PART II

COMPLEX ASYMPTOTIC METHODS

ASYMPTOTICS

Some functions are too complicated to be expanded e.g.

\text{e.g.} \text{unlabelled, non-plane trees}

\[ U(z) = e^z \exp(\frac{1}{2} U(z^2) + \frac{1}{3} U(z^3) + \cdots) \]

"Universality" phenomena are not apparent

- schemas applying to wide classes
- limit laws shared by - - -
Principle: Assign to the variable $z$ complex values.

View a GF as a geometric transformation from $C \to C$.

Modulus of GF of balanced trees.

Singularities matter!

The $\gamma$-singularity of the OGF of binary trees.

Analytic functions: The EGF of $\text{Ur}(n, \gamma \cdot (e^z - 1))$. 
CHAPTER 4: BASIC COMPLEX ASYMPTOTICS

- Notion of an analytic (holomorphic) function
- Meromorphic function
- Residue theorem; coefficient theorem
- Singularities and exponential order

FUNDAMENTAL EQUIVALENCE THEOREM

There is equivalence between the following properties

- \( f(z) \) is analytic at all points \( z_0 \in \mathbb{D} \)
- \( f(z) \) is complex-differentiable at all points \( z_0 \in \mathbb{D} \)

Also, say that \( f(z) \) is "regular".

DEFINITION: \( f(z) \) is analytic at \( z_0 \) iff it admits a locally convergent series expansion

\[
f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n
\]

(Note: such expansions converge in discos !)

\( f(z) \) is complex-differentiable (holomorphic) at \( z_0 \) iff the limit exists:

\[
\lim_{h \to 0} \frac{f(z_0+h)-f(z_0)}{h} \quad \text{or} \quad \frac{df}{dz} \bigg|_{z_0} \ldots
\]

Let \( f(z) \) be defined from \( D \) to \( E \)

(\( \overline{D} \) = connected open set)
Analyticity (series expansion) is what interests combinatorialists; a priori.

Complex differentiability makes easier the development of the theory.

\[ \frac{\Delta f}{\Delta z} \] calculation, ...

Closure under \( \pm, \times, 1/ \) (if denom \( \neq 0 \))

Composition, inversion (cond. on derivatives \( \neq 0 \))

* The function \( \sqrt{z} \)

\[ \sqrt{e^{i\theta}} = \sqrt{e^{i\theta}} \]

can only be made continuous in \( \mathbb{C} \backslash (-\infty, 0) \)

\[ \log z, \log(e^{i\theta}) = \log(p + i\theta) \]

\[ \frac{e^z}{\sqrt{1-z}} \]

is regular in \( \mathbb{C} \)

\[ \frac{1 - \sqrt{1-z}}{2} \]

is regular in \( \mathbb{C} \backslash \{1 + i\infty\} \) etc.

Any polynomial \( z^2 + 19z + 5 \)

is regular in whole of \( \mathbb{C} \)

A rational function \( \frac{z^3}{(z-1)(z^2-2)} \)

is regular in \( \mathbb{C} \) minus poles

The exponential function is regular in \( \mathbb{C} \)

\[ \frac{e^{-z}}{1-z} \]

regular in \( \mathbb{C} \backslash \{1, 3\} \)
**INTEGRATION**

Define integration along curves \( \int_y f(z) \, dz \).

**FUNDAMENTAL INTEGRAL THEOREM**

Let \( f \) be analytic in \( D \), that is, \( f \) be contractible to a single point in \( D \). Then

\[
\int_D f(z) \, dz = 0
\]

\( \Rightarrow \int_A f(z) \, dz \) does not depend on path.

**DEFINITION:** \( g(z) \) is meromorphic in \( D \) iff

\[
\text{near any } z_0 \text{, one has } g(z) = \frac{A(z)}{B(z)} \text{ with } A(z), B(z) \text{ analytic at } z_0.
\]

A point \( z_0 \) such that \( B(z_0) = 0 \) while \( A(z_0) \neq 0 \) is called a pole. Its order is the multiplicity of the zero/root \( z_0 \) of \( B(z) \).

