COUNTING WITH $AB-BA=I$

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“COMBINATORIAL MODELS OF ANNIHILATION-CREATION”; IN PREP. (2010)
From Quantum Physics (QED):
- \( A = \text{Annihilate} \) a “random” particle
- \( B = \text{give Birth} \) to a new particle

\[ A = + + \]

\[ B = + + \]
Annihilation and Creation

- **A** = Annihilate a “random” particle
- **B** = give Birth to a new particle

\[ AB\mathcal{F} = A(\Box\mathcal{F}) = (A\Box)\cdot\mathcal{F} + \Box\cdot(A\mathcal{F}) \]

\[ BA\mathcal{F} = \Box\cdot(A\mathcal{F}) \]

\[ (AB - BA)\mathcal{F} \cong \mathcal{F} \]

Theorem (Partial commutation)

**A** and **B** must satisfy:

\[ AB - BA = 1 \]
Can be viewed as a rewrite rule: \[ AB \leftrightarrow 1 + BA : \]

\[
\begin{align*}
B \rightarrow A & \quad \rightarrow B \\
BAB & \quad \rightarrow \quad BA \\
BB & \quad \rightarrow \quad BB \\
\end{align*}
\]

[Algebra: Polynomials/Ideal \( \mathbb{C}\langle A, B \rangle/(AB - BA - 1) \).]

Leads to Normal form: \( \mathcal{N}(f) \), with all \( B \)'s before all \( A \)s.

**Lemma**

The **differential interpretation** \( A \leftrightarrow D, \ B \leftrightarrow X \) is **faithful**.

\[
X \cdot f(x) := xf(x), \quad D \cdot f(x) := \frac{d}{dx} f(x).
\]

\[
DX f - XD f = f
\]

Agrees with classical combinatorial analysis of \( X, D \).

\[
D \equiv \text{“choose and delete”;} \quad X \equiv \text{“append atom”}.
\]
1. Gates and Diagrams

~~A first combinatorial model~~

[Diagram of an electronic circuit]
A gate of type \((r, s)\) is a one-vertex digraph with \(r\) incoming legs (edges) and \(s\) outgoing legs. Legs are ordered.

A diagram is an acyclic assembly of gates with interconnections. A labelled diagram has vertices that bear integer labels.

An increasing diagram is labelled and such that labels are increasing along directed paths.
A diagram has **basis** $\mathcal{H} \subset \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ iff all the gates that it comprises have type in $\mathcal{H}$.

Let $\mathcal{C}[\mathcal{H}]$ be the class of (increasing) diagrams with basis $\mathcal{H}$; size is number of vertices.

**Theorem (Błasiak, Penson, et al., 2000++)**

Consider the (normalized) operator $\mathfrak{h} = \sum_{(r,s) \in \mathcal{H}} w_{r,s} X^r D^s$. Then the normal form of $\mathfrak{h}^n$ is given by

$$\mathcal{N}(\mathfrak{h}^n) = \sum_{a,b} c_{n,a,b} X^a D^b,$$

where $c_{n,a,b} := \# \{ \text{diagrams in } \mathcal{C}[\mathcal{H}] \text{ of size } n, \text{ with } a \text{ inputs and } b \text{ outputs} \}$. 
Example: \((X + D)^2\)

\[
(X + D)^2 = XX + DD + XD + DX
\]

\[\mathcal{H}[(X + D)^2] = X^2 + D^2 + 2XD + 1.\]

1, 2, 5, 14, ... ?????
Example: \((X + D)^2\)

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\[\mathcal{H}[(X + D)^2] = X^2 + D^2 + 2XD + 1.\]

OEIS #5425: 1, 2, 5, 14, 43, 142, 499, 1850,...
monomial $X^r D^s$ $\leftrightarrow$ gate
polynomial $\mathfrak{h}$ in $X, D$ $\leftrightarrow$ weighted basis $\mathcal{H}$ of gates
$\mathfrak{h}^n$ $\leftrightarrow$ labelled diagrams of size $n$ on $\mathcal{H}$
$e^{z \mathfrak{h}}$ $\leftrightarrow$ generating function of all diagrams
$(z, X, D)$ $\leftrightarrow$ (size, \#outputs, \#inputs).

