Chapter 9: Multivariate asymptotic and limit distributions.
Singularities and Random Combinatorial Structures

**Eulerian distribution** rises in permutations

\[ \Pr(\text{\# rises} = k) \]

\[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-w^2/2} \, dw \]

**Why is the binomial distribution asymptotically Gaussian?**

- **De Moivre**: Stirling's approximation: \( \frac{1}{\sqrt{\pi n}} (\begin{pmatrix} n \end{pmatrix}) \)
- **Laplace**: As a sum of a large number of RV's

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De Moivre  
Laplace  
Gauss
Singularities and Random Combinatorial Structures

**Theorem:**

\[ f_n(u) = \frac{[z^n]}{[z^n]} C(z, u) \]

*Typical case:* \( f_n(u) \) arises from a bivariate generating function.
Proof involves

- Continuity Theorem for characteristic functions $\equiv$ Lévy's Theorem.

$\phi_X(t) : = \mathbb{E}(e^{itX})$ characteristic function

$= \text{PGF}(e^{it})$ for discrete RV's.

Continuity theorem: If $\phi_{X_n}(t) \to \phi_Y(t)$ pointwise

for all $X_n$, then $X_n \overset{d}{\to} Y$

i.e. for cumulative distribution functions [cdf]

$\forall x : \, F_{X_n}(x) \to F_Y(x)$ [at points of continuity].

+ Berry-Esseen: if characteristic functions are close, then cumulative distribution functions are also close. [Feller]

The supercritical sequence $F = \text{Seq}(G)$

$F(z,u) = \frac{1}{1-uG(z)}$

Assume $G(p) > 1$ where $r = \text{radius of conv. of } G$.

**Theorem:** The # of $G$-components in a large $F$-structure is asympt. Normal

Proof: Let $p \in (0,r)$ be such that $G(p) = 1$.

- Equation $1-uG(z) = 0$ has root $p_u(z)$ where $p_u(z)$ depends analytically on $u$ for $u$ near 1

- $F(z,u)$ with $u$ a param. has simple pole

  with $[z^n] F(z,u) \sim c(u) p_u^{-(n-1)}$

  Quasi-Powers Theorem applies!
Applications:

- Integer compositions of all parts
- Subtrees aka preferential arrangements.

- Any structure "driven" by a sequence in a supercritical way (i.e., $G(r) > 1$).

For a large collection of combinatorial classes & parameters, we have a functional equation

$$\Phi(z, y, u) = 0$$

In the counting case ($u=1$), get a singular expansion

$$y(z, u) = \cdots (1 - z/p(x))^\alpha + \cdots$$

A perturbation of $u$ near 1 will often induce a direct perturbation of the expansion of $y(z, u)$, e.g.,

movable singularity

$$y(z, u) = \cdots (1 - z/p(x))^\alpha + \cdots$$

movable exponent

$$y(z, u) = \cdots (1 - z/p(x))^{a(u)} + \cdots$$

with $p(x)$ or $a(x)$ analytic at 1

$$\Rightarrow$$ Asymptotic normality $\Rightarrow$ Quasi-Powers

Ornella Valletti's theorem

$$\Rightarrow$$ Asymptotic Normality

Local configurations in random structures are almost always normally distributed

- The path-m-graphs (aka automata) framework $\Rightarrow$ Singularity moves analytically

$$\frac{\Gamma(z_n)}{B(z_n)} \Rightarrow$$ (cf. also Markov chain theory)

e.g., # occurrences of a fixed pattern

in words

- For algebraic systems, e.g., sample families of trees and local parameters

(e.g., # leaves) $\Rightarrow$ Singularity moves

exponent $\leq \frac{1}{2}$
**Singularities and Random Combinatorial Structures**

**EXAMPLE 1** Polynomials over finite fields

- are a sequence of coefficients
  \[ \Rightarrow \text{GF has a pole} \]
  Coeffs grow like \( q^n \)

- are a set of irreducibles (primes)
  \[ \Rightarrow \text{GF has a LOG sing.} \]
  Coeffs grow like \( \frac{q^n}{n} \)

A Prime Number Theorem: The density of irreducibles is \( \frac{1}{n} \).

- Bivariate Analytic Schema
  \[ \exp(u \log) \]
  \[ \Rightarrow \text{Gaussian law} \]

An Erdős-Kac Theorem: The number of irreducibles is asymptotically Gaussian.