Combinatorial Models and Algebraic Continued Fractions

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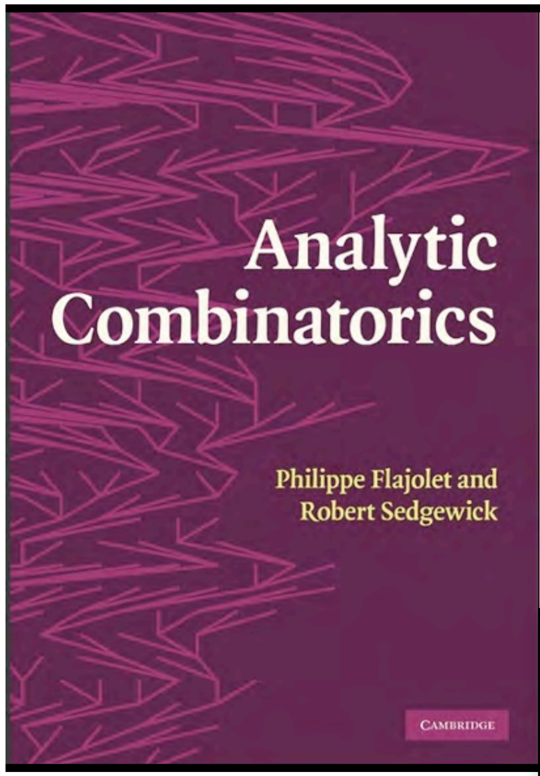
Orthogonal Polynomials, Special Functions and Applications
Leuven, Begium, July 2009



ANALYTIC COMBINATORICS,

by P. Flajolet & R. Sedgewick
Cambridge 2009, 824p
free download: `algo.inria.fr/flajolet/`

= exactly solvable models + asymptotics

CF

Continued fractions associated with power series are tightly linked to lattice paths

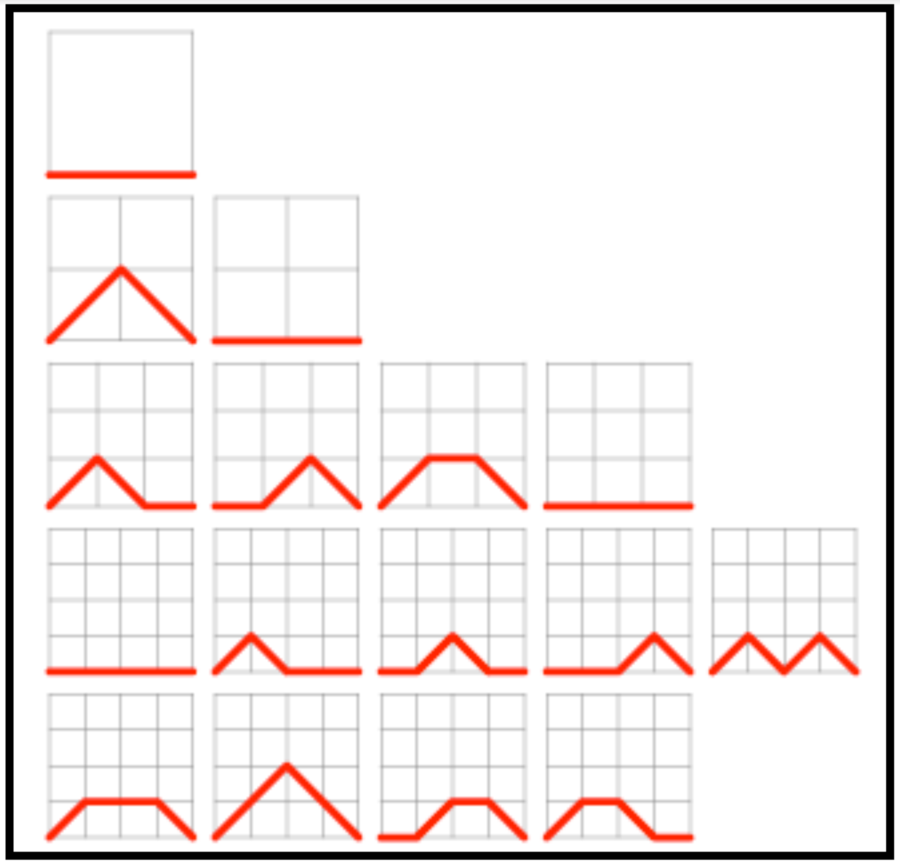
- Read off CF identities from combinatorics
- Solve combinatorial & discrete probabilistic models via CFs and Orthogonal Polynomials

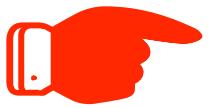
OPS

LATTICE PATHS

- are comprised of steps
  - Ascents ($a : \nearrow$);
  - Descents ($b : \searrow$);
  - Levels ($c : \rightarrow$);
- never go below horizontal axis.

Excursions start and end at 0–altitude.





Main theorem: *Equivalence between:*

— **Sum of all excursions** *encoded with altitudes;*

— **Universal continued fraction** *of Jacobi type.*

FORMALITIES (i)

- *Quasi-inverses*. Let $\mathcal{A}$ be a **ring**; by *telescoping*:

$$(1-f)(1+f+\cdots+f^n) = 1-f^{n+1}.$$

If $f^n \to 0$, *then* $\dfrac{1}{1-f} = 1+f+f^2+f^3+\cdots.$

- *Distributivity*: $(x+y)^n = \sum_{|w|=n} w$ (all words of length n).

Corollary A. If $a = \nearrow,\ b = \searrow,\ c = \rightarrow$, then [e.g., $n \equiv 3$]

$$(c+ab)^n = \overbrace{\rightarrow\rightarrow\rightarrow + \nearrow\searrow\rightarrow\rightarrow + \rightarrow\nearrow\searrow\rightarrow + \nearrow\searrow\nearrow\searrow\nearrow\searrow + \cdots}^{n \text{ blocks}}.$$

Corollary B. Combining with the *sum of a geometric progression*

$$\frac{1}{1-c-ab} = \sum \overbrace{(\rightarrow\nearrow\searrow\rightarrow\rightarrow\nearrow\searrow)}^{\text{any }\#\text{ blocks}}.$$

FORMALITIES (ii)

Recall: $\displaystyle \frac{1}{1-c-ab} = \sum \rightarrow \nearrow\searrow \rightarrow\rightarrow \nearrow\searrow.$

Substitute further $ab \mapsto a\frac{1}{1-d}b$. Then:

Corollary C. *With* $a = \nearrow$, $b = \searrow$, $c = \rightarrow$, $d = \Rightarrow$, *in* $\mathbb{C}[\![a,b,c,d]\!]$:

$$\frac{1}{1-c-a\,\dfrac{1}{1-d}\,b} = \sum (\text{all diagrams of height} \le 1)$$

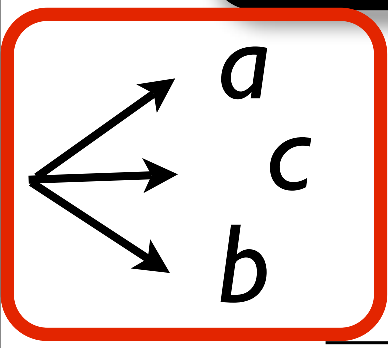
$$= \sum \rightarrow \nearrow^{\Rightarrow\Rightarrow\Rightarrow} \searrow \rightarrow\rightarrow \nearrow^{\Rightarrow} \searrow .$$

☺

FORMALITIES GIVE A THEOREM...

