Mellin vu du ciel

Mellin, seen from the sky

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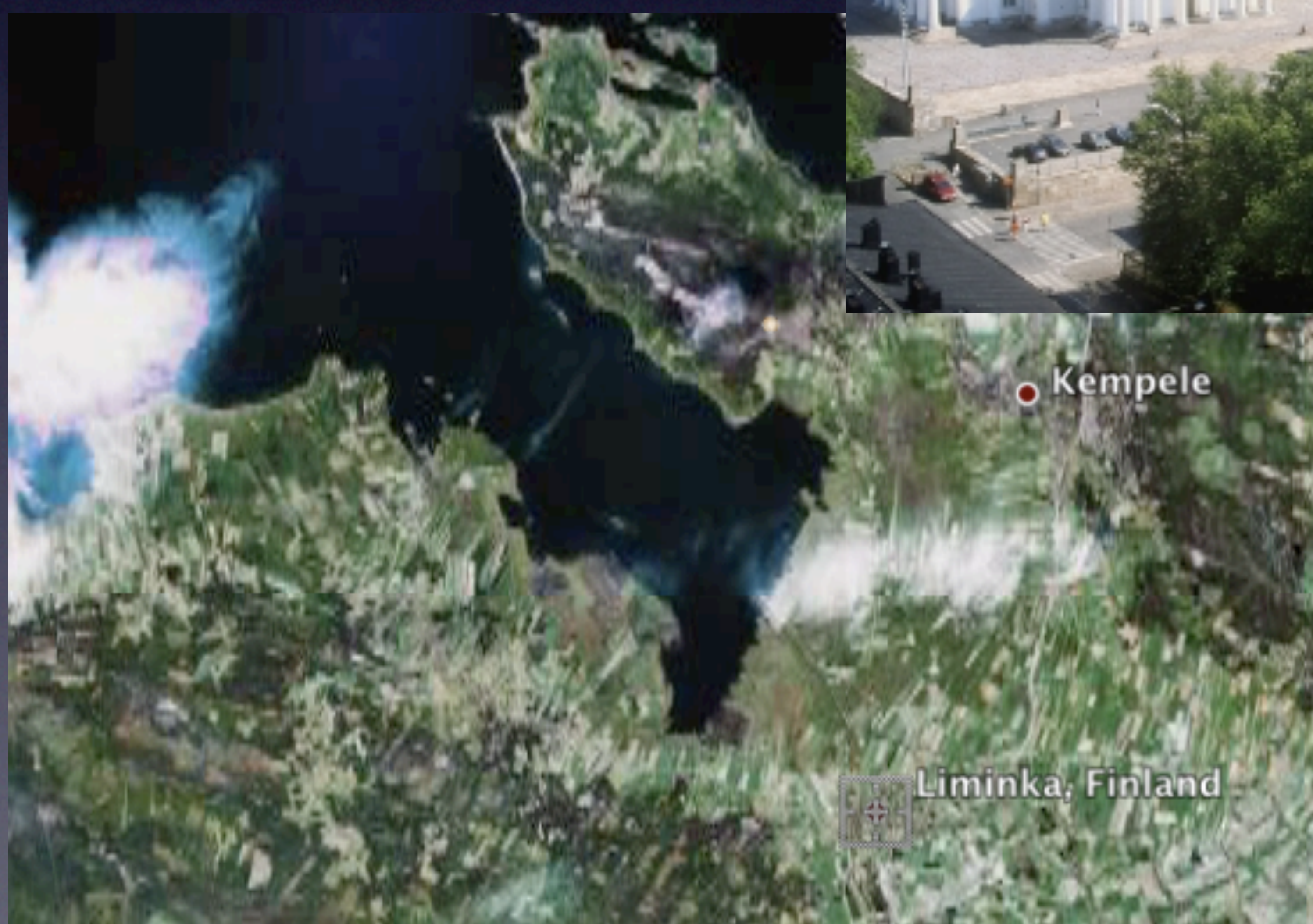
Hjalmar MELLIN 1854--1933



I. INTRODUCTION

Mellin transform.

$$f(x) \longmapsto f^*(s) = \int_0^\infty f(x)\, x^{s-1}\, dx$$

$(x \in \mathbb{R}_{\geq 0})$ $\qquad$ $(s \in \mathbb{C})$

NB: $x = e^t$: bilateral Laplace transform.

($x = e^{it}$: Fourier transform.)

WHY?

