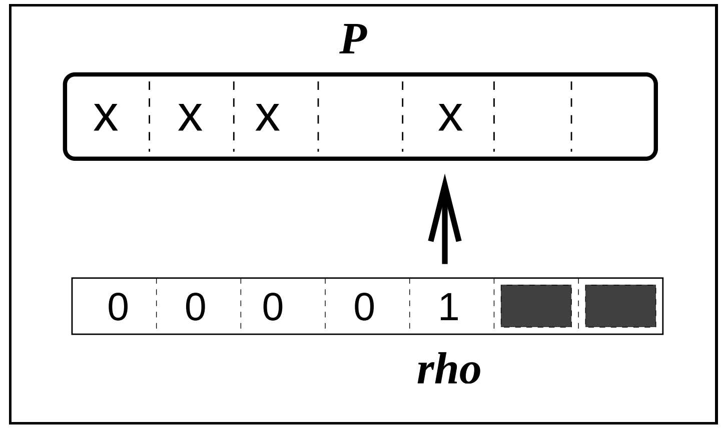
danskers, distract, fine, fra, immediately, loses, martiall, organe, pas-seth, pendant

$\boxed{\texttt{Google} = 8}$, all are to Shakespeare's Hamlet $\leadsto$ *mice*, later!

2.2 Probabilistic Counting

Second Counting Algorithm

for *cardinalities* :



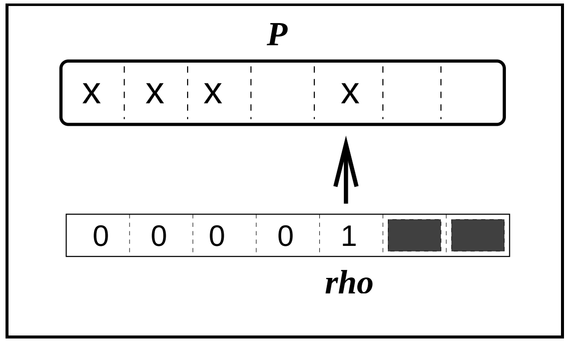
Algorithm: Probabilistic Counting

Input: a stream S; Output: cardinality $|S|$

For each $x \in S$ do /* $\rho \equiv$ position of leftmost 1-bit */

Set BITMAP$[\rho(\mathbf{hash}(w))] := 1$; od;

Return P where P is position of *first* 0.



Lemma: P estimates $\log_2(\varphi n)$, with $\varphi \doteq 0.77351$.

— Straight averaging over m trials, $\text{Ave} = \frac{1}{m}[P_1 + \cdots + P_m]$; return $\frac{1}{\varphi}2^{\text{Ave}}$; expect error $O(1/\sqrt{m})$.

— Stochastic averaging = *one* hash function and $O(1)$ per record:

> Split (mentally) stream: e.g., $S \mapsto (S_{000}, \ldots, S_{111})$, for $m = 8$;
> Work out each $P_j := P(S_j)$ separately; /* cost $O(1)$ per element */
> Let $\text{Ave} := \frac{1}{m}[P_1 + \cdots + P_m]$; /* used to estimate $\dfrac{n}{m}$ */
> Return $\frac{m}{\varphi}2^{\text{Ave}}$.

Theorem [FM85]: Define *magic constant* φ as

$$\varphi = \frac{e^{\gamma}}{\sqrt{2}} \prod_{n\geq 2}^{\star} n^{\epsilon(n)}, \qquad \epsilon(n) := (-1)^{\sum \mathrm{bits}(n)}.$$

— Probabilistic Counting is *asymptotically unbiased* (up to 10^{-5} fl.).

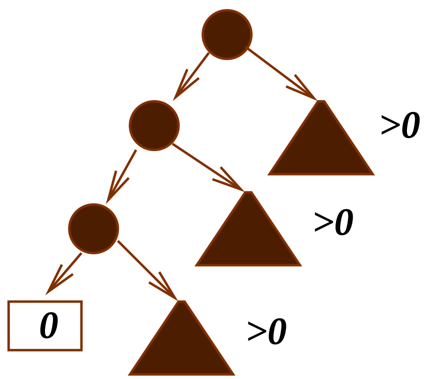
— Accuracy is $\frac{0.78}{\sqrt{m}}$ for m Words of size $\log_2 N$.

—- $\exists$ asymptotic form of distributions w/ exponential tails.