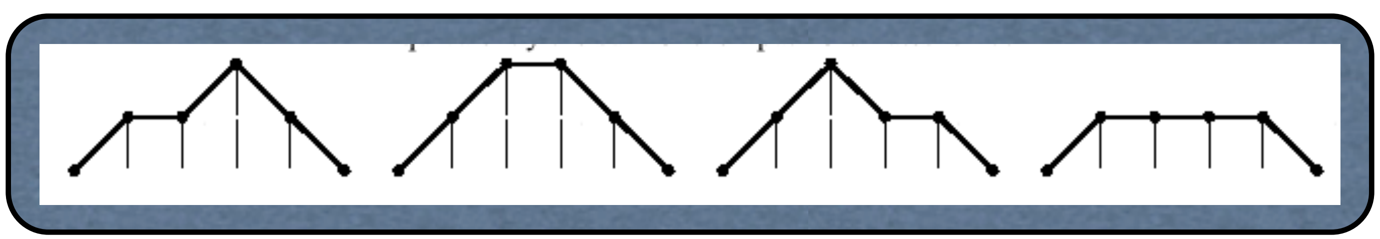
Theorem [The main continued fraction theorem]

$$\cfrac{1}{1 - c_0 - \cfrac{a_0 b_1}{1 - c_1 - \cfrac{a_1 b_2}{1 - c_2 - \ddots}}} = \sum (\text{all lattice paths})$$



$$\cfrac{1}{1 - c_0 - \cfrac{a_0 b_1}{1 - c_1 - \cfrac{a_1 b_2}{\cfrac{\ddots}{1 - c_2 - \cfrac{}{1 - c_h}}}}} = \sum \left(\begin{array}{c} \text{all lattice paths} \\ \text{with height} \leq h \end{array} \right)$$

$=P_h/Q_h$

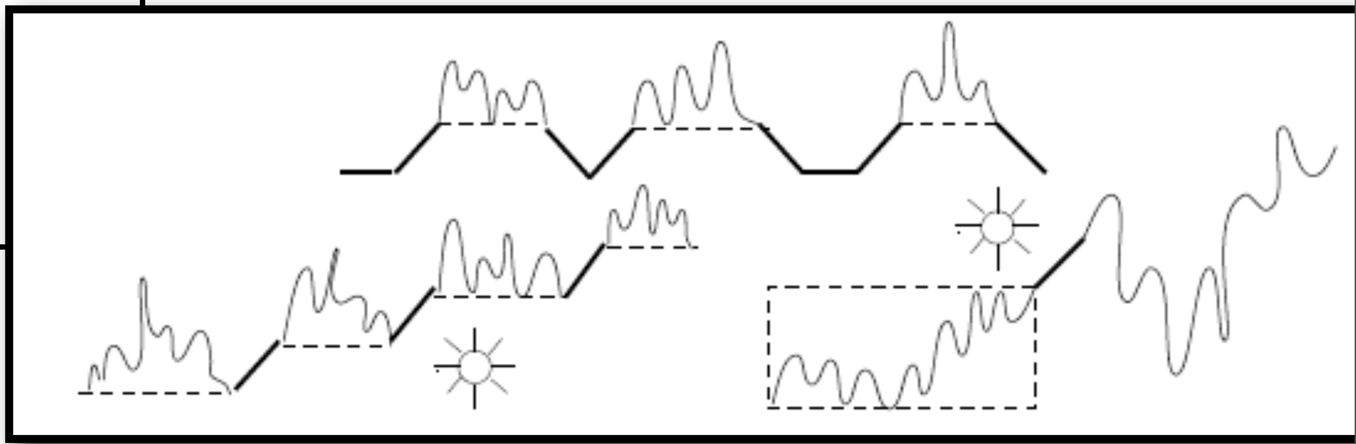


Equivalently, with **weighting rules**, $a_j \mapsto \alpha_j z$, $b_j \mapsto \beta_j z$, $c_j \mapsto \gamma_j z$:

$$\cfrac{1}{1-\gamma_0 z - \cfrac{\alpha_0\beta_1 \cdot z^2}{1-\gamma_1 z - \cfrac{\alpha_1\beta_2 \cdot z^2}{1-\gamma_2 z - \ddots}}} \equiv \sum_{\pi:\ \text{excursion}} z^{|\pi|}\, \text{weight}(\pi)$$

universal J–fraction $\equiv$ generating function of [weighted] excursions

Excursions :	J–fraction
—, bounded height :	$\frac{P_h}{Q_h}$ (convergent)
Paths ending at k :	$Q_k J - P_k$
—, traversing k–strip :	$\frac{1}{Q_k}$



- J. Touchard [1952]: chord diagrams (i)
- I.J. Good [1958]: discrete birth-death processes
- A. Lenard [1961]: statistical physics
- G. Szekeres [1968]: Rogers-Ramanujan identities
- D. Jackson [1978]: Ising model
- R. Read [1979]: chord diagrams (ii)
- P. Flajolet [1978-80]: "File histories". *Discr. Math.*

What next?

- 1. Simple applications
- 2. Convergents and orthogonal polynomials
- 3. Arches
- 4. Snakes
- 5. Addition formulae
- 6. Elliptic matters

1. Simple applications (ballots, coins)

A SOLUTION (?!) TO THE BALLOT PROBLEM

"Two candidates, Alice and Bob, with (eventually) each n votes. What is the probability that Alice is always ahead or tied?"

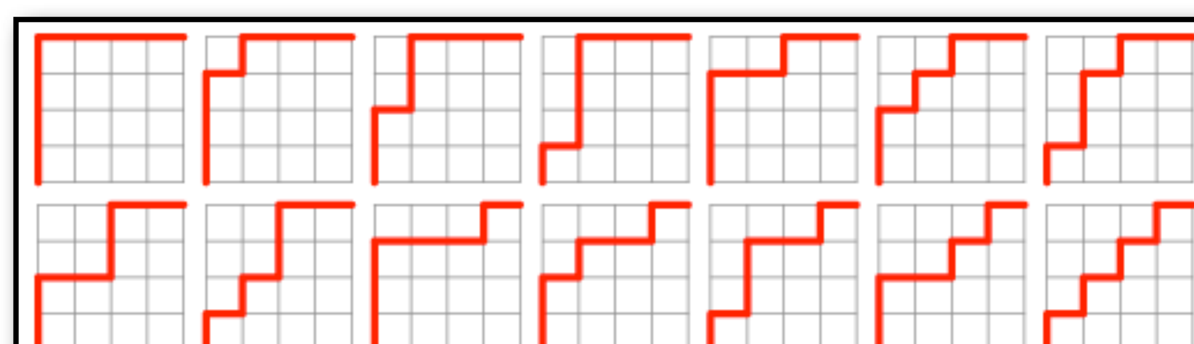
Do $\boxed{a \mapsto z;\ b \mapsto z;\ c \mapsto 0}$. By main theorem:

$$C := \cfrac{1}{1 - \cfrac{z^2}{1 - \cfrac{z^2}{\ddots}}} = \sum_{\beta \text{ ballot sequence}} z^{|\beta|} = \sum_n C_n z^{2n}.$$

