COUNTING WITH $AB-BA=1$

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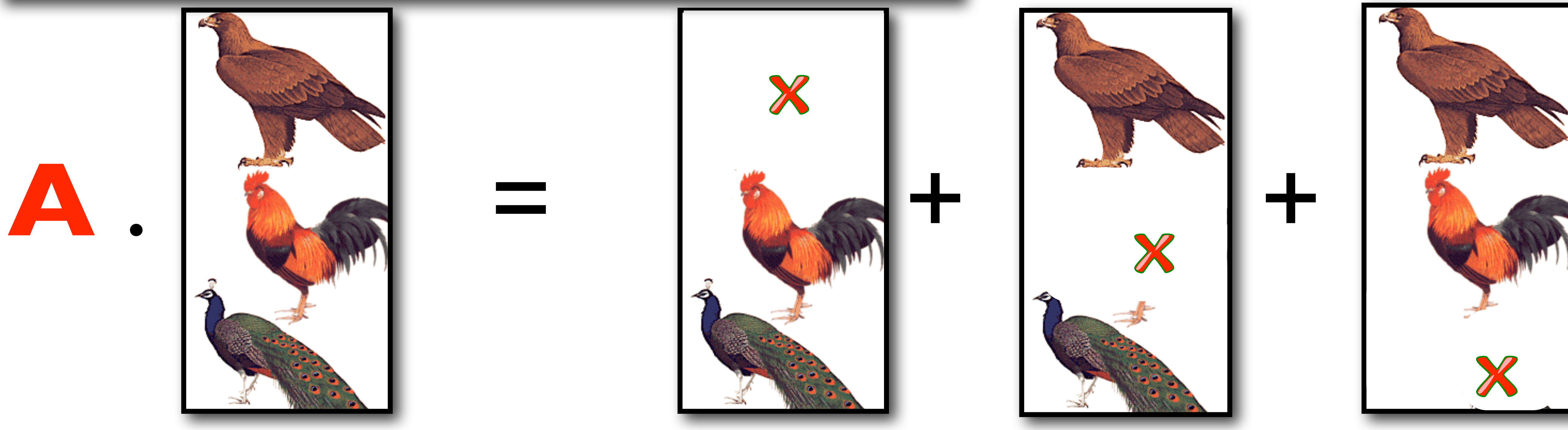
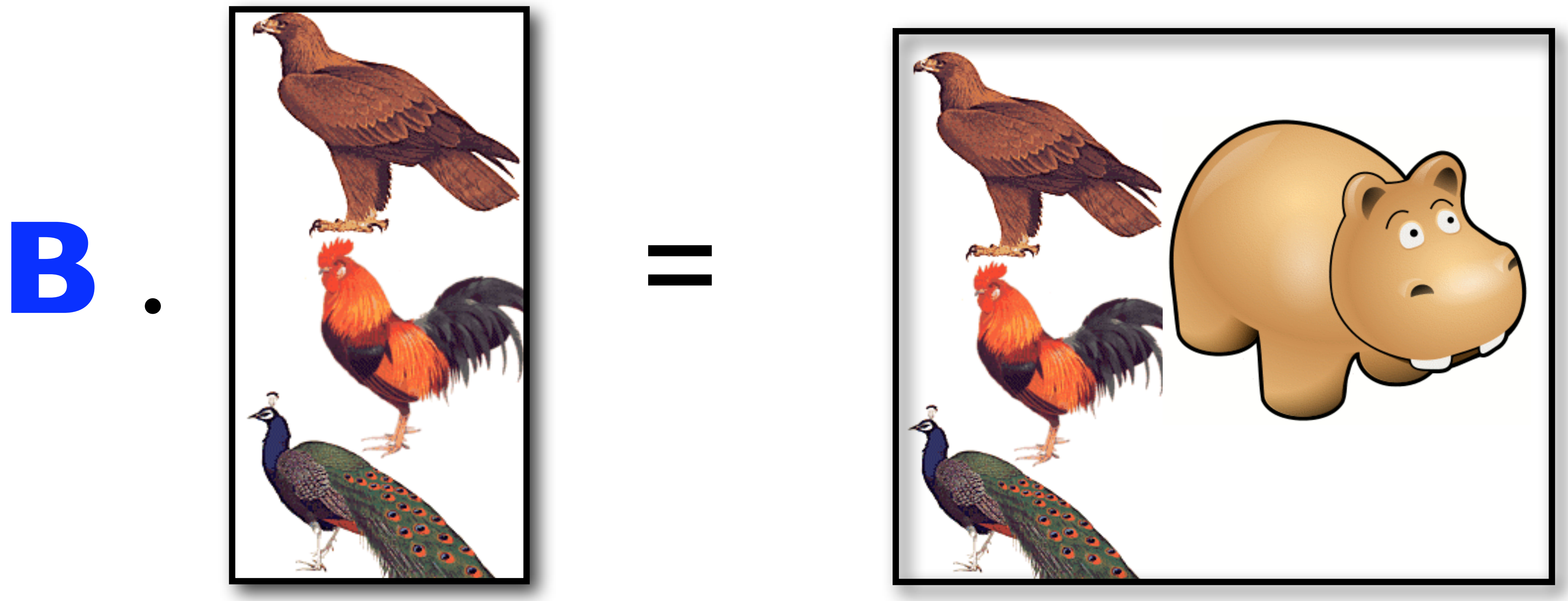
"COMBINATORIAL MODELS OF ANNIHILATION-CREATION"; IN PREP. (2010)