- The exponential generating function (EGF) of all diagrams

  with basis $\mathcal{H}$ is $e^{z \mathfrak{h}}$, with

  $$e^{z \Gamma} := \sum \frac{\Gamma^n z^n}{n!}.$$
Proof of Theorem

- Conceptually(!) for combinatorialists: symbolic methods, theory of species, ... 

\[ D \equiv \text{"choose and delete"}; \quad X \equiv \text{"append atom"}. \]

Otherwise, by recurrence on number of gates, based on

\[
(X^r D^s)(X^a D^b) = \sum_{t=0}^{s} \binom{s}{t} \binom{a}{t} t! X^{r+a-t} D^{s+b-t}.
\]

The coefficient is also \# ways of attaching a new gate to an already reduced diagram.

Cf also: Viennot-Leroux, ...
2. The (X+D) forms

Involutions & such...
The form \((X + D)\) and involutions

\[
\begin{align*}
\mathcal{N}[(X + D^1)] &= X + D \\
\mathcal{N}[(X + D)^2] &= X^2 + 2XD + D^2 + 1 \\
\mathcal{N}[(X + D)^3] &= \cdots 14 \text{ terms} \cdots \\
\end{align*}
\]

2 terms (instead of 2)

5 terms (instead of 4)

14 terms (instead of 8)

- **Involutions** \((\sigma^2 = \text{Id})\) with bicoloured singetons:

\[
\exp(2z + z^2/2) = 1 + 2 \frac{z}{1!} + 5 \frac{z^2}{2} + 14 \frac{z^3}{3!} + 43 \frac{z^4}{4!} + \cdots.
\]

- **Normal forms of** \((\alpha X + \beta D)\):

**Proposition (Linear forms and involutions)**

\[
\mathcal{N}((\alpha X + \beta D)^n) = \sum_{\ell,m} \frac{n!}{2^{(n-\ell-m)/2}((n-\ell-m)/2)!\ell!m!} \alpha^{n-m} \beta^{n-\ell} X^\ell D^m.
\]
Evolution equations

### Definition
An equation for $F \equiv F(x, t)$, such as

$$\frac{\partial}{\partial t} F = \Gamma \cdot F, \quad F(x, 0) = f(x),$$

is known as an **evolution equation** (with initial value, or Cauchy, conditions), based on the “spatial” operator $\Gamma \in \mathbb{C} [x, \frac{\partial}{\partial x}]$.

### Proposition

**Evolution equation is formally solved by** $F(x, t) = e^{t\Gamma} \cdot f$

**Solving** $\frac{\partial}{\partial t} F = \frac{\partial}{\partial x} F + xF$

- Use **normal form** $F = e^{t(X+D)} = e^{t^2/2} \cdot e^{tx} \cdot e^{tD} \cdot f$.
- **Taylor** “means” $e^{tD} \cdot f(x) = f(x + t)$.
- Conclude:

$$F(x, t) = e^{t^2/2+xt}f(x + t).$$
Extensions to \((X^2 + D), (X + D^2), \ldots\)

The model is that of a **COMB**.

\[
e^{z^3/3 + z^2 + z}
\]

\[
\mathcal{M} \left[ e^{z(D^2 + X)} \right] = e^{z^3/3 + zX} \cdot e^{zD^2 + z^2 D}.
\]

- Generalization to \((a(X) + D)\) or \((a(D) + X)\);
- **PDE:** solve \(\frac{\partial}{\partial t} F = \frac{\partial}{\partial x} F + a(x)F\).

[cf method of characteristics; heat kernel...]