We get Catalan numbers [Euler-Segner 1750; Catalan 1850]

$$C = \frac{1}{1 - z^2 C} \implies C = \frac{1 - \sqrt{1 - 4z^2}}{2z^2} \implies C_n = \frac{1}{n+1}\binom{2n}{n}.$$

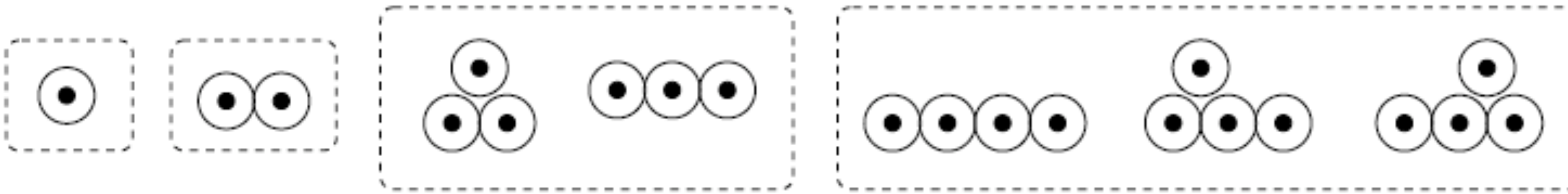
The probability is $\dfrac{C_n}{\binom{2n}{n}} = \dfrac{1}{n+1}$.



COIN FOUNTAINS

[Odlyzko-Wilf, 1988]

$$C(q) = 1 + q + q^2 + 2q^3 + 3q^4 + 5q^5 + 9q^6 + 15q^7 + 26q^8 + \cdots$$



Greetings from The On-Line Encyclopedia of Integer Sequences!

Search: **1, 2, 3, 5, 9, 15, 26, 45**

Displaying 1-2 of 2 results found. pag

Format: long | short | internal | text Sort: relevance | references | number Highlight: on | off

A005169 Number of fountains of n coins.
(Formerly M0708)

1, 1, **1, 2, 3, 5, 9, 15, 26, 45**, 78, 135, 234, 406, 704, 1222, 2120, 3679, 6385, 11081, 19232, 33379, 57933, 100550, 174519, 302903, 525734, 912493, 1583775, 2748893, 4771144, 8281088, 14373165, 24946955, 43299485, 75153286, 130440740 (list; graph; listen)

OFFSET 0,4

COMMENT A fountain is formed by starting with a row of coins, then stacking additional coins on top so that each new coin touches two in the previous row.

Do: $\boxed{a_j \mapsto 1,\ b_j \mapsto q^j}$. Then:

$$C(q) = \cfrac{1}{1 - \cfrac{q}{1 - \cfrac{q^2}{1 - \cfrac{q^3}{\cdots}}}} = \frac{\sum (-1)^n \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)}}{\sum (-1)^n \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}}.$$

Number of coin fountains: $C_n \sim 0.31 \cdot 1.73566^n$.

Ramanujan's fraction:

$$\cfrac{1}{1 + \cfrac{e^{-2\pi\sqrt{5}}}{1 + \cfrac{e^{-4\pi\sqrt{5}}}{1 + \cfrac{e^{-6\pi\sqrt{5}}}{\cdots}}}} = e^{2\pi/\sqrt{5}} \left(\frac{\sqrt{5}}{1 + \sqrt[5]{\frac{5^{3/4}}{\left(1/2 + 1/2\sqrt{5}\right)^{5/2}} - 1}} - \frac{1 + \sqrt{5}}{2} \right)$$

$$= 0.\mathbf{999999}2087 \cdots .$$

$$\boxed{e^{\pi\sqrt{163}} = 262537412640768743.999999999999\cdots}$$

2. Convergent polynomials

- Revisiting the ballot problem
- Orthogonality

"Two candidates, Alice and Bob, with (eventually) each n votes. If Alice is always ahead (or tied), what is the probability that she never **leads** *by more than h?"*

The number of favorable cases has generating function (GF), with $z^2 \mapsto z$:

$$C^{[h]}(z) = \cfrac{1}{1 - \cfrac{z}{1 - \cfrac{\ddots}{1 - \cfrac{}{1 - z}}}} \quad \Bigg\} \; h \text{ stages.}$$

$$\frac{1}{1}, \quad \frac{1}{1-z}, \quad \frac{1-z}{1-2z}, \quad \frac{1-2z}{1-3z+z^2}, \quad \cdots, \quad \frac{F_{h+1}(z)}{F_{h+2}(z)},$$

where $F_{h+2} = F_{h+1} - zF_h$ are Fibonacci polynomials.

- "Constant-coefficient" recurrence; Lagrange inversion.
- Roots are $1/(4\cos^2\theta)$, $\theta = \frac{k\pi}{h}$; partial fractions.

+ *Chebyshev*

Lagrange [1775] & Lord Kelvin & De Bruijn, Knuth, Rice [1973]

$$C_n^{[h]} = \sum_k \cdots 4^n \cos^{2n}\left(\frac{k\pi}{h}\right) = \sum_k \cdots \binom{2n}{n-kh}.$$

partial fraction / Lagrange inversion

- Related to Kolmogorov–Smirnov tests in statistics:
  Compare $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$? "Sort and vote!"
- Pólya [1927]: **totally elementary proof of elliptic-theta transformation**:

$$\sum_{\nu=-\infty}^{\infty} e^{-\nu^2 t^2} = \sqrt{\frac{\pi}{t^2}} \sum_{\nu=-\infty}^{\infty} e^{-\pi^2\nu^2/t^2}.$$

= *Do multisection of $(1+z)^{2n}$, with $h = t\sqrt{n}$, in two ways!*

+ *asymptotics!*

Orthogonal polynomials

Linear fractional transformations [homographies] get composed like 2×2 matrices:

$$\frac{ax+b}{cx+d} \qquad \mapsto \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

- Convergent polynomials $\dfrac{P_h(z)}{Q_h(z)}$ satisfy a three-term recurrence with numers/denoms of the continued fraction.
- Reciprocals of convergent polynomials are **orthogonal** with respect to $\langle f, g \rangle = \langle f \cdot g \rangle$, where moments $\langle z^n \rangle$ are coefficients in the expansion of the continued fraction.