From Quantum Physics (QED):

— **A** = **A**nnihilate a "random" particle

— **B** = give **B**irth to a new particle





Annihilation and Creation

— **A** = **A**nnihilate a "random" particle
— **B** = give **B**irth to a new particle ⟶ $\square$

$$\begin{aligned} \mathbf{AB}\mathcal{F} &= \mathbf{A}(\square\mathcal{F}) = (\mathbf{A}\square)\cdot\mathcal{F} + \square\cdot(\mathbf{A}\mathcal{F}) \\ - \quad \mathbf{BA}\mathcal{F} &= \square\cdot(\mathbf{A}\mathcal{F}) \end{aligned}$$

$$(\mathbf{AB}-\mathbf{BA})\mathcal{F} \cong \mathcal{F}$$

Theorem (Partial commutation)

A *and* **B** *must satisfy:*

$$\boxed{\mathbf{AB}-\mathbf{BA}=\mathbf{1}}$$

Annihilation and Creation: $\mathbf{AB - BA = 1}$

- Can be viewed as a rewrite rule: $\boxed{\mathbf{AB \mapsto 1 + BA}}$:

$$\mathbf{B\overline{AB}BA}$$

$$\mathbf{BB\check{\ }A} \swarrow \qquad \searrow \mathbf{BBABA}$$

[Algebra: Polynomials/Ideal $\mathbb{C}\langle A, B\rangle/(AB - BA - 1)$.]

- Leads to $\mathfrak{N}$ormal form: $\mathfrak{N}(f)$, with all **B**'s before all **A**s.

Lemma

The differential interpretation $\mathbf{A} \mapsto D$, $\mathbf{B} \mapsto X$ *is faithful.*