Pole of order \( m \): \( g(z) = \frac{c_m}{(z - z_0)^m} + \ldots + \frac{c_1}{2 - z_0} + c_0 + c_1(z - z_0) + \ldots \)

D is simply connected.

not a loop
**Cauchy Residue Theorem**

\( \text{Re} \left( \frac{1}{z-z_0} \right) \)

\[ \text{Res}(f(z))_{z=z_0} = \lim_{z \to z_0} (z-z_0)f(z) \]

\[ f(z) = \sum_{n \to m} c_n (z-z_0)^m \]

\[ \Rightarrow \text{Res}(f(z)) = c_{-1} \]

**Proof:** Termwise "local" integration

**Cauchy Coefficient Theorem**

\[ \text{Coeff} \left[ z^n \right] f(z) = \frac{1}{2\pi i} \oint_{|z|=r} f(z) \frac{dz}{z^{n+1}} \]

\[ \Rightarrow \text{Coeff} \left[ z^n \right] f(z) = \frac{1}{2\pi i} \oint_{|z|=1} f(z) \frac{dz}{z^{n+1}} \]

\[ \Rightarrow \text{Coeff} \left[ z^n \right] f(z) = \frac{1}{2\pi i} \oint_{|z|=1} f(z) \frac{dz}{z^{n+1}} \]

**Complex Analysis: Local & Global**

**Computing Integrals**

\[ \int_{-\infty}^{+\infty} \frac{dx}{1+x^4} = \lim_{R \to \infty} \int_{-R}^{R} \frac{dx}{1+x^4} \]

\[ = 2\pi \sum_{\omega \in \{ \exp(\pm i\pi/4), \exp(\pm 3\pi/4) \}} \frac{1}{\omega} \]

\[ = \frac{\pi}{\sqrt{2}} \]

**Estimating Coefficients**

\[ D_n = \frac{1}{2\pi i} \oint_{|z|=1} \frac{e^{-z}}{z^{n+1}} \]

\[ d_n := \text{Pr} \{ \text{derangement}/\binom{n}{n} \} \]

\[ d_n = \frac{1}{2\pi i} \oint_{|z|=1} \frac{e^{-z}}{1-z} \]

\[ \frac{dz}{z^{n+1}} \]

**Evaluate Instead on \( |z|=2 \)**

\[ J_n = \frac{1}{2\pi i} \oint_{|z|=2} \frac{e^{-z}}{1-z} \]

\[ \frac{dz}{z^{n+1}} = O(z^n) \]

\[ = \text{Re} \left( \oint_{|z|=2} \frac{dz}{z^{n+1}} \right) + \text{Re} \left( \oint_{|z|=0} \frac{dz}{z^{n+1}} \right) \]

\[ = d_n - e^{-1} \]

**Asymptotic Estimate**

\[ \Rightarrow d_n = e^{-1} + O(z^{-n}) \]
Singularities
- What they are
- Why they are important

\[ f(z) \text{ has a singularity at border point } \sigma \text{ iff } \]
\[ f(z) \notin \mathbb{C}, \sigma \in \mathbb{R}, \sigma \notin \mathbb{C}, \sigma \notin \mathbb{C}. \]

E.g.: \( f(\sigma) = \infty, f'(\sigma) = \infty, \text{ other causes} \)

**FACT**
A series \( f(z) = \sum f_n z^n \) always has at least one singularity on its circle of convergence but none inside.

**THEM:** Convergence radius = analyticity radius.

Help: **Pringsheim's Theorem**

Circle of convergence

\[ \sum f_n z^n \]

With nonnegative coefficients,

\[ R = \rho \]
Factor \( f(z) \) be analytic at 0 with radius of convergence exactly \( R \). Then for any \( \epsilon > 0 \):

- \( f_n \left( R - \epsilon \right)^n \to 0 \)
- \( f_n \left( R + \epsilon \right)^n \) is unbounded.

In other words, \( \limsup |f_n(z)|^{1/n} = R \).

The exponential order of growth of \( f_n \) is \( R \). \( f_n \approx R^n \).