Monday, January 25, 2010
3. The (XD)- forms

Set partitions & such...
The form \((XD)\) and set partitions

- Well-known connections in analysis, difference calculus, and combinatorics(!)

\[ G = \text{SET}(u \text{SET}_{\geq 1}(\mathbb{Z})) \quad \implies \quad G(z, u) = e^{u(e^z - 1)}. \]

\[ \mathcal{N} \left[ e^{z(XD)} \right] = \sum_{k \geq 0} \frac{1}{k!} (e^z - 1)^k X^k D^k. \]

\[ \sim \quad \text{PDE} \quad \frac{\partial}{\partial t} F = x \frac{\partial}{\partial x} F \quad \text{is solved by} \quad F(x, t) = f(xe^t). \]
The forms \((X^2D^2)\) and such, after Blasiak, Penson et al.

\[
\begin{align*}
(X^2D^2)^2 &= X^2D^2 & & 1 \\
(X^2D^2)^3 &= 2X^2D^2 + 4X^3D^3 + X^4D^4 & & 7 \\
(X^2D^2)^n &= \cdots \text{87 terms} \cdots & & 87
\end{align*}
\]

![Diagram](image)

bi-labelled structures:

\[
e^u G(z, x) = \sum_k (k(k - 1))^n \frac{x^k z^n}{k! n!}
\]

\[
\begin{align*}
\{\begin{array}{c}
k \end{array}\}^{2,2}_n &= \frac{1}{k!} \sum_j (-1)^{k-j} \binom{k}{j} (k(k - 1))^n; \\
\omega^{(2,2)}_n &= e^{-1} \sum_{\ell} \frac{(\ell(\ell - 1))^n}{\ell!}.
\end{align*}
\]

- Count matrices with two ones per line, no null column.
- Coupon collector with group drawings [DuFlRoTa]:

\[
\mathbb{E}_m(T) = \frac{m(m - 1)}{2m - 1} \left[ H_m + 12m - 1 - \frac{(-1)^m}{(m + 1)(\binom{2m-1}{m+1})} \right] \sim \frac{1}{2} \log m.
\]

- Also set partitions with constrained contiguities.
4. The \((X^2+D^2)\)-form

Permutations & such
The “circle” form \((X^2 + D^2)\)

- Get two types of gates: CUPS \((X^2)\) and CAPS \((D^2)\)

- These assemble into chains that are either open or closed.
- As we go along a chain, label values alternate.
- There are symmetry factors since we enter from left or right.

Alternating (zigzag) perms have EGF \(\tan(z), \sec(z)\) [André 1881]
Alternating cycles have EGF \(\log \cos(z)\).
We must take \textbf{Sets} of these.

\[
G(x, y, z) = \frac{1}{\sqrt{\cos(2z)}} \exp \left( \left( \frac{x^2}{2} + \frac{y^2}{2} \right) \tan(2z) + xy(\sec(2z) - 1) \right).
\]
The general quadratic form \((\alpha X^2 + \beta D^2 + \gamma XD + \delta DX)\)

**Principle:** similarly follow the spaghetti!
— Get the famous peaks, troughs, double rises, double falls.
— Cf: Carlitz; Françon–Viennot, \ldots

- Use **tree decomposition** of perms:

- Make use of “max-rooting \(\xrightarrow{\sim} \int\)”.

\[
A = (B^\Box \ast C) \implies A(z) = \int_0^z (\partial_t B(t)) \cdot C(t) \, dt
\]

Contrast with Lie algebra or *ad hoc* computations.

Ordering of the exponential of a quadratic in boson operators. I. Single mode case

C. L. Mehta
5. The tree form ($X^{\sim 2D}$) ... and trees
The special form \((X^2D)\)

- The unique gate is a ‘\(Y\)’.

- Get all permutations as connected components.
- Take Sets of these.
- GFs are variants of \(\exp\left(\frac{z}{1 - z}\right)\).

\[
\mathcal{N} \left[ e^{zXD^2} \right] = \exp \left( z \frac{X^2}{1 - zX} D \right).
\]
The semilinear form $\phi(X)D$

Gives rise to **increasing trees**, with $\phi(X)$ the basic constructor.

- Case $(XD)$: threads (unary trees) $\leadsto$ set partitions.
- Case $(X^2D)$: binary trees.
- Case $(X^rD)$: $r$–ary trees, ...