$$X \cdot f(x) := xf(x), \qquad D \cdot f(x) := \frac{d}{dx} f(x).$$

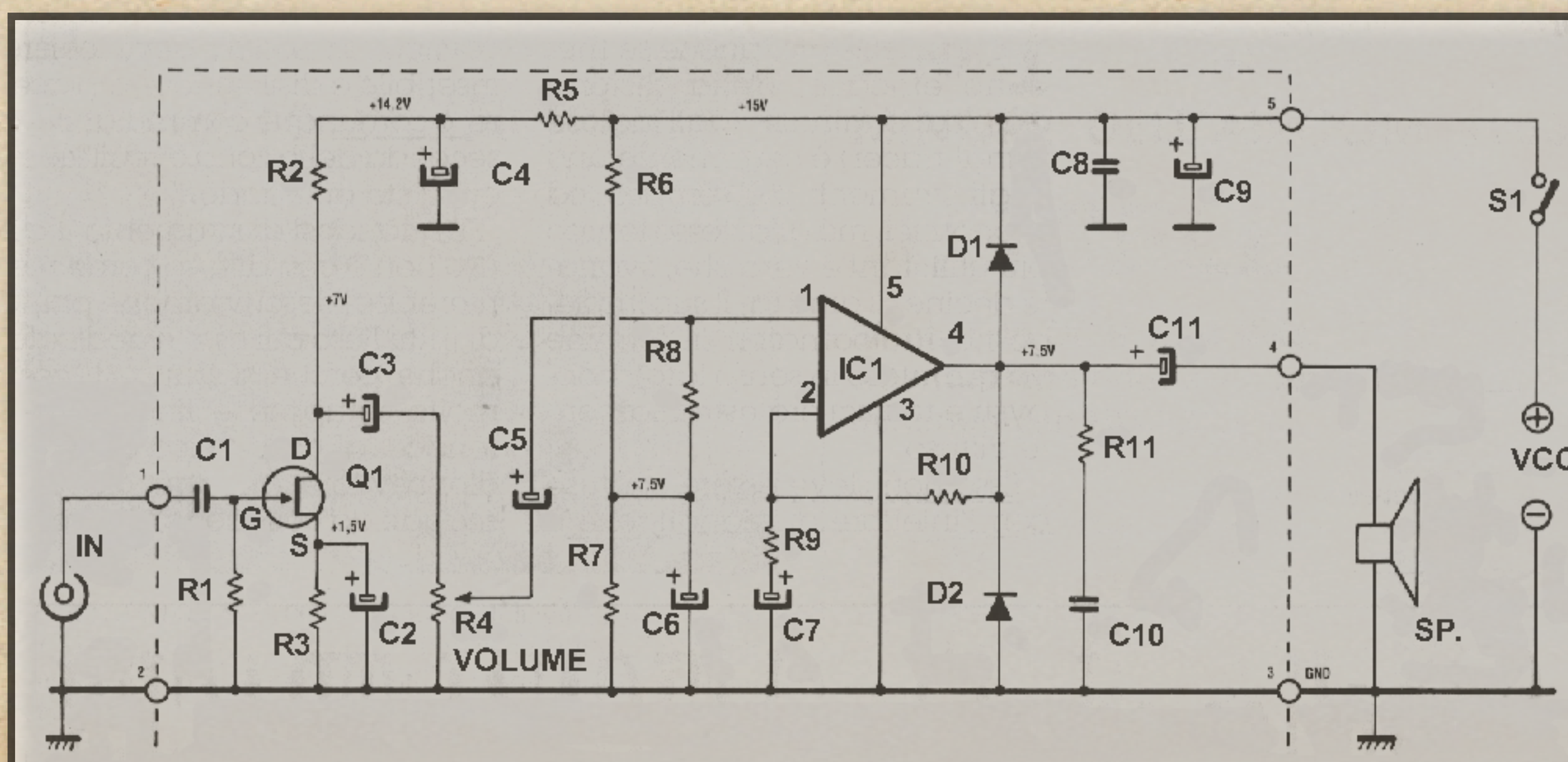
$$DX\,f - XD\,f = f$$

- Agrees with classical combinatorial analysis of X, D.

$D \equiv$ "choose and delete"; $\qquad X \equiv$ "append atom".