- Expected height of Dyck paths / gen. Catalan trees. [BB-K-R-19

$$S_n = \sum_{k\geq 1} d(k) \frac{\binom{2n}{n-k}}{\binom{2n}{n}}$$

- Mean node depth of a digital tree, aka "trie". [DB-K, 1965]

$$T_n = \sum_{k\geq 1} 1-\left(1-\frac{1}{2^k}\right)^n$$

Subtle: _ d(k) oscillates but S_n has smooth asymptotics

summands are smooth, but T_n oscillates -

- Height of trees $\rightsquigarrow$

$$\hat{S}(x) = \sum d(k) e^{-k^2 x^2} \qquad (x \to 0)$$

- Trie levels $\rightsquigarrow$

$$\hat{T}(x) = \sum \left[1 - e^{-x/2^k}\right] \qquad (x \to \infty)$$

Elementarily: $\quad S_n \approx \hat{S}(1/\sqrt{n})\,; \quad T_n \approx \hat{T}(n)$

TWO PROPERTIES: define harmonic sums

$$F(x) = \sum_k \lambda_k f(\mu_k x)$$

λ_k → amplitudes; μ_k → frequencies

- **Factorization of harmonic sums**

$$F^*(s) = f^*(s) \times \left(\sum_k \lambda_k \cdot \mu_k^{-s} \right)$$

$f^*(s)$ → transform of base function

$\sum_k \lambda_k \mu_k^{-s}$ → generalized Dirichlet series

- **Mapping properties**

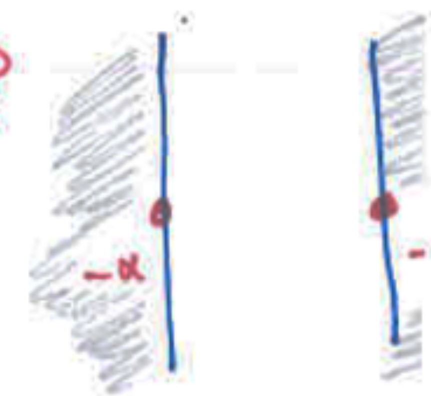
Asympt. of $g(x)$ at $0, +\infty$ ⟷ Singularities of $g^*(s)$

II. BASICS

$$f(x) \stackrel{\mathcal{M}}{\longmapsto} f^*(s) = \int_0^\infty f(x)\, x^{s-1}\, dx$$

- Assume that $f(x)$ is locally $\left\{\begin{matrix}\text{Riemann} \\ \text{Lebesgue}\end{matrix}\right\}$ integrable $/ \mathbb{R}_{\geq 0}$

- $f(x) = \begin{cases} O(x^\alpha), & x \to 0 \\ O(x^\beta), & x \to \infty \end{cases} \implies$ $f^*(s)$ exists in STRIP $-\alpha < \mathrm{Re}(s) < -\beta$



Examples:

$$e^{-x} \longmapsto \Gamma(s) \quad \text{on} \quad \langle 0, +\infty \rangle$$

$$\frac{1}{1+x} \longmapsto \frac{\pi}{\sin \pi s} \quad \text{on} \quad \langle 0, 1 \rangle.$$

Functional properties

$$\Rightarrow f(\lambda x) \mapsto \lambda^{-s} f^*(s)\,; \qquad f(x^\rho) \mapsto \frac{1}{\rho} f^*\left(\frac{s}{\rho}\right);$$

$$x^\nu f(x) \mapsto f^*(s+\nu)\,; \qquad f(x)\log x\,, \quad \frac{d}{dx} f(x)\,, \ldots$$

E.g., $e^{-x^2} \mapsto \frac{1}{2}\Gamma\left(\frac{s}{2}\right)$; $\quad e^{-4x^2} \longmapsto 2^{-s} \times \frac{1}{2}\Gamma\left(\frac{s}{2}\right)$;

$$e^{-4x^2}\log x \longmapsto \frac{d}{ds}\left(2^{-s} \times \frac{1}{2}\Gamma\left(\frac{s}{2}\right)\right).$$

HARMONIC SUMS

$$\sum \lambda_k f(\mu_k x) \longmapsto f^*(s) \cdot \left(\sum \lambda_k \mu_k^{-s}\right)$$

Inversion

Theorem I:

$$f(x) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} f^*(s)\, x^{-s}\, ds$$

{ $\underline{c}$ in fundamental strip

Riemann: $f(x)$ continuous at x (Bounded Variation ok)

Lebesgue: true almost everywhere.

$$e^{-x} = \frac{1}{2i\pi} \int_{1-i\infty}^{1+i\infty} \Gamma(s)\, x^{-s}\, ds\,; \qquad \frac{1}{1+x} = \frac{1}{2i\pi} \int_{1/2-i\infty}^{1/2+i\infty} \frac{\pi}{\sin \pi s}\, x^{-s}\, ds$$

Mappings

Theorem M: $f(x) \sim \sum c \cdot x^{\xi} (\log x)^{k} \quad (x \to 0)$

$\Longleftrightarrow$ (COND.) $f^*(s)$ has pole at $-\xi$ $\quad \dfrac{c}{(s+\xi)^k} \times (-1)^k k!$

- ξ exponent of x (@ 0) in $f(x)$ $\Longrightarrow$ $-\xi$ location of a pole of $f^*(s)$.
- k exponent of $(\log x)$ $\Longrightarrow$ k is multiplicity

■ Same at $(x \to \infty)$, with a minus sign.