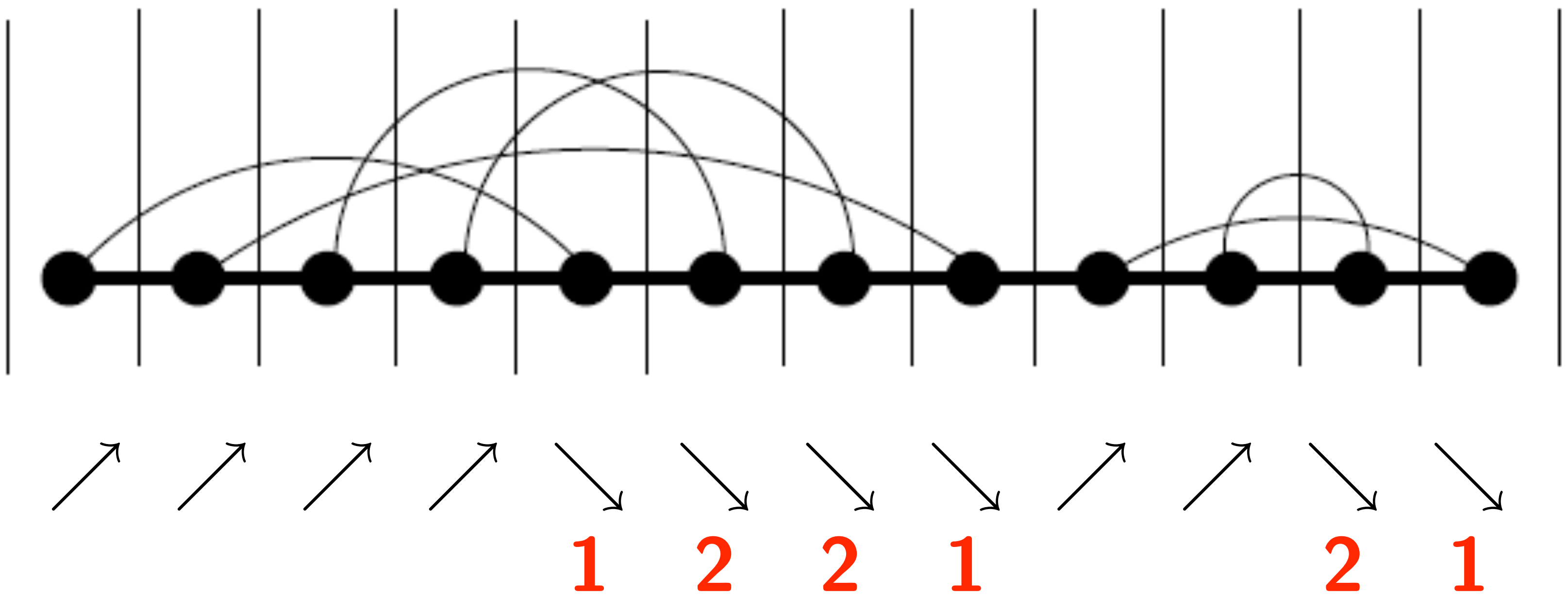
In all generality:

- Orthogonal polynomials must appear in counting of *paths of bounded height* and in "equivalent" structures.

3.Arches and such

- Colouring rules
- Hermite polynomials

In how many ways can one join $2n$ points on the line in pairs?



A descent from altitude j has j possibilities: $d_j \mapsto jz, \quad a_j \mapsto z$.

$$\sum_{n\geq 1}(1\cdot 3\cdots(2n-1))z^{2n} = \cfrac{1}{1-\cfrac{1\cdot z^2}{1-\cfrac{2\cdot z^2}{1-\cfrac{3\cdot z^2}{\ddots}}}}.$$

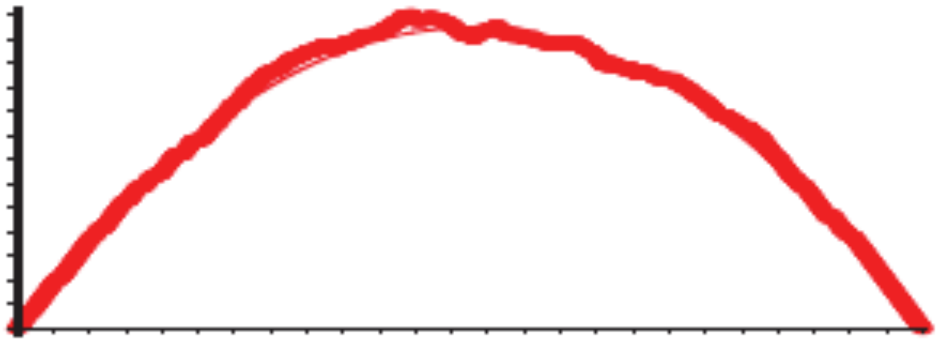
Gauss !!

Lagarias-Odlyzko-Zagier [1985]: Which capacity do we need to arrange pairwise connections between 2n points, with high probability?

- The answer lies in the zeroes of **Hermite polynomials**.

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) \cdot g(x)\, e^{-x^2/2}\, dx.$$

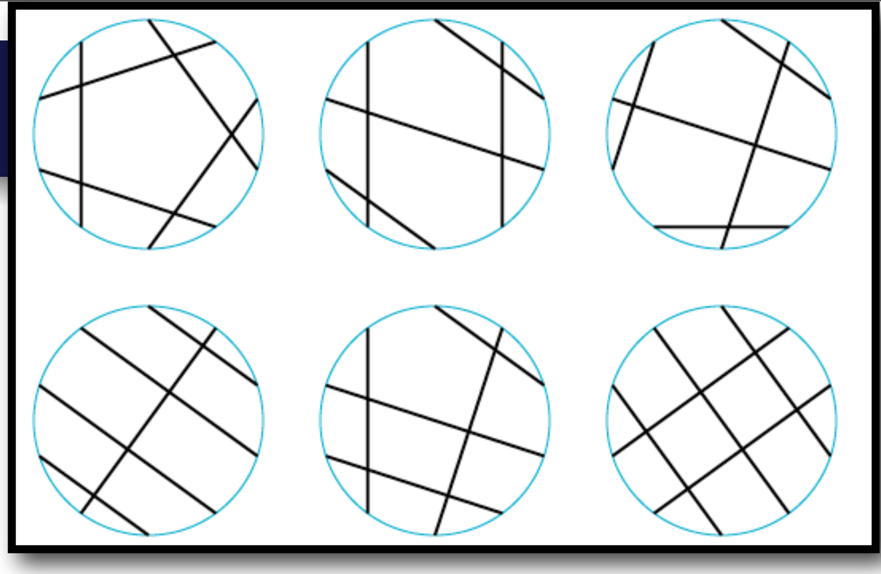
Proof. **height** h:
$$\left.\cfrac{1}{1-\cfrac{1\cdot z^2}{1-\cfrac{2\cdot z^2}{\ddots\, \cfrac{}{1-h\cdot z^2}}}}\right\} \quad h \text{ levels.}$$



Louchard & Janson: a Gaussian process = deterministic parabola + Brownian noise.

+ Airy connection

Chord systems



Join $2n$ points on circle by chords. How many crossings?

Sweeping: $T(z,q) \equiv \sum_{\sigma} z^{|\sigma|} q^{\#xings(\sigma)} = \cfrac{1}{1 - \cfrac{[1]\cdot z^2}{1 - \cfrac{[2]\cdot z^2}{1 - \cfrac{[3]\cdot z^2}{\ddots}}}}, \quad [n] = \frac{1-q^n}{1-q}.$

Theorem [Touchard]. *Number of crossings has generating function*

$$T(z(1-q),q) = \sum_{k\geq 0} q^{\binom{k+1}{2}} (-z^k) C^{2k+1}; \quad C := \tfrac{1}{2z}\left(1-\sqrt{1-4z}\right).$$