1. Gates and Diagrams

~~A first combinatorial model~~

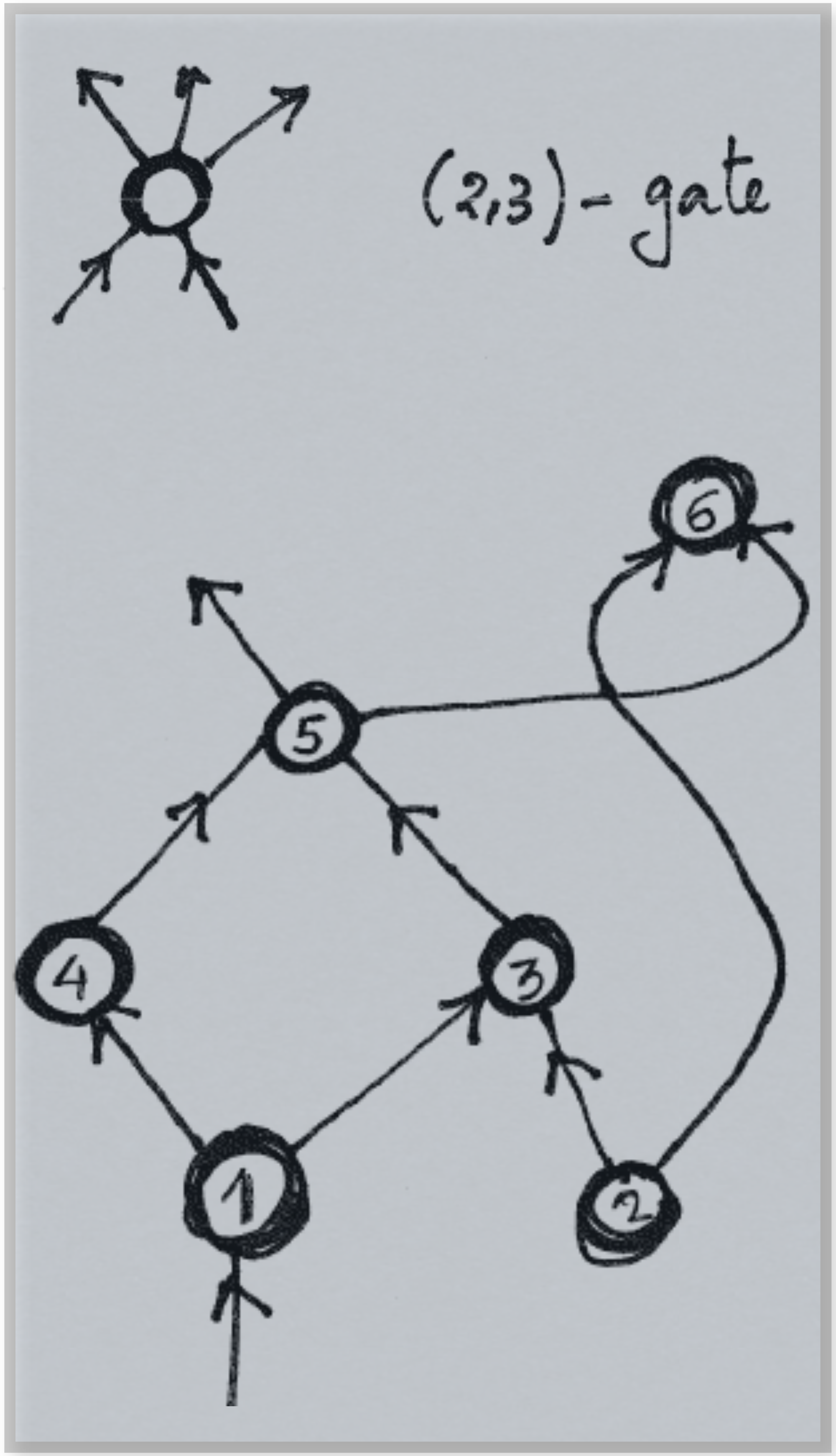


Gates and Diagrams (def.)

- A **gate** of type (r, s) is a one-vertex digraph with r incoming legs (edges) and s outgoing legs. Legs are *ordered*.

- A **diagram** is an acyclic assembly of gates with interconnections. A labelled diagram has vertices that bear integer labels.

- An **increasing diagram** is labelled and such that labels are increasing along directed paths.



• A diagram has **basis** $\mathcal{H} \subset \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ iff all the gates that it comprises have type in $\mathcal{H}$.

• Let $\mathcal{C}[\mathcal{H}]$ be the class of (increasing) diagrams with basis $\mathcal{H}$; size is number of vertices.