Proof of direct mapping (e^{-x})

$$\int_0^\infty e^{-x} x^{s-1}\,dx = \int_0^1 e^{-x} x^{s-1}\,dx + \underbrace{\int_1^\infty e^{-x} x^{s-1}\,dx}_{\text{is entire}}$$

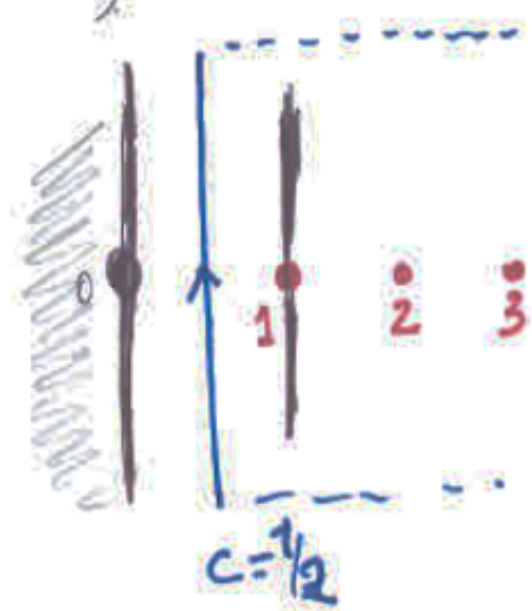
$$\underbrace{\int_0^1 (e^{-x}-1)\, x^{s-1}\,dx}_{\exists \text{ in } \mathrm{Re}(s)>-1} + \underbrace{\int_0^1 x^{s-1}\,dx}_{=1/s}$$

{ Constant term of e^{-x} gives pole at 0 of $\Gamma(s)$ }

Proof of converse mapping ($\pi/\sin\pi s$)

$$\frac{1}{2i\pi}\int_{1/2-i\infty}^{1/2+i\infty} \frac{\pi}{\sin\pi s}\, x^{-s}\,ds = -\sum_{j=1}^{m} \mathrm{Res}\left(\frac{\pi}{\sin\pi s}\, x^{-s}\right)_{s=j} + O(x^{-m-1/2})$$

$$\sim \left[x^{-1} - x^{-2} + x^{-3} - \cdots\right].$$



III. HARMONIC SUMS

Asymptotics

MAPPINGS

Exponentials.

<0,+oo>

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \cdots \quad \Longleftrightarrow \quad \Gamma(s) \asymp \left(\frac{1}{s}\right)_{s=0} - \left(\frac{1}{s+1}\right)_{s=-1} + \left(\frac{1}{s+2}\right)_{s=-2} -$$

$$\frac{e^{-x}}{1-e^{-x}} = \frac{1}{x} - \frac{1}{2} + \frac{x}{12} - \frac{x^3}{720} \quad \Longleftrightarrow \quad \Gamma(s)\zeta(s) \asymp \left(\frac{1}{s-1}\right)_{s=1} - \frac{1}{2}\left(\frac{1}{s}\right)_{s=0} +$$

<1,+oo>

— $\zeta(s)$ is meromorphic in $\mathbb{C}$;

— $\zeta(s) \sim 1/(s-1)$ at $s = 1$; $\zeta(0) = -1/2$;

— $\zeta(-2) = \zeta(-4) = \cdots = 0$; $\zeta(-1),\ \zeta(-3),\ \cdots =$ Bernoulli #s.

Mellin asymptotic summation

$$\sum_k \lambda_k f(\mu_k x) \sim \pm \sum_{s \in \mathcal{H}} \mathrm{Res}\,[f^\star(s) \cdot \Lambda(s)]$$

Example 1. Harmonic numbers $H_n = \sum_{k=1}^{n} \frac{1}{k}$:

- *harmonic sum*

$$h(x) = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+x} \right) = \sum \frac{1}{k} \cdot \frac{x/k}{1+x/k}.$$

- *Mellin* $h^\star(s) = -\frac{\pi}{\sin \pi s} \zeta(1-s)$

<-1,0>

- *Poles & asymptotics*: $H_n \sim \log n + \gamma + \frac{1}{2n} + \cdots$.

Example 2. Stirling's formula.

$$\ell(x) := \log\Gamma(x+1) - \gamma x = \sum_{n\ gel} \frac{x}{n} - \log\left(1 + \frac{x}{n}\right)$$

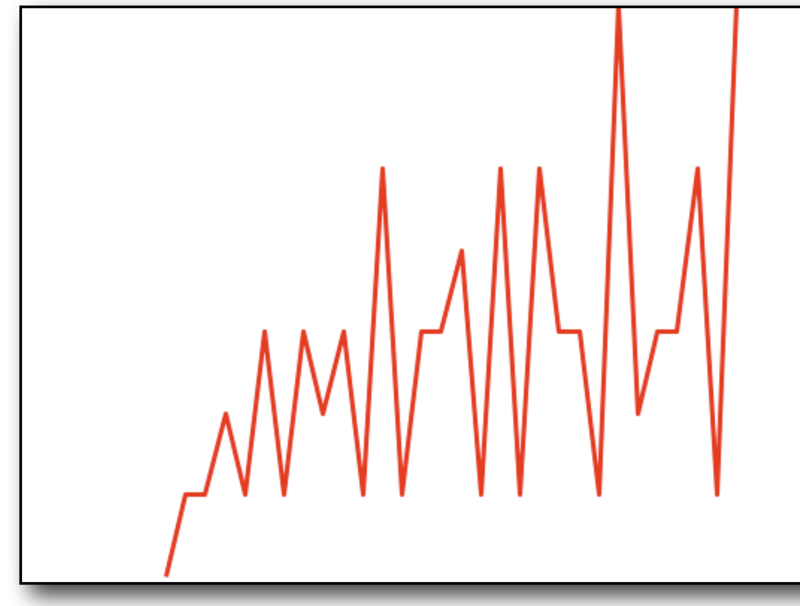
$$\ell^{\star}(s) = -\zeta(-s)\frac{\pi}{s \sin \pi s}.$$