E.g. 1,000W = 4kbytes $\leadsto$ 2.5% accuracy.

Proof:

trie analysis



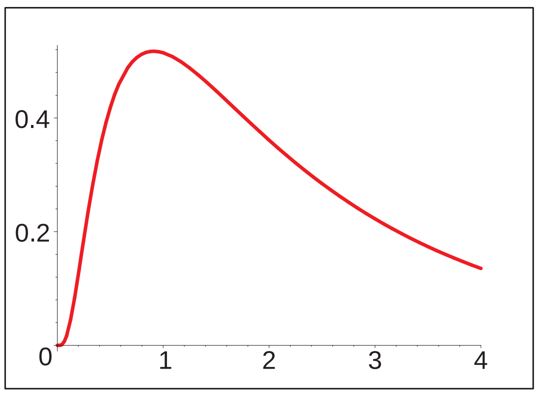
$$\mathbf{1}\cdot(e^{x/8}-1)(e^{x/4}-1)(e^{x/2}-1)$$

$$(1-q)(1-q^2)(1-q^4)\cdots=\sum_n \overbrace{(-1)^{\sum \text{bits}(n)}}^{\epsilon(n)} q^n.$$

Distribution:

$$Q(x) := e^{-x/2}\prod_{j=0}^{\infty}(1-e^{-x2^j})$$

$$\mathbb{P}_n(X=\ell)\sim Q\left(\frac{n}{2^\ell}\right)$$



+ Mellin requires $N(s) := \sum_{n\geq 1}\frac{\epsilon(n)}{n^s}$. One finds $\log_2\varphi \equiv -\Gamma'(1)-N'(0)+\frac{1}{2}$, &c.

Application: Data mining of the Internet graph

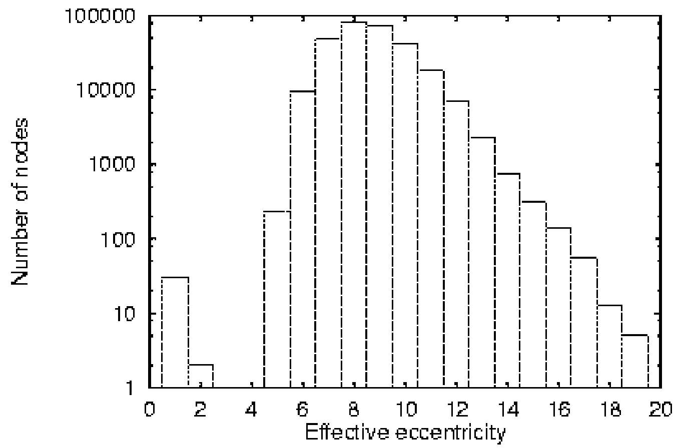
(Palmer, Gibbons, Faloutsos^2, Siganos 2001)

Internet graph: 285k nodes, 430kedges.

For each vertex v, define ball $B(v; R)$ of radius R.

Want: histograms of $|B(v, R)|$ $R = 1..20$

Get it in minutes of CPU rather than a day (400$\times$ speedup)



b) Histogram of diameters

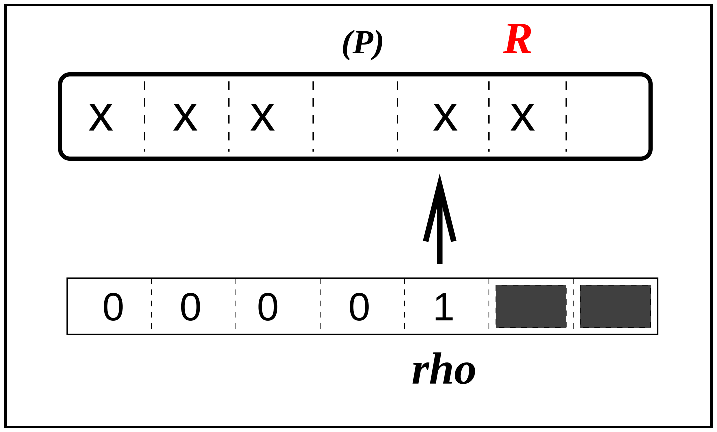
+ Sliding window usage (Motwani et al).