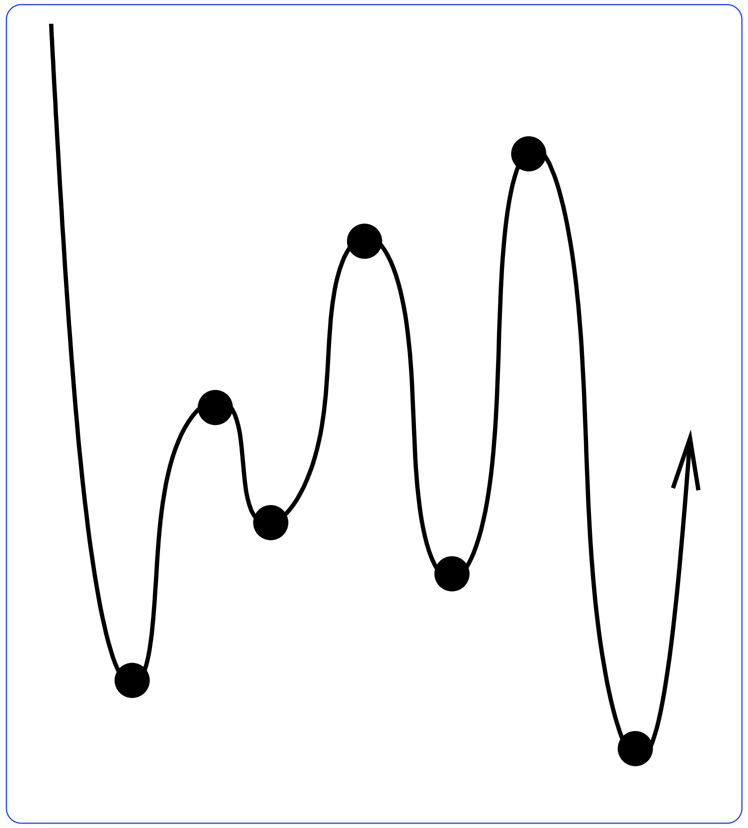
Corollary [F-Noy 2000]: *# crossings is asympt. Gaussian.*

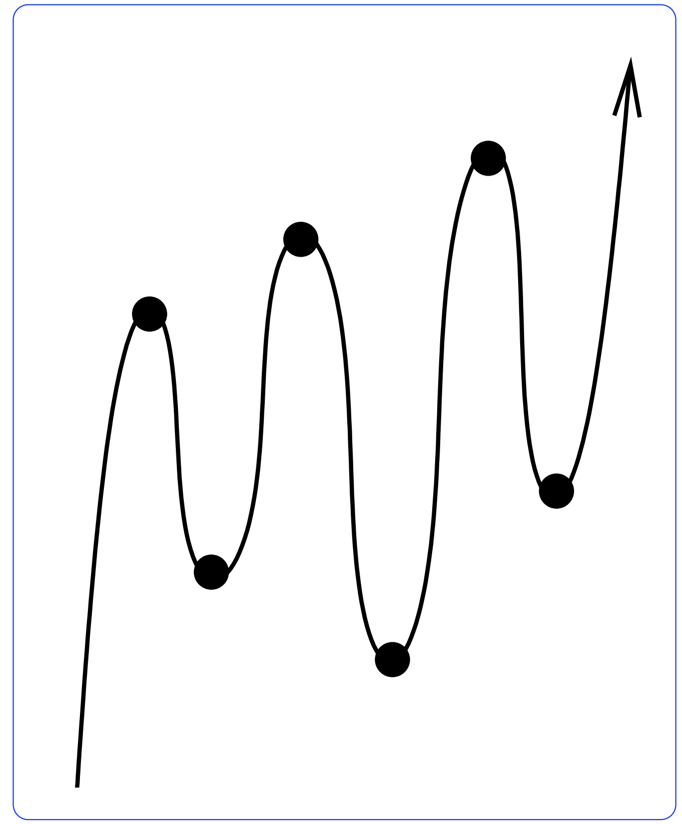
Cf [Ismail, Stanton, Viennot 1987] for nice combinatorics (q-Hermite).

4. Snakes and curves

- Arnold's snakes
- Stieltjes' fraction
- Postnikov's Morse links

Arnold [1992]: How many topological types of "smooth" functions?

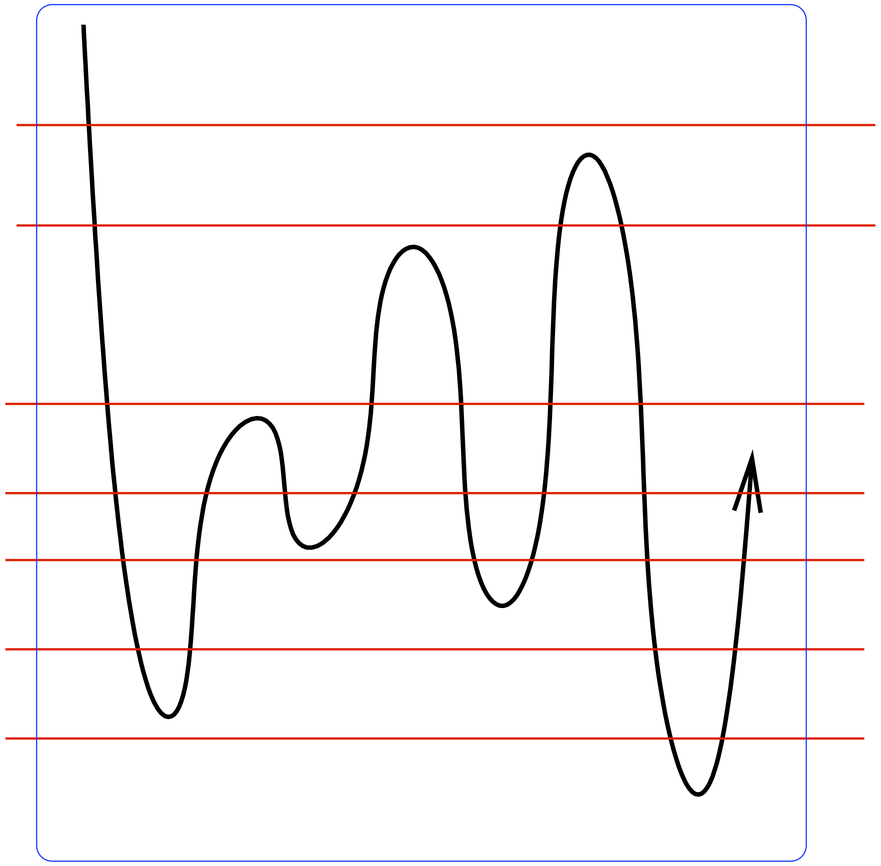


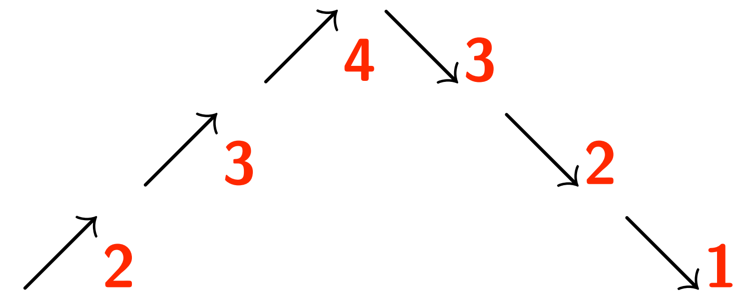


D. André [1881]: alternating perms = the coefficients of tan(z) and sec(z).

$$
\begin{cases}
\tan z = \mathbf{1}\frac{z}{1!} + \mathbf{2}\frac{z^3}{3!} + \mathbf{16}\frac{z^5}{5!} + \mathbf{272}\frac{z^7}{7!} + \cdots \\
\sec z = \mathbf{1} + \mathbf{1}\frac{z^2}{2!} + \mathbf{5}\frac{z^4}{4!} + \mathbf{61}\frac{z^6}{6!} + \cdots .
\end{cases}
$$

Proof. Decompose according to minimum and get ODE: $Y' = 1 + Y^2$ via recurrence.





$$\sum (\tan)_{2n+1} z^{2n+1} = \left\langle\!\left\langle \int_0^\infty e^t \tan(zt)\, dt \right\rangle\!\right\rangle = \cfrac{z}{1 - \cfrac{1 \cdot 2\, z^2}{1 - \cfrac{2 \cdot 3\, z^3}{\cdots}}}.$$

1, 2, 16, 272, ...