<-2,-1>

$$\log(x!) \sim \log\left(x^{x} e^{-x} \sqrt{2\pi x}\right) + \cdots$$

Example 3. Divisor sum.

$$D(x) := \sum_{k \geq 1} d(k) e^{-kx}$$



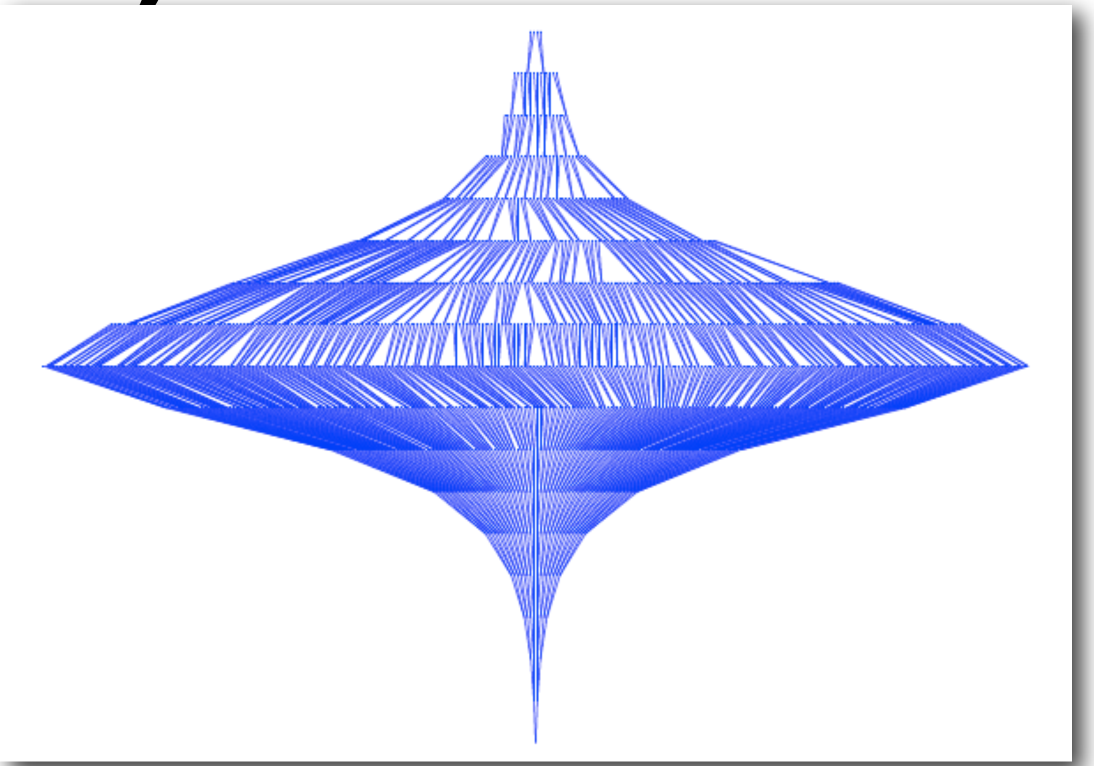
$$D^{\star}(s) = \zeta(s)^2 \Gamma(s)$$

$$D(x) \underset{x \to 0}{\sim} \frac{1}{x}\left(\log \frac{1}{x} + \gamma\right) + \frac{1}{4} + \sum \text{“Bernouilli}^2\text{''} x^{2k+1}.$$

NB: $\sum d(k) e^{-k^2 x} \rightsquigarrow \zeta(2s)^2 \Gamma(s) \rightsquigarrow$ **exponentially small error terms**.

Digital trees (tries)

Digital trees aka "tries"



- Set up recurrence and generating function
- Solve and get a sum
- Approximate (equiv. Poisson approx.)
- Analyse harmonic sum

- *Recurrence:* $f_n = t_n + \sum \frac{1}{2^n}\binom{n}{k}(f_k + f_{n-k})$.

- *Generating function:* $f(x) := \sum_n f_n \frac{x^n}{n!}$

$$\boxed{f(x) = t(x) + 2e^{x/2} f\left(\frac{x}{2}\right)}$$

- Iteration & expansion $\rightsquigarrow$

$$f(x) = t(x) + 2e^{x/2}t(x/2) + 4e^{x/2}e^{x/4}t(x/4) + \cdots = \sum_{k\geq 0} 2^k e^{(1-1/2^k)x} t(x/2^k).$$

E.g.: pathlength $t_n = n$ and $t(x) = x(e^x - 1)$; size $t_n = 1\,[\![n > 1]\!]$ etc

$$PL_n = n \sum_{k\geq 0}\left[1 - \left(1 - \frac{1}{2^k}\right)^{n-1}\right]$$

TRIES

- Need to estimate $PL_n = n \sum_{k \geq 0} \left[1 - \left(1 - \frac{1}{2^k}\right)^{n-1} \right]$
- Use: $(1-a)^n \rightsquigarrow e^{-na}$. Consider

$$\boxed{G(x) = \sum_{k \geq 0} \left(1 - e^{-x/2^k}\right).}$$

– One can show elementarily (Knuth, vol 3, p131) that $PL_n = n\,G(n) + O(\sqrt{n})$.