2.3 LogLog Counting

Third Counting Algorithm for *cardinalities*: (Durand-F, 2003/DFFGM, 2006)

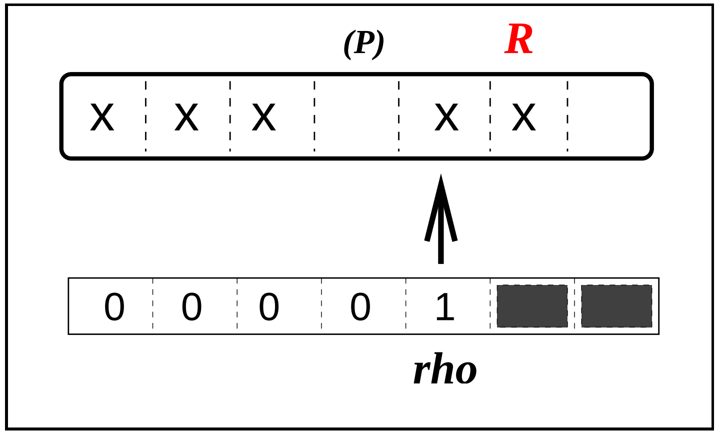
Claim: WAS/IS the best algorithm on the market!

- Hash and get $\rho(h(x))$:= position of rightmost 1-bit $\cong$ a geometric RV.
- To set S associate $R(S) := \max_{v \in S} \rho(h(v))$.



- Max of geometric RVs are well-known (Prodinger*).

$R(S)$ estimates $\sim \log(\widehat{\varphi}\,\mathrm{card}(S))$, with $\widehat{\varphi} := e^{-\gamma}\sqrt{2}$.



Algorithm LogLog Counting:

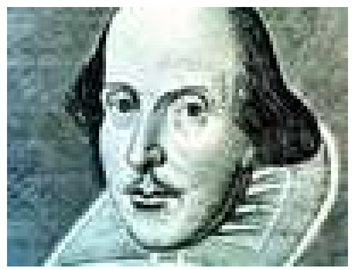
- Use rightmost **1**–bit as "observable".
- Do stochastic averaging with $m = 2^\ell$. E.g., $S \cong \langle S_{00}, S_{01}, S_{10}, S_{11} \rangle$.
- Return: $\boxed{\frac{m}{\widehat{\varphi}} 2^{\text{Average}}}$, where $\widehat{\varphi} = e^{-\gamma}\sqrt{2}$.

$+$ Switch to Hit Counting for small cardinalities.

$++$ Optimize by pruning discrepant values $\rightsquigarrow$ superLogLog or better by harmonic means $\rightsquigarrow$ FFGM's HyperLogLog ($\ll$ Chassaing-Gerin, 2006)

Theorem. LogLog is *asymptotically unbiased*.
— It needs m "bytes", each of length $\log_2 \log N_{\max}$.
— Standard error (accuracy) is: $\frac{1.30}{\sqrt{m}}$, where $1.30 \doteq \sqrt{\frac{1}{12}\log^2 2 + \frac{1}{6}\pi^2}$.
— Distribution is approximately Gaussian.

Whole of Shakespeare:



$m = 256$ small "bytes" of 4 bits each $\equiv$ 128bytes

```
ghfffghfghgghggggghghheehfhfhhgghghghhfgffffhhhiigfhhffgfiihfhhh
igigighfgihfffghigihghigfhhgeegeghgghhhgghhfhidiigihighihehhhfgg
hfgighigffghdieghhhgggghhfghhfiiheffghghihifgggffihgihfggighgiiif
fjgfgjhhjiifhjgehgghfhhfhjhiggghghihigghhihihgiighgfhlgjfgjjjmfl
```

Estimate $n^\circ \approx 30,897$ against $n = 28,239$ distinct words
Error is $+9.4\%$ for 128 bytes(!!)

Proof: Trie-like analyses: need $\mathrm{coeff}[z^n] \left(e^z \sum_k 2^{k/m} \left[e^{-z/2^{k-1}} - e^{-z/2^k} \right] \right)^m$.