Examples (singularities):

- \( \frac{e^{-z}}{1 - z} \) is singular at \( z = 1 \) [derangements]
- \( \frac{1}{\sqrt{1 - \frac{2z}{z}}} \) is regular at \( z = \frac{1}{2} \) [binary mappings]
- \( 1 - \sqrt{1 - \frac{z}{2}} \) is singular at \( z = \frac{1}{4} \) [Catalan trees]

- Words
  \[
  \frac{1}{1 - z} \quad \Rightarrow \quad W_n \sim 2^n
  \]

- Derangements
  \[
  \frac{e^{-z}}{1 - z} \quad \Rightarrow \quad D_n \approx \frac{D_n^n}{m^n}
  \]

- \((1,2)\) coverings
  \[
  \frac{1}{1 - (z + z^2)} \quad \Rightarrow \quad C_n \sim \phi^n \quad \phi = \frac{1 + \sqrt{5}}{2} \approx 1.61
  \]

- General trees
  \[
  \frac{1 - \sqrt{1 - 4z}}{2} \quad \Rightarrow \quad G_n \approx 4^n
  \]

In fact

\[
G_n = \frac{1}{n} \binom{2n - 2}{n - 1}
\]

\[
\sim \frac{4^{n+1}}{\sqrt{n}} \quad [\text{by Stirling}]
\]
Exercise: Unary-binary trees $U$

$U(z) = \frac{1 - 3 - \sqrt{1 - 2z^3 - z^6}}{2z}$

Show that $U_n \sim 3^n$.

**Asymptotic exponential order is computable automatically for positive functions constructions**

\[
\rho(f + g) = \min(\rho(f), \rho(g)) \\
\rho(f \cdot g) = \min(\rho(f), \rho(g)) \\
\rho\left(\frac{1}{1 - f}\right) = \min(\rho(f), \{\mathcal{B}\mid f(\mathcal{B}) = 1\}) \\
\rho(e^f) = \rho(f) \\
\rho(\log \frac{1}{1 - f}) = \sqrt{1/(1-f)}
\]

Recursive structures can be approached via Implicit function theorem.
CHAPTER 5

Rational and meromorphic function asymptotics.

Find subexponential factors

\[ f_n \propto R^{-n} \]

\[ f_n = \Theta(n) \cdot R^{-n} \]

where \( \Theta(n) \) is like \( n^\alpha \), \( \log n \), \( e^{\sqrt{n}} \), etc.

Theorem:

Each pole \( \zeta \) with multiplicity \( r \) contributes a term

\[ \zeta^{-n} P(n) \quad \text{with} \quad \deg(P) = r - 1. \]
Poles are arranged in order of increasing modulus. Dominant one matters exponentially.

\[ x_n \] \~ \text{exponential growth.} 

\[ x_j \]

**Problem:** Denumerants. How many integer solutions to \( \sum_{j=1}^{m} x_j a_j = n \) where \( \{a_j\} \) is fixed? Assume freely \( \gcd(\{a_j\}) = 1 \).

= Integer partitions with fixed summands
= **MONEY CHANGER**'s **PROBLEM**

\[ D = \text{Seq}(1^{S_1}) \times \text{Seq}(1^{S_2}) \times \cdots \times \text{Seq}(1^{S_m}) \]

\[ D(z) = \frac{1}{1-z^{S_1}} \times \frac{1}{1-z^{S_2}} \times \cdots \times \frac{1}{1-z^{S_m}} \]

**Denumerants** are ways of changing \( m \) to \( 1 \):

\[ D(z) = \frac{1}{(1-z^1)(1-z^2)(1-z^3)(1-z^6)} \]

\[ D(z) = \frac{1}{1-z^{S_1}} \times \frac{1}{1-z^{S_2}} \times \cdots \times \frac{1}{1-z^{S_m}} \]

Poles at various roots of unity with order \( \leq m \)

At \( z = 1 \) with order exactly \( m \)

\[ D_n \sim [z^n] D(z) \sim [z^n] \frac{1}{1-z^m} \frac{1}{m!} \]

\[ \sim \frac{n^{m-1}}{(m-1)!} \times \frac{1}{m!} \]

(Schur)