Theorem (Błasiak, Penson, *et al.*, 2000++)

Consider the (normalized) operator $\mathfrak{h} = \sum_{(r,s)\in\mathcal{H}} w_{r,s} X^r D^s$. Then the normal form of $\mathfrak{h}^n$ is given by

$$\mathfrak{N}(\mathfrak{h}^n) = \sum_{a,b} c_{n,a,b} X^a D^b,$$

where $c_{n,a,b} :=$
$\#\{$ diagrams in $\mathcal{C}[\mathcal{H}]$ of size $\underline{n}$, with $\underline{a}$ inputs and $\underline{b}$ outputs $\}$.

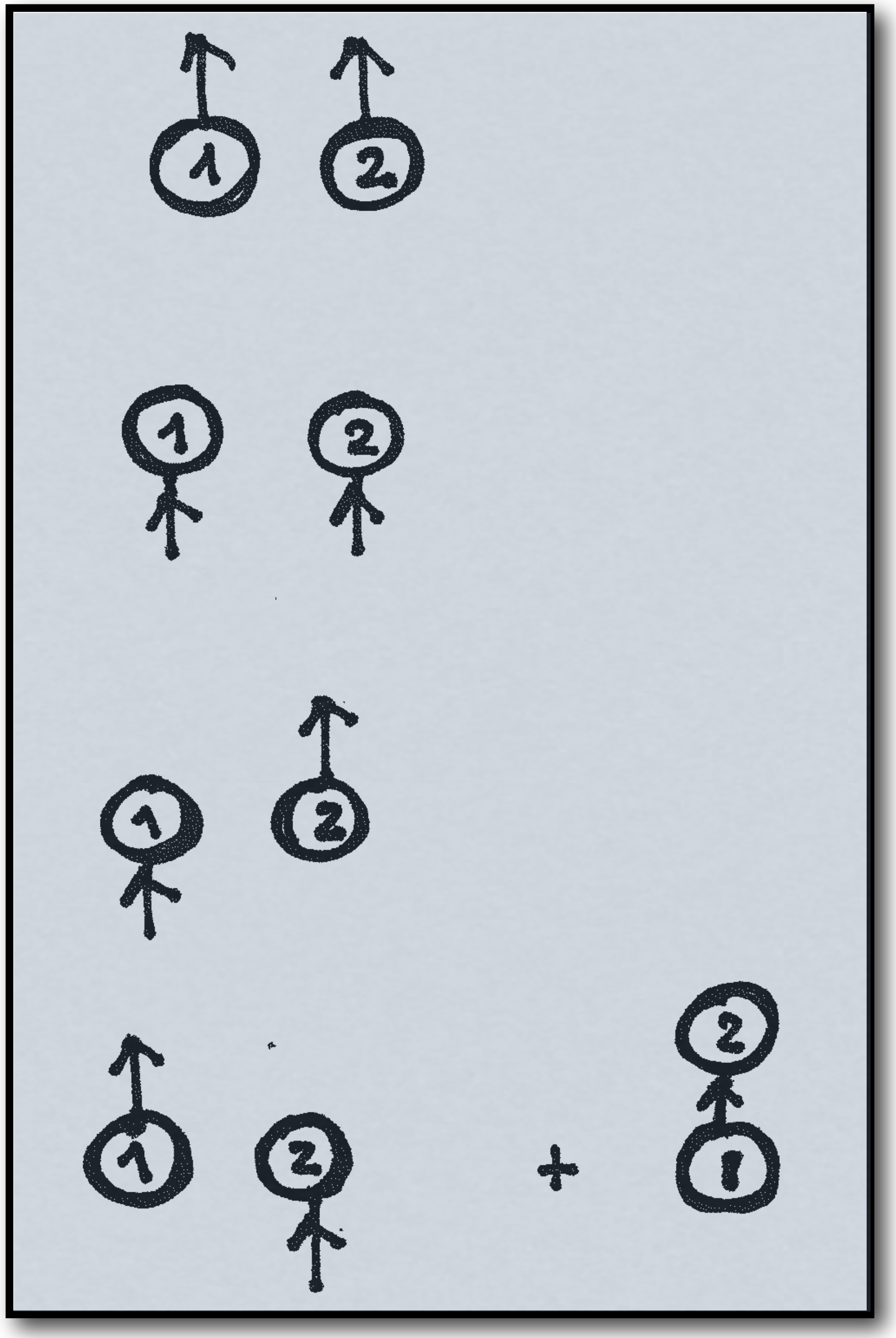
Example: $(X + D)^2$

$$(X + D)^2 = XX$$

$$+ DD$$

$$+ XD$$

$$+ DX$$

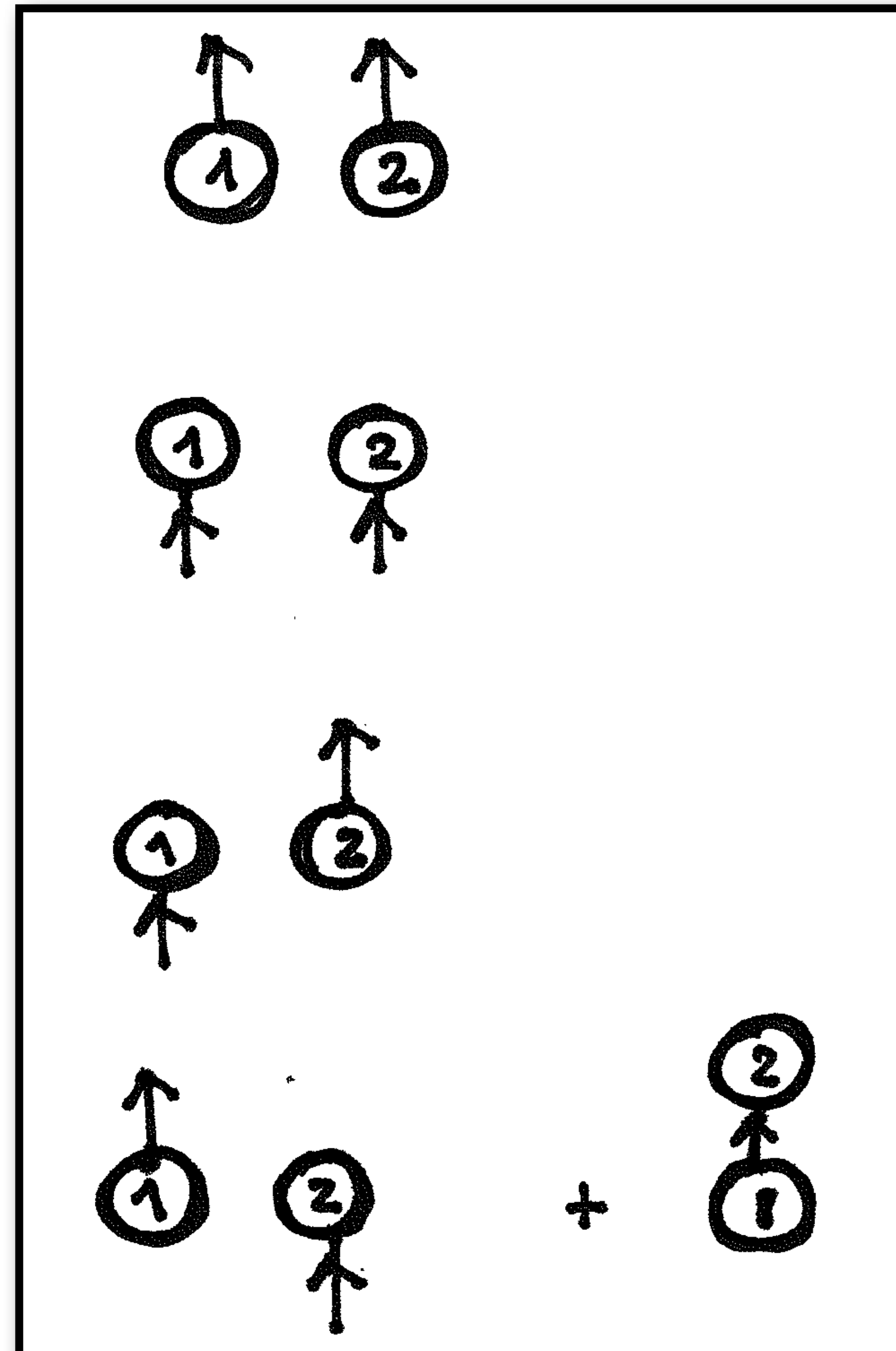


$$\mathfrak{N}\left[(X + D)^2\right] = X^2 + D^2 + 2XD + 1.$$

1, 2, 5, 14, ... ?????