Analytic depoissonization [JaSz95$^+$]

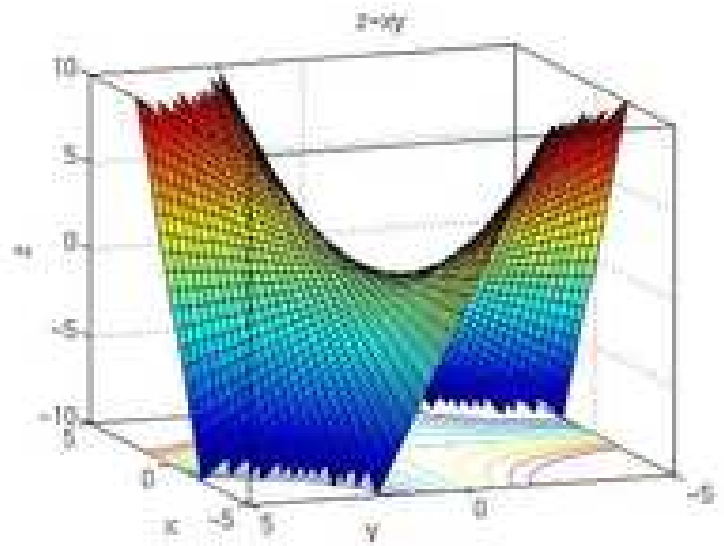
• Recover asymptotics of f_n from $\phi(z) := \sum_n f_n e^{-z} \frac{z^n}{n!}$ ≡Poisson GF?

• Intuition: with luck $\boxed{f_n \sim \phi(n)}$

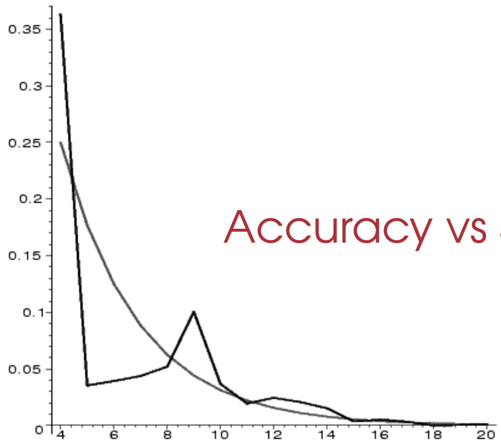
Here: "Luck" means good lifting of $\phi(z)$ to $\mathbb{C}$ ≡ Poisson flow of complex rate!

E.g.: ∃ *cone*. Inside: $\phi(z) \sim z^\alpha$. Outside: $\phi(z)$ is exponentially smaller.

$$f_n = \frac{n!}{2i\pi} \oint e^z \phi(z) \frac{dz}{z^{n+1}} \approx \phi(n)$$



Features: Errors $\approx$ *Gaussian*, seldom more than $3\times$ standard error.
Algorithm *scales down* and up (for small/large cardinalities).



Mahābhārata: 8MB, 1M words, 177601 diff.

HTTP server: 400Mb log pages 1.8 M distinct req.

m	2^6 (50by)	2^{10} (0.8kby)	2^{14} (12kb)	2^{18} (200kb)
Obs:	8.9%	2.6%	1.2%	0.32%
σ:	11%	2.8%	0.7%	0.36%

Summary of F_0 algorithms

$N = 10^8$ & 2% error

— Hit Counting: $\approx \frac{1}{10} N_{\max}$ = <u>1 Mbyte</u> + used for corrections

— **Adaptive Sampling** ($\epsilon = \frac{1.20}{\sqrt{m}}$): <u>16 kbytes</u> + *domain sampling, mice*

— **Probabilistic Counting** ($\epsilon = \frac{0.78}{\sqrt{m}}$) = <u>8 kbytes</u> + sliding window

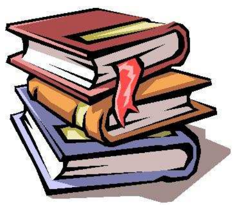
— Multiresolution bitmap (analysis??) = <u>5 kbytes?</u>

— MinCount ©Giroire = <u>4 kbytes</u> + sliding window

— **Loglog Counting** ($\epsilon = \frac{1.30}{\sqrt{m}}$) = <u>2 kbytes</u> + *elephants*

Refs: Hit Counting (Whang et al., 1990). MinCount (Bar-Yossef et al, Giroire+Fusy). Multiresolution bitmap (Estan-Varghese, 2001).

Document similarity



= An application of cardinality counts. For multisets A and B, define $\mathrm{sim}(A, B) := \dfrac{|A \cap B|}{|A \cup B|}$ (Broder).

Here: $|A| := \mathrm{card}(A)$. Let $\mathrm{Reg}(A)$ be signature of A, i.e., (LogLog) register dump of A, so that $|A| = \mathrm{estim}(\mathrm{Reg}(A))$.