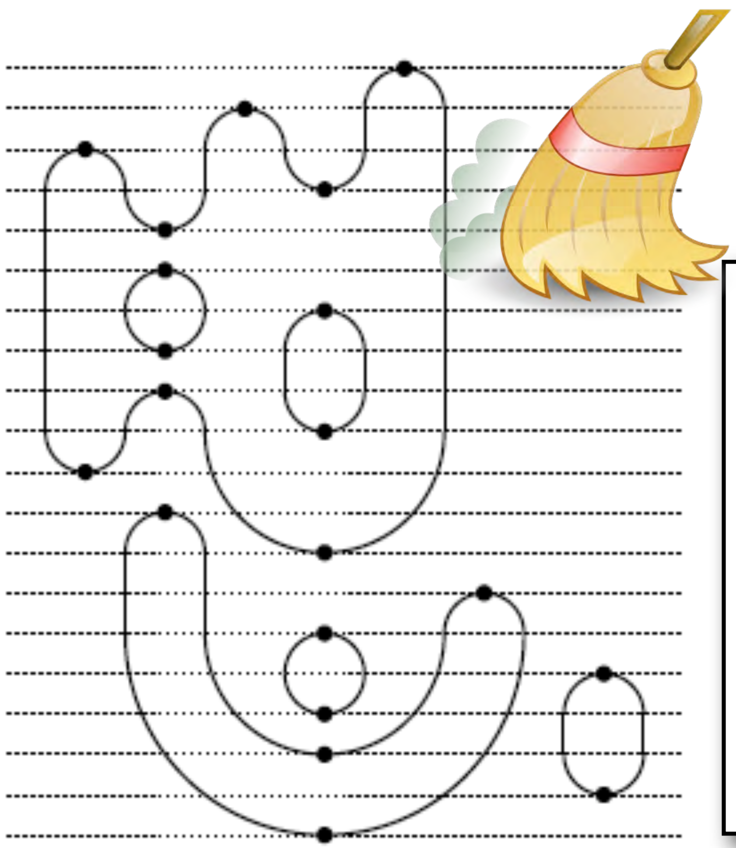
Related to a bijection of Françon and Viennot

= A continued fraction of Stieltjes

Ann. Fac. Sci. Toulouse, 1894

A continued fraction of Postnikov (2000)

= *Morse links (systems of closed Morse curves)*



$$\frac{\Gamma\left(x-\frac{1}{2}a+\frac{1}{4}\right)\Gamma\left(x+\frac{1}{2}a+\frac{1}{4}\right)}{\Gamma\left(x-\frac{1}{2}a+\frac{3}{4}\right)\Gamma\left(x+\frac{1}{2}a+\frac{3}{4}\right)}= \frac{4}{4x+\cfrac{1^2-4a^2}{8x+\cfrac{3^2-4a^2}{8x+\cfrac{5^2-4a^2}{8x+\cfrac{7^2-4a^2}{8x+\cdots}}}}}$$

Theorem [F.2008]. *The Morse–Postnikov numbers satisfy*

$$L_n \sim \widehat{L}_n, \qquad \textit{where} \quad \widehat{L}_n = \frac{1}{2}(2n-1)!\left(\frac{4}{\pi}\right)^{2n+1}.$$

E.g.: $\dfrac{L_4}{\widehat{L}_4} \doteq 0.99949.$

5. Addition formulae &c.

- Stieltjes-Rogers
- Addition formulae, paths, and OPs are all belong to a single family of identities
- Applications to "processes"...

The Stieltjes–Rogers Theorem

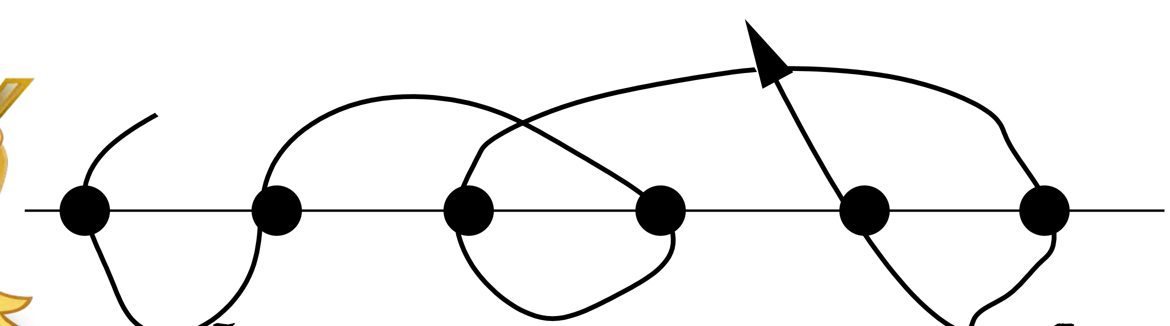
Definition. $\phi(z) = \sum_{n=0}^{\infty} \phi_n \frac{z^n}{n!}$ *satisfies an addition formula if*

$$\phi(x+y) = \sum_k \omega_k \phi_k(x)\phi_k(y), \text{ where } \phi_k(x) = \frac{x^k}{k!} + O(x^{k+1}).$$

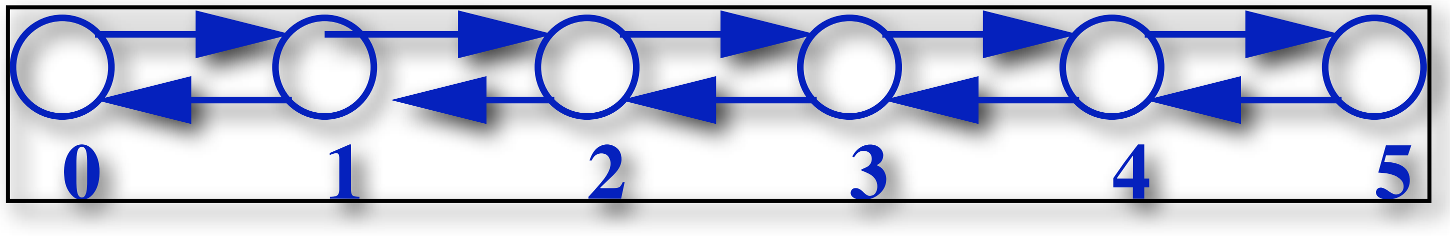
Theorem. *An addition formula gives automatically a continued fraction for* $f(z) = \sum_{n=0}^{\infty} \phi_n z^n = \langle\!\langle \int_0^\infty e^t \phi(zt)\, dt \rangle\!\rangle$.

$$\frac{1}{1-x-y} = \sum_k (k!)^2 \frac{x^k/k!}{(1-x)^{k+1}} \frac{y^k/k!}{(1-y)^{k+1}}$$

$$\sum n! z^n = \cfrac{1}{1 - z - \cfrac{1^2 z^2}{1 - 3z - \cfrac{2^2 z^2}{\ddots}}}$$



[Biane, Françon-Viennot]



Systems of paths and **birth–death processes**:
Number & probability of weighted paths from a to b;

- Discrete time processes: I.J. Good [1950's];
- Continuous time processes: Karlin–McGregor; F–Guillemin [AAP 2000];
- Combinatorial processes = "file histories", [F–Françon–Vuillemin–Puech, 1980+]

$\leadsto$ Paths from 0 to k have exp. gen. function φ_k of addition formula.