– Also $x\,G(x)$ is a Poisson expectation (the number of elements is Poisson(x)).

$G(x)$ is a **harmonic sum**:

- base function $(1-e^{-x}) \overset{\mathcal{M}}{\Longrightarrow} -\Gamma(s)$ in strip $\langle -1, 0 \rangle$
- amplitudes $= 1$, frequencies $= 2^{-k}$ $\Longrightarrow$ Dirichlet series is

$$\Lambda(s) = \sum_{k \geq 0} (2^{-k})^{-s} = \frac{1}{1-2^s}$$

in $-\infty < \mathrm{Re}(s) < 0$.

Summary: $G(x) = \sum_{k\geq 0} (1 - e^{-x/2^k})$ has Mellin transform

$$\boxed{G^*(s) = -\frac{\Gamma(s)}{1-2^s}}$$ with fundamental strip $-1 < \mathrm{Re}(s) < 0$.

Use mapping properties: for $x \to \infty$, what happens on the right matters.

- Double pole at $s = 0$ $\Longrightarrow$ $G(x) \sim \log_2 x + C + \cdots$
- Simple poles at $s = \chi_k$ $\Rightarrow$ a term of the form $\frac{1}{\log 2} \Gamma\left(\frac{2ik\pi}{\log 2}\right) x^{-2ik\pi/\log 2}$

  $\chi_k = \frac{2ik\pi}{\log 2}$ $(k \in \mathbb{Z}_{\neq 0})$

Thus:

$$G(x) \sim \log_2 x + C + Q(\log_2 x) + \text{exponentially small.}$$

where $Q(u) = \sum' c_k e^{-2ik\pi u}$ $\Leftarrow$ a Fourier series with amplitude $< 10^{-5}$ since $\Gamma(2ik\pi/\log 2) \approx \Gamma(8i) \approx 10^{-5}$.

QED

IV. SOME GOODIES

- A near(?) identity
- Möbius

A (near) identity ???

$$\sum_{n=1}^{100000} \frac{(-1)^n \left(\frac{99}{100}\right)^n}{1 + \left(\frac{99}{100}\right)^{2n}}$$

```
> evalf(%,220);
```

$-0.24999999999999999999999999999999999999999999999999\backslash$
$999999999999999999999999999999999999999999999999\backslash$
$999999999999999999999999999999999999999999999999\backslash$
$999999999999999999999999999999999999999999999999\backslash$
$999999999999999999999999999999998211016094$

Cf. Brigitte Vallée: binary Euclidean GCD

A (near) identity ???

$$\sum_{n=1}^{100000} \frac{(-1)^n \, 0.99^n}{1 + 0.99^{2n}}$$

```
> evalf(%,220);
```

$-0.2499999999999999999999999999999999999999999999999999\backslash$

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9999

$$\sum_{n=1}^{\infty} \frac{(-1)^n \mathrm{e}^{-n\,t}}{1 + \mathrm{e}^{-2\,n\,t}} = -\frac{1}{4} + \textit{exp. small}$$

WHY $-0.2499\ldots$

Take $\Phi(t) = \sum (-1)^n \varphi(nt)$ $\xrightarrow{\text{Mellin}}$ $\Phi^*(s) = \hat{\zeta}(s) \cdot \Gamma(s) \cdot L(s)$

$\varphi(t) = \frac{e^{-t}}{1+e^{-2t}}$

where: $\hat{\zeta}(s) = -(1-2^{1-s})\zeta(s)$; $L(s) = \frac{1}{1^s} - \frac{1}{3^s} + \frac{1}{5^s} - \cdots$

We have $\hat{\zeta}(-2) = \hat{\zeta}(-4) = \cdots = 0$ [by previous calculations]

$L(-1) = L(-3) = \cdots = 0$ [by similar process].