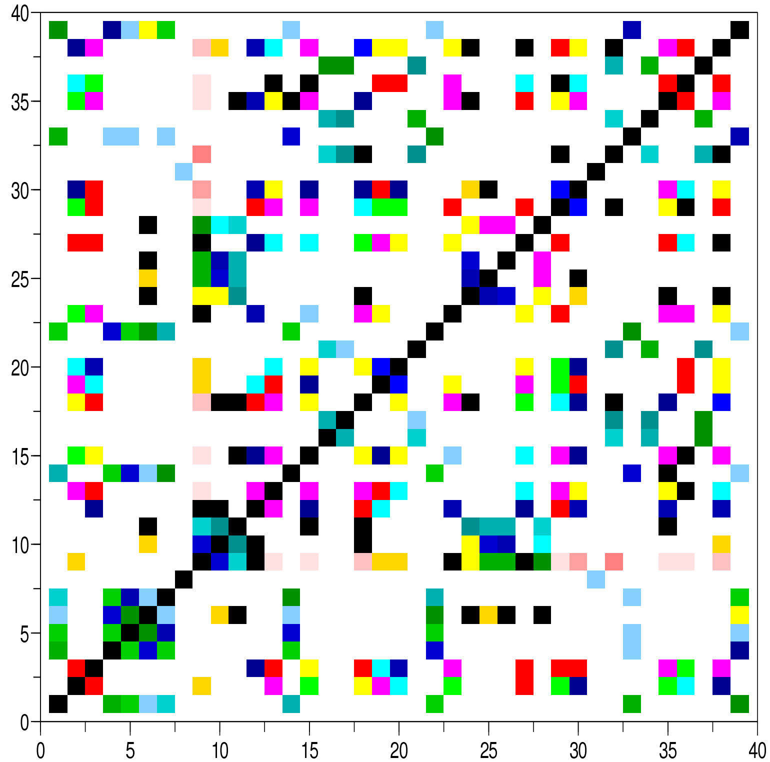
$$\begin{cases} |A| = \mathrm{estim}(\mathrm{Reg}(A)); \quad |B| = \mathrm{estim}(\mathrm{Reg}(B)) \\ |A \cup B| = \mathrm{estim}\left(\mathrm{Reg}(A) \bigoplus_{\max} \mathrm{Reg}(B)\right); \quad |A \cap B| = |A \cup B| - |A| - |B|. \end{cases}$$

• For r files, pairwise comparisons have cost $O(\sum |F_j|) + O(r^2)$, as opposed to $O(\sum |F_j|)^2$:
$\Longrightarrow$ For 10^5 files of size 10^5, work in minutes instead of days!

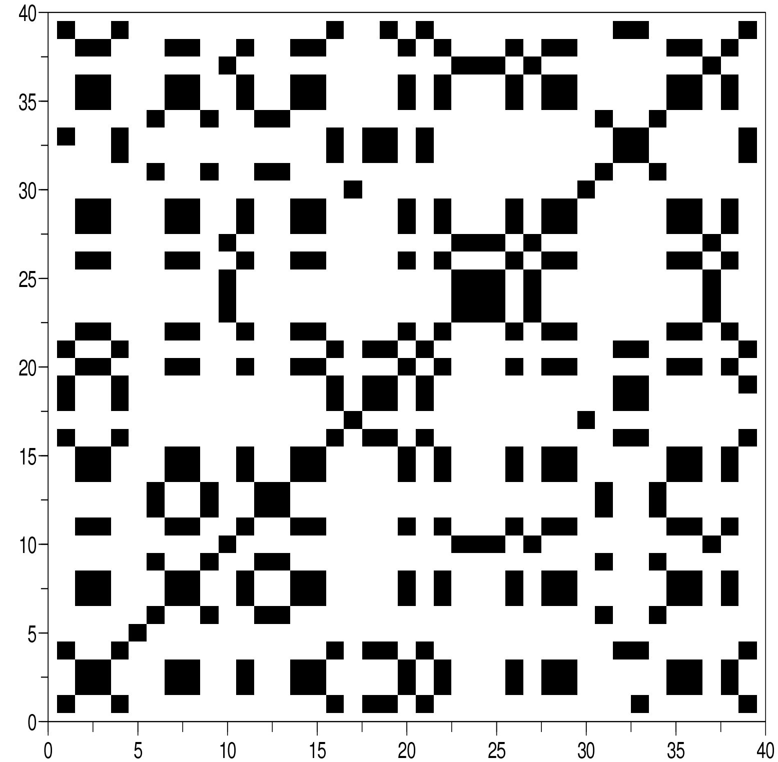
प्रणवः कश्यपः A blind test, by Pranav Kashyap

39 files of 20k words each, encrypted word-by-word.