Example: $(X+D)^2$

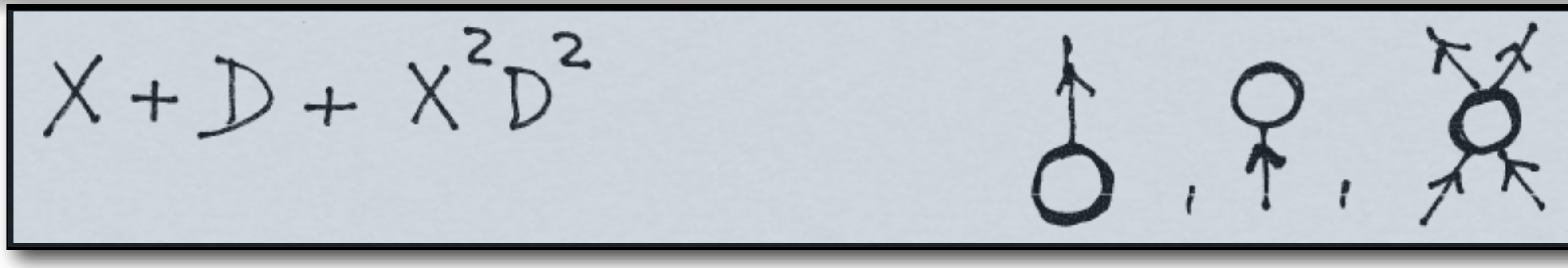
$$(X+D)^2 = XX + DD + XD + DX$$



$$\mathfrak{N}\left[(X+D)^2\right] = X^2 + D^2 + 2XD + 1.$$

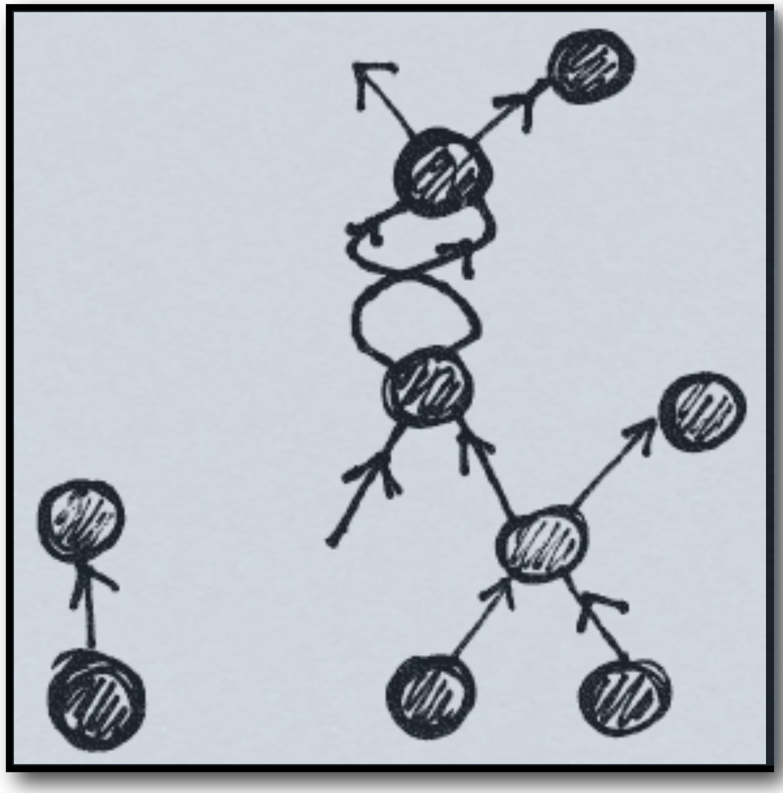
OEIS #5425: 1, 2, 5, 14, 43, 142, 499, 1850, ...