Thus $\Phi(t) \underset{t \to 0}{\sim} \mathrm{Res}\left(t^{-s}\hat{\zeta}(s) L(s) \Gamma(s)\right)_{s=0} +$ exponentially small.

$$\boxed{\Phi(t) = -\frac{1}{4} + \text{exp small}}$$

Q.E.D.

$$\sum_{n=1}^{2000} \mu(n)\, 0.99^{n} = -1.886785550$$

$$\sum_{n=1}^{10000} \mu(n)\, 0.999^{n} = -1.98810495$$

$$\sum_{n=1}^{200000} \mu(n)\, 0.9999^{n} = -1.9988015$$

1, -1, -1, 0, -1, 1, -1, 0, 0, 1, -1, 0, -1, . . .

$$\sum_{n=1}^{2000} \mu(n)\, 0.99^n = -1.886785550$$

$$\sum_{n=1}^{10000} \mu(n)\, 0.999^n = -1.98810495$$

$$\sum_{n=1}^{200000} \mu(n)\, 0.9999^n = -1.9988015$$

$$\sum_{n=1}^{\infty} \mu(n)\, e^{-n t}$$

```
> add(coeff(series(1/Zeta(s)*GAMMA(s)*t^(-s),s=-j),s+j,-1),j=0..3);
  evalf(%,10);
```

$$-2 + 12\,t - \frac{1}{2}\,\frac{t^2 \ln(t)}{\zeta(1,\,-2)} + \left(\frac{\frac{3}{4} - \frac{1}{2}\gamma}{\zeta(1,\,-2)} - \frac{1}{4}\,\frac{\zeta(2,\,-2)}{\zeta(1,\,-2)^2} \right) t^2 - 20\,t^3$$

$$-2. + 12.\,t + 16.42119333\,t^2 \ln(t) + 2.58027982\,t^2 - 20.\,t^3$$

$$\sum_{n=1}^{2000} \mu(n)\, 0.99^n = -1.886785550$$

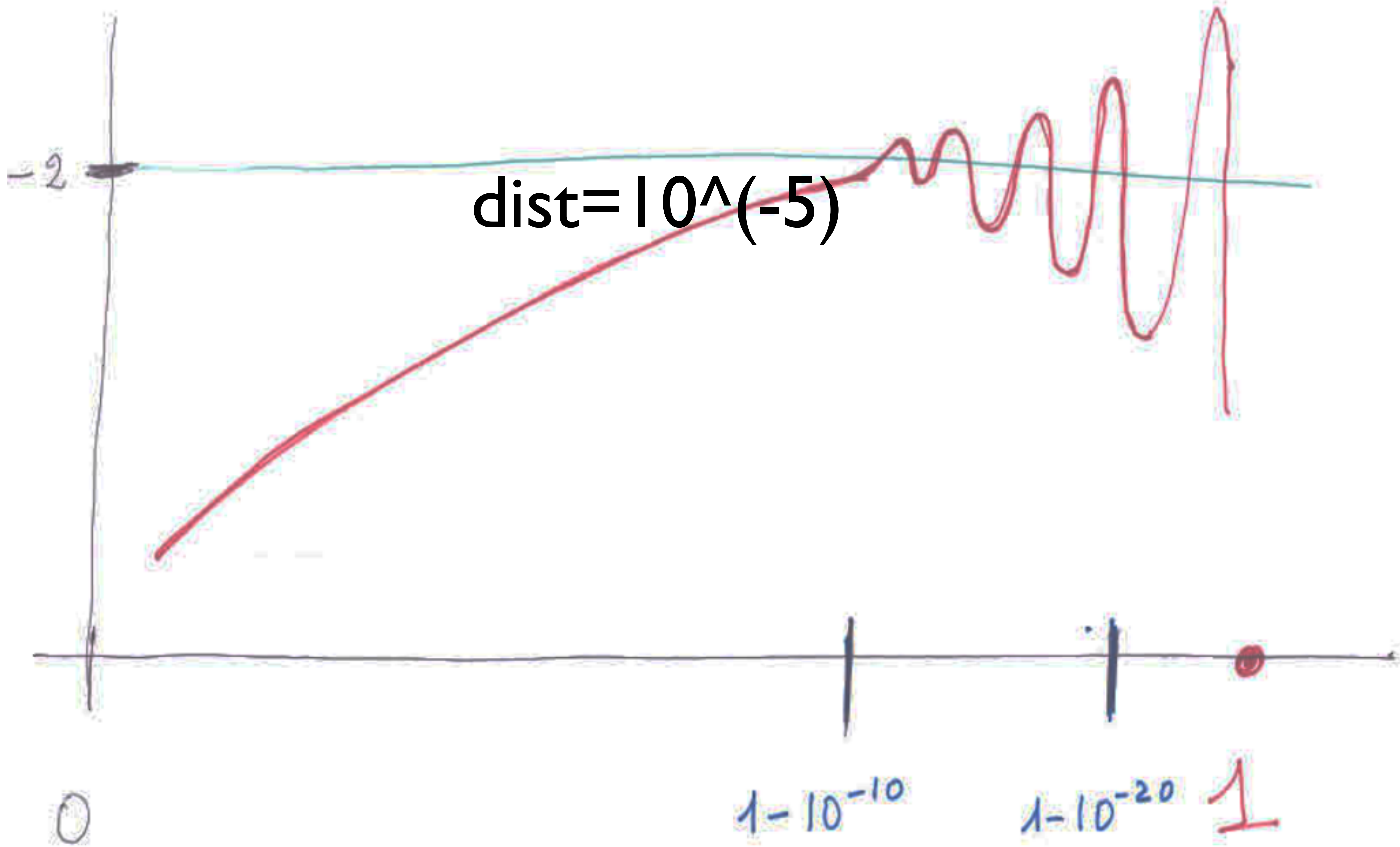
$$\sum_{n=1}^{10000} \mu(n)\, 0.999^n = -1.98810495\ldots$$

$$\sum_{n=1}^{200000} \mu(n)\, 0.9999^n = -1.9988\ldots$$

$$\sum_{n=1}^{\infty} \mu(n)\, e^{-n t}$$

Theorem: The sum tends to INFINITY like

$$\frac{5.70\ 10^{-10} \sin(14.1 \ln(t))}{\sqrt{t}}$$



$$\frac{10^{-10}}{\sqrt{t}} \sin(\log t)$$

[Clement+Flajolet-Vallee, 2000-2001]