Meixner's class of special OP's

Classical orthogonal polynomials appear to share many properties.

Theorem [Meixner 1934]: *If the exponential generating function satisfies a strong decomposability property,*

$$\sum_h \overline{Q}_h(z)\frac{t^n}{n!} = A(t)e^{zB(t)},$$

then there are only five possibilities.

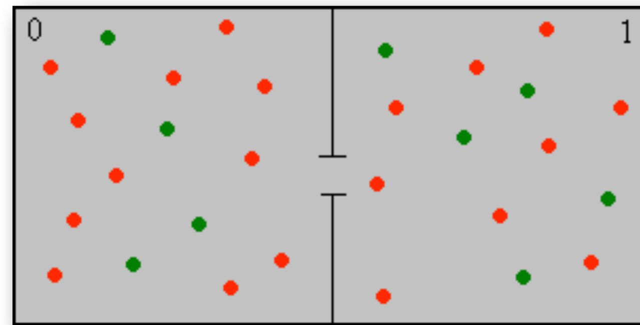
Laguerre	Hermite	Poisson-Charlier	Meixner I	Meixner II
Perms	Arcs	Set partitions	Snakes	Pref. arrang.
$\frac{1}{1-z}$	$e^{z^2/2}$	e^{e^z-1}	$\sec(z)$	$\frac{1}{2-e^z}$.

Computations for *linear possibilities* are automatic:

unified theory of "libraries", basic queueing systems

The Ehrenfest urn model

Particles switching chambers



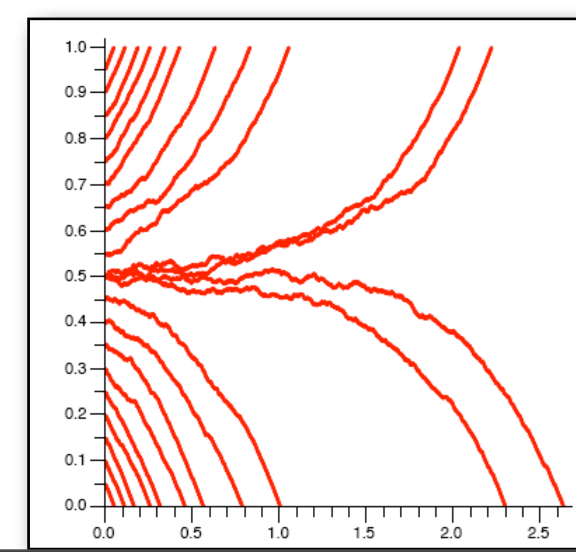
addition formula $\boxed{(\cosh(z))^N}$ & $$\cfrac{1}{1-\cfrac{1\cdot N z^2}{1-\cfrac{2\cdot(N-1)\,z^2}{\cdots}}}.$$

Stieltjes; Kac 1947;
Edelman-Kostlan 1994

The Mabinogion urn model

- Spread of influence in populations:
  $A \Longrightarrow (B \longrightarrow A),\ B \Longrightarrow (A \longrightarrow B).$

Theorem [F–Huillet 2008]. *Fair urn: absorption time is* $\sim \frac{1}{2} N \log N$, *with limit distribution of density* $\asymp e^{-t} e^{-e^{-2t}}$.



6. Some Elliptic matters

- Jacobian functions
- Dixonian functions
- Bacher's numbers

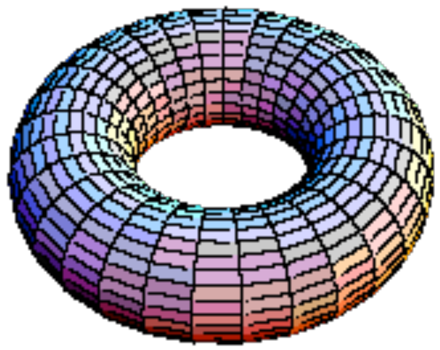
- *Pollaczek fractions* have coefficients that are polynomials in the level = a *mysterious* class!

- Includes *some* Hurwitz zeta; cf Stieltjes-Apéry

$$\zeta(3) = \cfrac{6}{\varpi(0) - \cfrac{1^6}{\varpi(1) - \cfrac{2^6}{\varpi(2) - \cfrac{3^6}{\ddots}}}},$$

- An interesting "*sporadic*" subclass appears to be related to *elliptic functions*

[Pollaczek, *Mem. Sc. Math*., 1956]

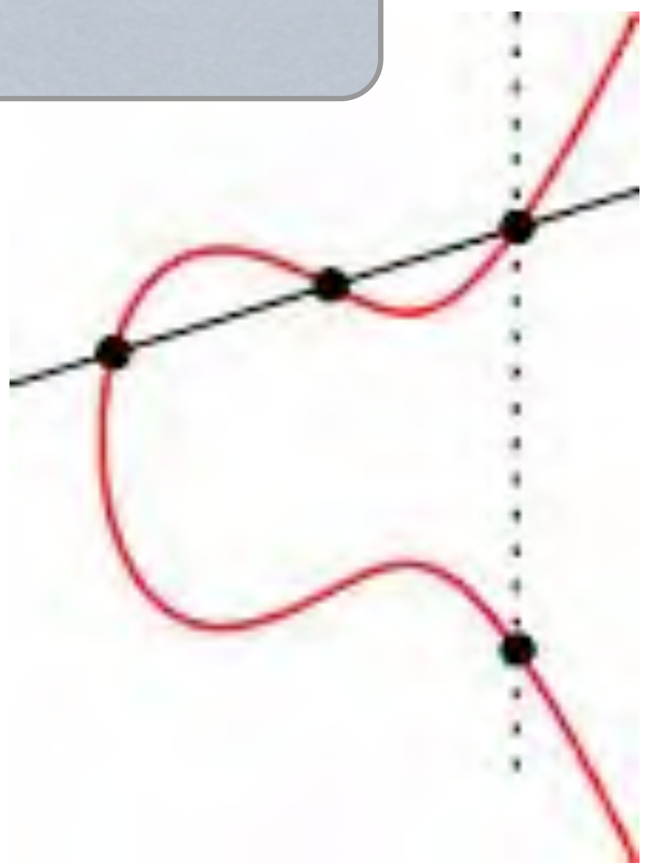


- **Algebraic curves of genus 1** are doughnuts. The integrals have two "periods". The inverse functions are **elliptic functions**; i.e., doubly periodic meromorphic.

- Weierstraß $\wp$ arises from $y^2 = P_3(z)$;
- Jacobian sn, cn arise from $y^2 = (1 - z^2)(1 - k^2 z^2)$;
- Dixonian sm, cm arise from $y^3 + z^3 = 1$.

They satisfy addition formulae!