- *How many languages? Which are which?*

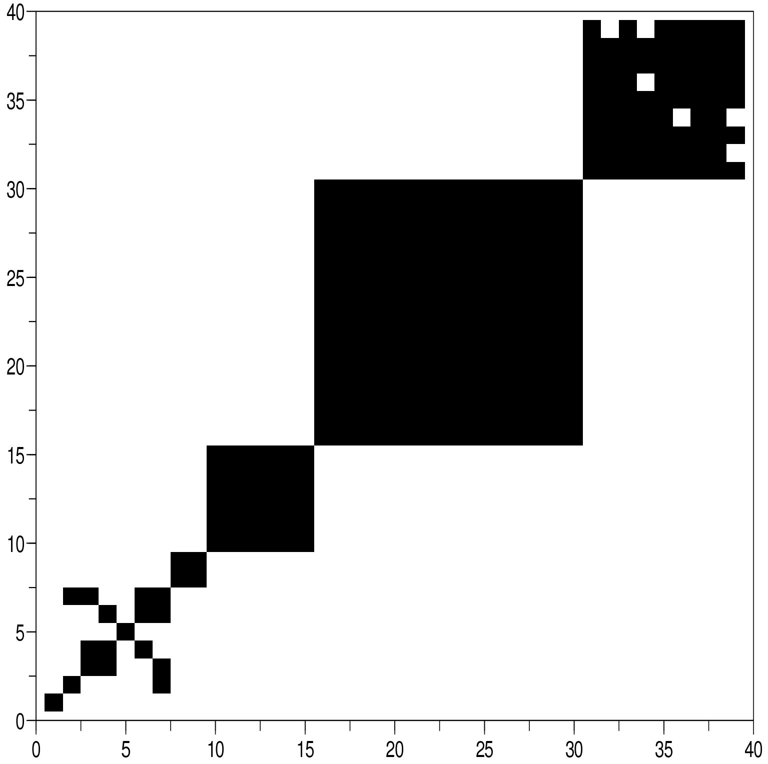


Raw comparison data

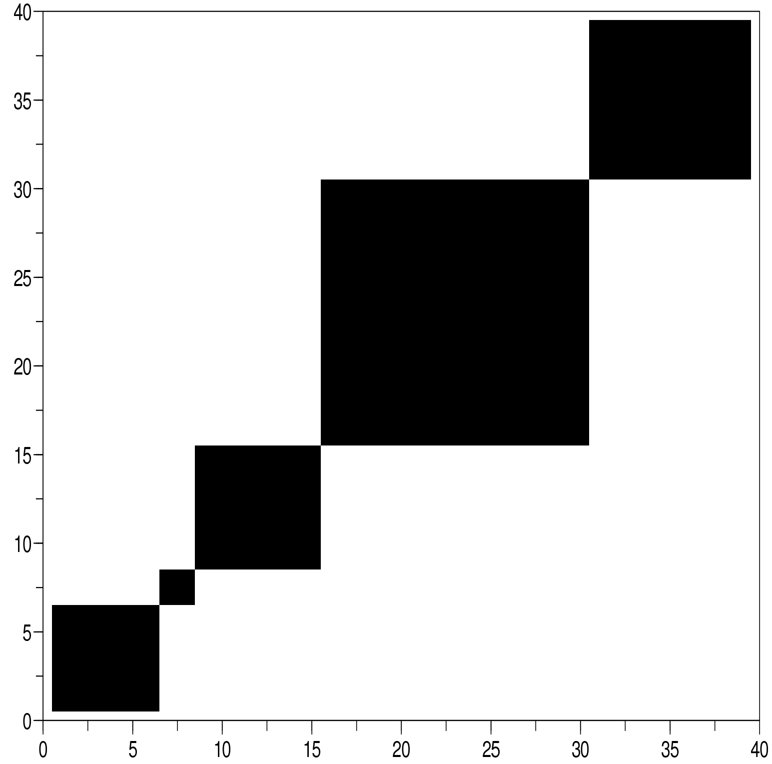


With thresholding ($\theta = 0.25$)

प्रणवः कश्यपः



Blind classification ($\theta = 0.25$)



Actual ($\theta = 0.20$)

3 MICE

Simply use **Adaptive Sampling** and keep running counts!