Theorem 6. *The expected cost of sorting* n *uniform real numbers given by their basic continued fraction representations is*

$$P(n) = K_0 n \log n + K_1 n + Q(n) + K_2 + o(1),$$

where K_0 *is Lévy's entropic constant,* K_1 *is Porter-like*

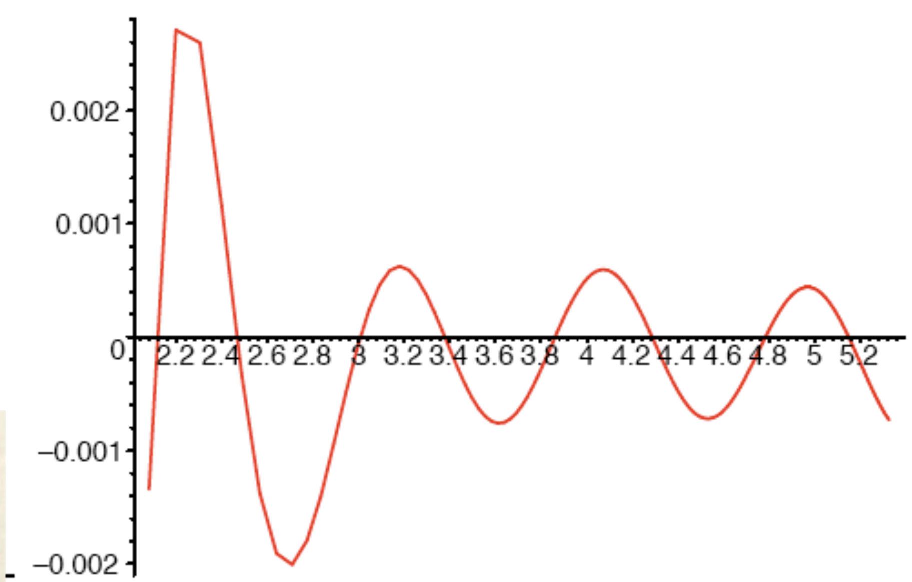
$$K_0 = \frac{6 \log 2}{\pi^2}, \qquad K_1 = 18\frac{\gamma \log 2}{\pi^2} + 9\frac{(\log 2)^2}{\pi^2} - 72\frac{\log 2 \zeta'(2)}{\pi^4} - \frac{1}{2}.$$

The function $Q(u)$ *is an oscillating function with mean value* 0*:*

$$Q(n) = O(u^{\delta/2}),$$

where δ *is any number such that*

$$\delta > \sup\{\Re(s) \mid \zeta(s) = 0\}.$$



V. Advanced techniques

- Perron's formulae
- Approximating Dirichlet series
- Poisson + Mellin = Newton + Nörlund (Rice)

Perron

$H(x) =$  $\Rightarrow H^*(s) = \int_0^1 x^{s-1}\,dx = \frac{1}{s} \quad (\mathrm{Re}(s) > 0).$

- We have a sequence (a_n) and its Dirichlet series $\alpha(s) = \sum_{n\geq 1} \frac{a_n}{n^s}$.
  How do we get back (a_n) from $\alpha(s)$?

Write $A(y) = \sum_{n<y} a_n = \sum_n a_n H\left(\frac{n}{y}\right) = \sum_n a_n H(nx) \quad [x = 1/y]$

$$\Rightarrow A^*(s) = \alpha(s) H^*(s) = \frac{\alpha(s)}{s}.$$

Thus, by Mellin inversion:

$$\boxed{A(y) = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} \alpha(s)\, y^s \frac{ds}{s}}$$

known as Perron's formula.

- Higher order: use $\sum\sum \Rightarrow$  and $\sum\sum\sum \Rightarrow$  and get better convergence (at $\pm i\infty$).

$$\sum\sum a_n \rightsquigarrow \int_{c-i\infty}^{c+i\infty} \alpha(s)\, y^s \frac{ds}{s(s+1)} \quad \text{etc.}$$

Cf. B. Vallée – analysis of Euclidean algorithms.

APPROXIMATE DIRICHLET SERIES

Say we want asymptotics of $F(x) = \sum e^{-n(n+1)x} \xrightarrow{\mathcal{M}} \Gamma(s) \cdot \Lambda(s)$, $\Lambda(s) \equiv \sum \frac{1}{[n(n+1)]^s}$.