**Proposition**

The EGF of increasing trees with “rule” $\phi$ is

$$T(z) = \text{Inv} \int_0^t \frac{dw}{\phi(w)}.$$ 

Some exactly solvable models

[Bergeron-Fl-Salvy, 1992]
Example: $XD^r$, after Blasiak, Penson, Solomon

\[ G(x, y, z) = \exp \left( \frac{xy}{(1 - \rho x^\rho z)^{1/\rho}} - xy \right), \quad \rho := r - 1. \]

- For $r = 3$, get $\exp \left( \frac{1}{\sqrt{1 - 2z}} - 1 \right)$, which is evocative of binary trees(?).
- Explicit binomial sums are available for $\mathcal{M}$ of powers of $XD^r$. 
6. The \((X^a+D^a)\) forms

Dyck path, histories, & such
In the **canonical basis** \((x^k)\), \(X\) and \(D\) become matrices:

\[
D = \begin{bmatrix}
0 & 1 & \cdots & \cdots & \cdots \\
\cdot & 0 & 2 & \cdots & \cdots \\
\cdot & \cdot & 0 & 3 & \cdots \\
\cdot & \cdot & \cdot & 0 & 4 \\
\cdot & \cdot & \cdot & \cdot & \cdot
\end{bmatrix}, \quad X = \begin{bmatrix}
0 & \cdots & \cdots & \cdots & \cdots \\
1 & 0 & \cdots & \cdots & \cdots \\
\cdot & 1 & 0 & \cdots & \cdots \\
\cdot & \cdot & 1 & 0 & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdot
\end{bmatrix}.
\]

Schematically, effect on basis \((x^k)\) is described by the “Weyl graph”:

**Proposition**

Consider a monomial \(\mathfrak{f}\) in \(X, D\). The constant term (of its normal form) is nonzero if and only if the associated path \(\pi(\mathfrak{f})\) in the Weyl graph, starting from vertex 0, returns to vertex 0. In that case, this constant term is equal to the multiplicative weight of the path \(\pi(\mathfrak{f})\). 

Monday, January 25, 2010
Constant Terms (C.T.) = weighted Dyck paths

**Theorem F-1980]:**

\[
F(a, d, \ell) = \frac{1}{1 - \ell_0 - \frac{a_0 d_1}{1 - \ell_1 - \frac{a_1 d_2}{1 - \ell_2 - \frac{a_2 d_3}{\ddots}}}}.
\]

Dyck paths

- **level steps**
- **ascents**
- **descents**
Proposition 2. The normal ordering of \((X + D)\) corresponds to the continued fraction expansion

\[
\text{C.T.} \left( \frac{1}{1 - z(X + D)} \right) = \sum_{n \geq 0} \frac{1 \cdot 3 \cdots (2n - 1)}{1 \cdot z^2} \cdot \frac{1 \cdot 2 \cdots (2n - 2)}{1 \cdot z^2} \cdots = \text{C.T.} \left( e^{z(X+D)} \right).
\]
Laplace transform \((L)\)

\((X^2 + D^2)\)

\[ \mathcal{L} \left[ \frac{1}{\sqrt{\cos(2z)}} \right] = \frac{1}{1 - \frac{1 \cdot 2 \cdot z^2}{1 - \frac{1 \cdot 2 \cdot 3 \cdot z^2}{1 - \frac{3 \cdot 4 \cdot z^2}{1 - \frac{5 \cdot 6 \cdot z^2}{\ldots}}}} } \]

\[ \Phi_3(z) = \frac{1}{1 - \frac{1 \cdot 2 \cdot 3 \cdot z^2}{1 - \frac{4 \cdot 5 \cdot 6 \cdot z^2}{1 - \frac{7 \cdot 8 \cdot 9 \cdot z^2}{\ldots}}}} = 1 + 6z^2 + 756z^4 + 458136z^6 + 76534113z^8 + \ldots \]

**Identify?** Cf Dixonian elliptic functions; Apéry’s \(z(3)\)
Openings

Partial difference equations, q-analogues,...
• Relation with PDEs?
  Eg. *Duchon’s Clubs* and the *cubic oscillator*.

• *Difference equations* and q-analogues

• Relation with *Rook Polynomials* [Varvak]...

• Relation with various *tableaux*? exclusion...?
  *Cf* Viennot, Corteel, Josuat-Verges, ...