— Hamlet: catch the frequency profile of **mice**:

danskers^{1}, *distract*^{1}, *fine*^{9}, *fra*^{1}, *immediately*^{1}, *loses*^{1}, *martiall*^{1}, *organe*^{1}, *passeth*^{1}, *pendant*^{1}.

— With *cache size =100*, get a sample of 79 elements.

$$\mathbf{1^{50}, 2^{14}, 3^{4}, 4^{2}, 5^{1}, 6^{1}, 9^{1}, 13^{1}, 15^{1}, 28^{1}, 43^{2}, 128^{1}}.$$

	1-Mice	2-Mice	3-Mice
Estimated	63%	17%	5%
Actual	60%	14%	6%

The 10 most frequent words in Hamlet are: *the, and, to, of, i, you, a, my, it, in*. They account for $>$ 20% of all text. With 20 words, capture 30%; with 50, get 44%. **70 words capture 50% of all occurrences!**.

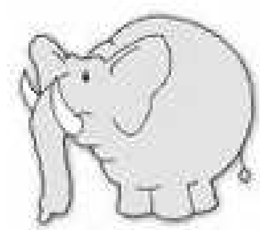
4 ICEBERGS

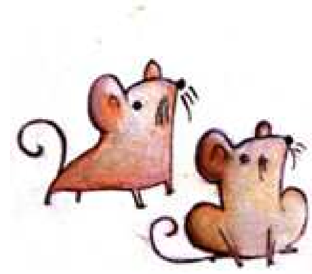
Definition: *A k–iceberg is present in proportion $> 1/k$.*

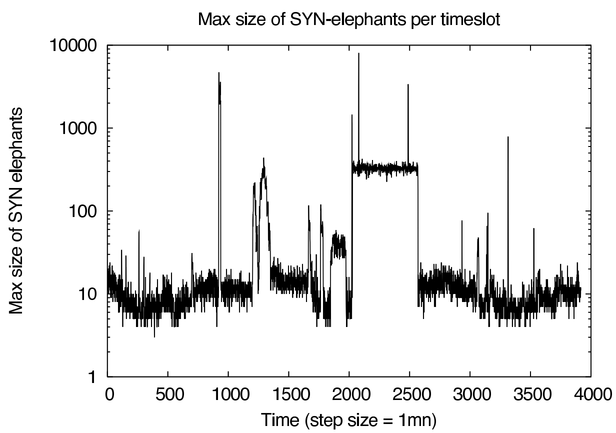
Algorithm Majority/Icebergs: /* For $k = 2$, uses 1 registers */
— Trigger a gang war: equip each individual with a gun.
— Two guys from different gangs shoot and kill one another. Majority gang is only surviving one.
— Adapt to $k \geq 2$ with $\mathbf{k} - \mathbf{1}$ registers. Supplement with straight sampling, say, with $m = 7k$.

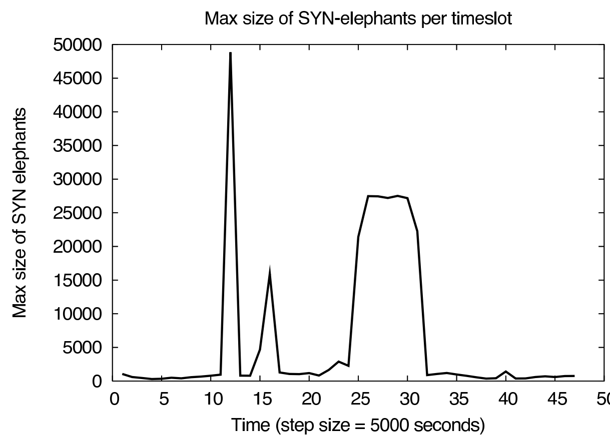
(Karp, Shenker, Papadimitriou + Bar-Yossef et al. 2001-2002)

5 ELEPHANTS









DoS attacks

©Yousra Chabchoub

Here: largest elephants (= to given destination) as function of time.