**Eq:** \( \# \{ \sum_{j} j x_j = m \} \sim \frac{n^m}{(m-1)! \cdot m!} \).
**Largest run in a random string**

\[
\begin{aligned}
\text{bbbbb} & \quad \text{ aabb } \quad \text{ aA } \quad \text{ abbb } \quad \text{ aAabbb } \\
\text{Seq}_{<m}(b) \times \text{Seq}_{<m}(a) \times \text{Seq}_{<m}(b) & \\
\frac{1-z^m}{1-z} \times \frac{1}{1-z} \times \frac{1}{1-z} & = \frac{1-z^m}{1-2z+2z^2} \\
\end{aligned}
\]

- Dominant pole at \( p_m \approx \frac{1}{2} \)

\[
p_m \approx \frac{1}{2} \left( 1 + \left( \frac{1}{2} \right)^{m+1} \right)
\]

- Check error from dominant pole is good

\[
\Pr(\text{Longest run} \leq m) \approx \left( \frac{3}{2} p_m \right)^n \approx e^{-n/2^{m+1}}
\]

- Coefficients of meromorphic FNS.

**Assumption:** \( g(z) \) is meromorphic in \( |z| < R \) and analytic on \( |z| = R \)

- Pole

**Theorem**

Each pole \( \xi \) with multiplicity \( r \) contributes a term

\[
\xi^{-n} P(n) \quad \text{with} \quad \deg(P) = r-1
\]

AND ERROR TERM \( O(R^{-n}) \)

**Proof:**

- Let \( h(z) \) gather contributions of poles.

\[
|g(z) - h(z)|
\]

Then \( |g(z) - h(z)| \) is analytic in \( |z| < R \)

Cauchy coeff. formula + trivial bounds

- Estimate \( \frac{1}{2\pi i} \int g \) on \( |z| = R \) by residues.
A WORKED OUT EXAMPLE

DERANGEMENTS

\[ D = \text{Set (Cycle } (\mathbb{Z}, \text{card } > 2)\text{)} \]

\[ D(z) = \exp \left( \log \frac{1}{1-z} - z \right) \]

\[ D(z) = \frac{e^{-z}}{1-z} \]

\[ D(z) \sim \frac{e^{-1}}{1-z} \text{ at singularity } z = 1 \]

\[ \Rightarrow \frac{D_n}{n!} \sim \left[ z^n \right] \frac{e^{-1}}{1-z} = e^{-1} \]

**Prop.** A perm is a derangement with probability \( e^{-1} = 0.367 \ldots \)

**Generalized derangements**

\[ D^*(z) = \frac{e^{-z} - \frac{z^2}{2}}{1-z} \]

\[ \frac{D_n^*}{n!} \sim e^{-3/2} \]

in general, get

\[ \text{prob. } \sim e^{-\lambda x} \text{ of all cycles of length } > \lambda \]

**Theorem:** Any «path in graph» model *finite automaton*

leads to a rational generating function

that has a unique dominant pole, no \( f_n \sim c \cdot p^n \)

provided [Perron Frobenius theory]

- graph is strongly connected.

- aperiodicity condition \( \exists N \geq N_0 \text{ such that } \exists \text{ paths of length } n \text{ from source to destination} \)

- only containing a fixed pattern (all)
APPLICATION: Supercritical schema (sequences)

Assume \( \mathcal{F} = \text{Seq}(G) \Rightarrow F(z) = \frac{1}{1 - G(z)} \)

- \( H_1: G(z) \) reaches 1 before it becomes singular
- \( H_2: \) the schema is aperiodic: \( F_n > 0 \) for all \( n \geq n_0 \)

Then \( F_n \sim C \cdot n^{-n} \) \( p = G^{(\ast)}(z) \)

Also: number of \( G \)-components in random \( G \)-structure

mean \( \sim \ln(n) \); variance \( \sim \sigma^2 n \Rightarrow \text{concentration} \)

Preferential arrangements / selections

\( R = \text{Seq}(\text{Set}_2(z)) \) \( R(z) = \frac{1}{1 - (e^z - 1)} = \frac{1}{z - e^z} \)

Pole at \( z = \log 2 \); others at \( \log 2 \pm 2 \pi i, \log 2 \pm 3 \pi i, ... \)

\( \frac{R_n}{n!} \sim C \cdot (\log 2)^{-n} \) \( (C = \frac{1}{2 \cdot \log 2}) \)