<u>Idea</u>: $\sum \frac{1}{[n(n+1)]^s} = \sum \frac{1}{n^{2s}} \left(1+\frac{1}{n}\right)^{-s} \cong \sum \frac{1}{n^{2s}} - s \sum \frac{1}{n^{2s+1}} +$ REMAINDER

$\zeta(2s)$ ↗ $\quad \zeta(2s+1)$ ↗

We get the poles of $\Lambda(s)$ from those of $\zeta(s)$!

Works for $\Lambda(s) = \sum P(n)^{-s}$ with P a polynomial, and other cases.

<u>Dirichlet depoissonization</u>: We have $F(x) = \sum \left[1 - \left(1 - \frac{1}{2^k}\right)^x\right]$, a harmonic sum. Use

$$\Lambda(s) = \sum \left(\log \frac{1}{1-2^{-k}}\right)^{-s} \cong \sum \left(2^{-k} - \frac{1}{2} 2^{-2k} + \cdots\right)^{-s} \cong \frac{1}{1-2^s} - \frac{s}{2} \frac{1}{1-2^{s-1}} + \text{REM.}$$

Thus $\underset{(x\to\infty)}{F(x)} \sim \log_2 x + C + \varphi(\log_2 x) + \frac{1}{x} + c_1 + \varphi_1(\log_2 x) + \cdots$

↖ Fourier series (oscillations). ↗

Poisson + Mellin = Newton + Nörlund

$(f_n) \longrightarrow$ Poisson GF: $\varphi(x) = \sum f_n \frac{x^n}{n!} e^{-x}$.

How to go back from $\varphi(x)$ to f_n. We expect often that $f_n \approx \varphi(n)$.

$$\xrightarrow{\mathcal{M}} \varphi^*(s) = \sum f_n \frac{\Gamma(s+n)}{n!} = \Gamma(s)\left[f_0 + f_1 \frac{s}{1!} + f_2 \frac{s(s+1)}{2!} + \cdots \right]$$

Thus $\boxed{\mathcal{N}(s) = \frac{\varphi^*(s)}{\Gamma(s)}}$ is a Newton series (interpolation at $0, -1, -2, \ldots$).

- Algebraically: exact expression $\boxed{f_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \mathcal{N}(-k)}$.
- Analytically $\boxed{f_n = \frac{\pm 1}{2i\pi} \oint \mathcal{N}(-s) \frac{n!}{s(s+1)\cdots(s+n)}}$ ← Nörlund-Rice integrals.

Can be estimated like Mellin by residues. {Useful for digital search trees, tries, data compression, etc.}

VI. More goodies

- Magic Duality and the Golden Triangle

With $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$:

$$2\sum_{n=1}^{\infty} H_{n^2-1}\frac{(-1)^n}{n^2} = \frac{7}{360}\pi^4 - \frac{1}{6}\pi^2 + \sum_{n=1}^{\infty}\frac{\pi}{n\sqrt{n}\sinh(\pi\sqrt{n})}.$$

Magic duality

Principle If $F(x) = \sum_{n\geq 1} c_n(-x)^n$ (as $x \to 0$) then

$$F(x) = \begin{cases} -\sum_{n\geq 1} c_{-n}(-x)^{-n} & (\text{as } x \to +\infty) \\ + E(x) \end{cases}$$

where $E(x) = \sum_{(\alpha,k)} x^\alpha (\log x)^k$ is an "elementary" function.

This assumes that the "c_n" are given by a law involving, e.g., binomial coefficients, harmonic numbers, rational fractions, etc ...

The secret: Let $\phi(s)$ extrapolate c_n to $\mathbb{C}$, and be "mild" towards $\pm i\infty$:

$$F(x) = \frac{1}{2i\pi} \int_{-1/2-i\infty}^{1/2+i\infty} \phi(s)\, x^s \frac{\pi}{\sin \pi s}\, ds .$$

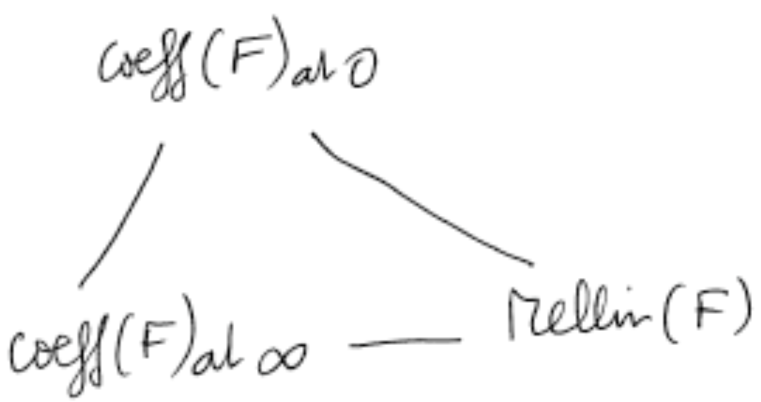
(Proof: residues on the left)

Then we move the integration line to the right and get the expansion at ∞.

The proof also shows that $\phi(s)$ is essentially the Mellin transform of $F(x)$.

GOLDEN TRIANGLE : Under suitable growth conditions, the same "law" governs the

- the coefficients of expansion of a function at 0
- the coefficients at ∞
- the Mellin transform.



(First observed empirically (??) by RAMANUJAN.)