RAP @ Inria: Bloom filters (Azzana, Chabchoub, Ph. Robert)

E.g. Think of 1Billion records; 10Million are 1-mice; Others are 100–10,000 elephants. No implied locality. What to do with memory = 1kWords?

Counting elephants via cardinality algorithms

= A cute idea of O. Gandouet and A. Jean-Marie @ Montpellier.

Algorithm GJM's ElephantCount

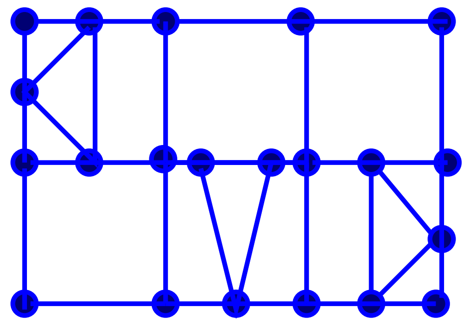
— Let S be stream composed (say) of 1-mice and ≥ 100 elephants.

— Estimate $N := |S|$ and $N_0 := |S_0|$, with S_0 a prob. p-sample of S.

— Solve system $\begin{cases} N & = & N_m + N_e \\ N_0 & = & p \cdot N_m + 0.999^+ \cdot N_e \end{cases}$, with $p = \frac{1}{10}$ (say).

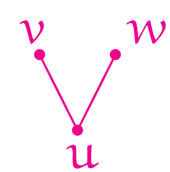
Counting triangles in graphs

Suggested by (Bar-Yossef, Kumar, Sivakumar, 2002) who propose to use F_2 (!)



Consider graph Γ of max–degree D given by adjacency list.

— Define a *vee* $(\vee)$ as any triple $\{u, v, w\}$ such that



— Make a stream of all vee's with cost $O(nD^2)$.

— Isolated vee's are 1-mice; triangles are 3-elephants.

— Use cardinality on top of p-sampling: e.g., $p = \frac{1}{2}$ gives

$$N = N_m + N_e, \qquad N_0 = \frac{1}{2}N_m + \frac{7}{8}N_e.$$

6 FREQUENCY MOMENTS: F_2

Recall: Alon, Matias, Szegedy (STOC 1996)*** $F_2 := \sum_v (f_v)^2$,

where f_v is frequency of value v. An elegant idea: flip$(x) \equiv \epsilon(x)$= ± 1 based on hash(x) = "reproducible randomness" .

ALG: F2;
Initialize Z:=0;
For each x in S do $Z := Z + \mathbf{flip}(x)$.
Return Z^2.
Collect m Z-values and average, with T-transform.

$$\mathbb{E}(Z^2) = \mathbb{E}\left(\sum_{x \in S} \epsilon(v)\right)^2 = \mathbb{E}\left(\sum_j f_j \cdot \epsilon(j)\right)^2 = \sum_j (f_j)^2.$$

Indyk's F_p algorithm

A beautiful idea of Piotr Indyk (FOCS 2000)*** for F_p, $p \in (0,2)$.

• Stable law of parameter $p \in (0,2)$: $\mathbb{E}(e^{itX}) = e^{-|t|^p}$.
No second moment; no 1st moment if $p \in (0,1)$.

$c_1X_1 + c_2X_2 \overset{\mathcal{L}}{\cong} \mu X$, with $\mu := (c_1^p + c_2^p)^{1/p}$.

ALG: Fp;
Initialize Z:=0;
For each x in S do $Z := Z + \mathbf{Stable}_\alpha(x)$.
Return Z.
Estimate F_p parameter from m copies of Z-values.

Remark: Use of $\log(|Z|)$ to estimate seems better than median(?)

Conclusions

For streams, using practically $O(1)$ storage, one can:
— Sample distinct values;
— Estimate F_0, F_1, F_2, F_p $(0 < p \leq 2)$ even for *huge* data sets;
— Estimate icebergs, # of mice and elephants.

♡ Need **virtually no assumption on nature of data**.

♡♡♡♡♡♡

The algorithms are based on randomization $\mapsto$ Analysis fully applies
— They work **exactly** as predicted on real-life data;
— They often have a wonderfully elegant structure;
— Their analysis involves beautiful methods for AofA: "Symbolic modelling by generating functions, Singularity analysis, Saddle Point and analytic depoissonization, Mellin transforms, stable laws and Mittag-Leffler functions, etc."

That's All, Folks!