(≠Stieltjes-Rogers)



- **Theorem [F; Dumont 1980].** *Jacobian elliptic functions count alternating perms w/parity of peaks.*

- **Theorem [Conrad+F, 2006].** *Dixonian functions have continued fractions*

$$\int_0^{\infty} \mathrm{sm}(u) e^{-u/x}\, du = \cfrac{x^2}{1 + b_0 x^3 - \cfrac{1 \cdot 2^2 \cdot 3^2 \cdot 4\, x^6}{1 + b_1 x^3 - \cfrac{4 \cdot 5^2 \cdot 6^2 \cdot 7\, x^6}{\cdots}}};$$

★ $\equiv$ *levels in trees and an urn model (≈Yule process), &c*

- **Theorem [Bacher+F, 2006].** *Pseudofactorials* $a_{n+1} = (-1)^{n+1} \sum \binom{n}{k} a_k a_{n-k}$ *have a CF*

$$\sum a_n z^n = \cfrac{1}{1 + z + \cfrac{3 \cdot 1^2\, z^2}{1 - z + \cfrac{2^2\, z^2}{1 + 3z + \ddots}}}.$$

A098777 a(0)=0, a(n+1)=(-1)^(n+1)*sum('binomial(n,k)*a(k)*a(n-k)','k'=0..n), n>=0.

1, -1, -2, 2, 16, -40, -320, 1040, 12160, -52480, -742400, 3872000, 664
411136000, -8202444800, 58479872000, 1335009280000, -10791497728000, -2

Pseudo-factorials: $a_{n+1} = (-1)^{n+1} \sum \binom{n}{k} a_k a_{n-k}$.

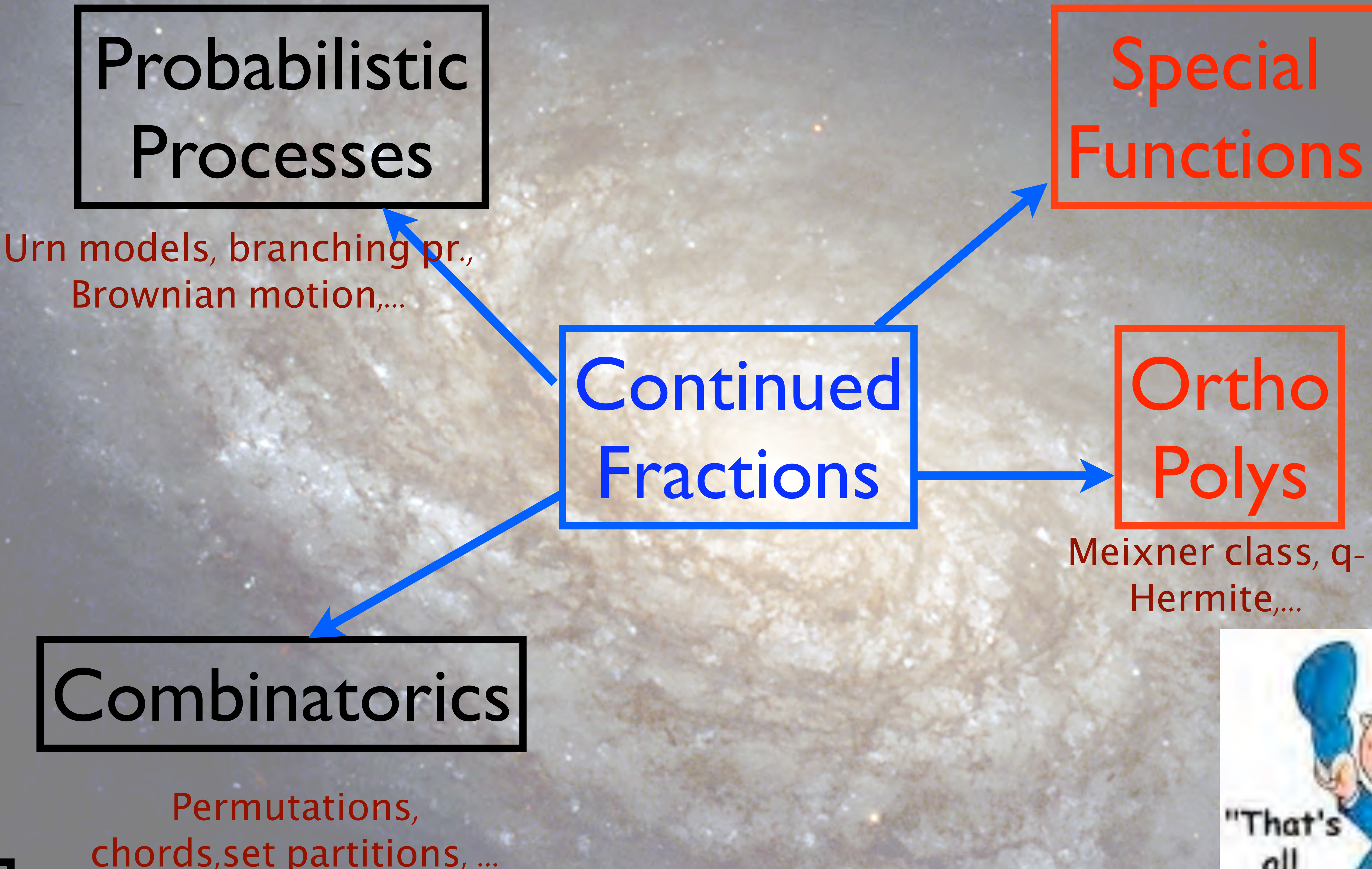
Theorem [Bacher+F, 2008]. *The exponential generating function of the orthogonal polynomials attached to* (a_n) *is*

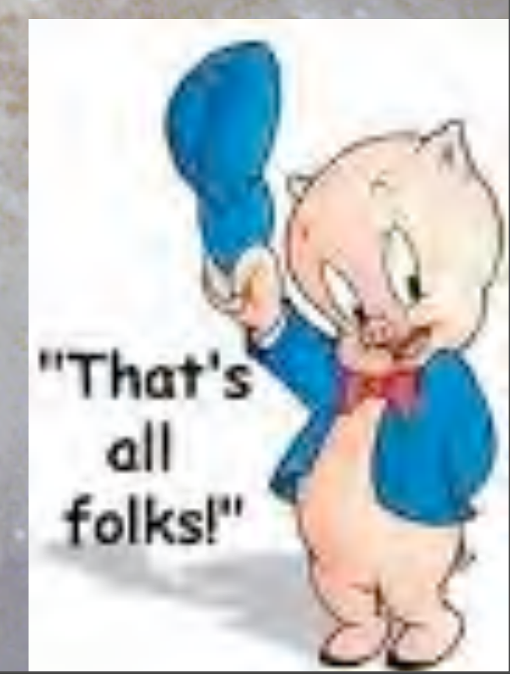
$$\eta(t) \cosh(zJ(t)) + \chi(t) \sin(zJ(t)),$$

where $J(t) := \int_0^t \frac{du}{\sqrt{1 - 3u^2 + 3u^4}}$ *and* η, χ *are algebraic functions.*

$\neq$ Carlitz 1960+; Ismail & Masson 1999; Lomont & Brillhart 2001; cf. Gilewicz *et al* 2006 for "sm".

Cf also: Flajolet--Bacher (an octic fraction, unpub.); Rivoal (deg=12(!)) relative to $\Gamma(1/3)^3$


Probabilistic Processes
Urn models, branching pr., Brownian motion,...
Special Functions
Continued Fractions
Ortho Polys
Meixner class, q-Hermite,...
Combinatorics
Permutations, chords,set partitions, ...


"That's all folks!"