From algebra to combinatorics: a dictionary



monomial $X^r D^s$	$\leftrightsquigarrow$	gate
polynomial $\mathfrak{h}$ in X, D	$\leftrightsquigarrow$	weighted basis $\mathcal{H}$ of gates
$\mathfrak{h}^n$	$\leftrightsquigarrow$	labelled diagrams of size n on $\mathcal{H}$
$e^{z\mathfrak{h}}$	$\leftrightsquigarrow$	generating function of all diagrams
(z, X, D)	$\leftrightsquigarrow$	(size, #outputs, #inputs).

- The exponential generating function (EGF) of all diagrams with basis $\mathcal{H}$ is $\boxed{e^{z\mathfrak{h}}}$, with $\boxed{e^{z\Gamma} := \sum \Gamma^n \frac{z^n}{n!}}$.



Proof of Theorem

- Conceptually(!) for combinatorialists: symbolic methods, theory of species, ...

D ≡ "choose and delete"; **X** ≡ "append atom".

- Otherwise, by recurrence on number of gates, based on

$$(X^r D^s)(X^a D^b) = \sum_{t=0}^{s} \binom{s}{t} \binom{a}{t} t! X^{r+a-t} D^{s+b-t}.$$

The coefficient is also # ways of attaching a new gate to an already reduced diagram.

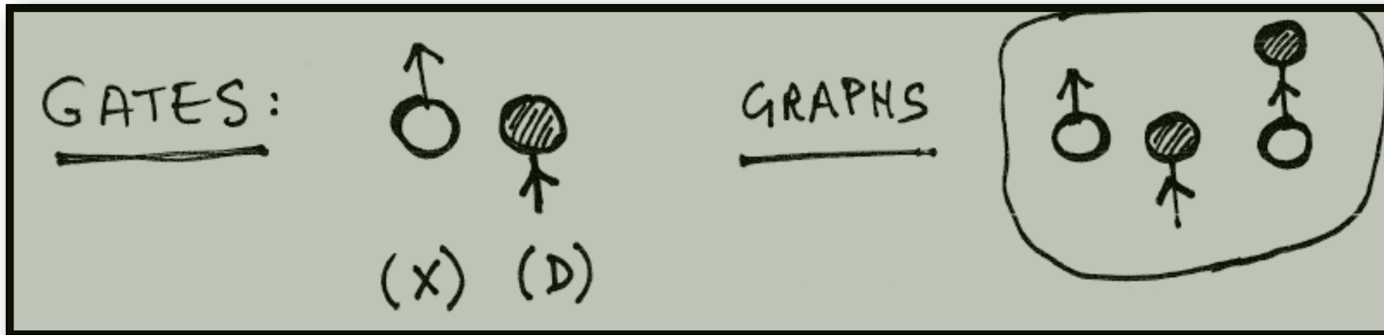
Cf also: Viennot-Leroux, ...

2. The (X+D) forms

Involutions & such...

The form $(X + D)$ and involutions

$\mathfrak{N}[(X+D^1)]$	$=$	$X + D$	**2** terms (instead of 2)
$\mathfrak{N}[(X+D)^2]$	$=$	$X^2 + 2XD + D^2 + 1$	**5** terms (instead of 4)
$\mathfrak{N}[(X+D)^3]$	$=$	$\cdots$ 14 terms $\cdots$	**14** terms (instead of 8)



- **Involutions ($\sigma^2 = \mathrm{Id}$)** with bicoloured singetons:

$$\exp(2z + z^2/2) = 1 + 2\frac{z}{1!} + 5\frac{z^2}{2} + 14\frac{z^3}{3!} + 43\frac{z^4}{4!} + \cdots .$$

- **Normal forms of $(\alpha X + \beta D)$:**

Proposition (Linear forms and involutions)

$$\mathfrak{N}((\alpha X + \beta D)^n) = \sum_{\ell,m} \frac{n!}{2^{(n-\ell-m)/2}((n-\ell-m)/2)!\ell!m!} \alpha^{n-m}\beta^{n-\ell}\mathbf{X}^\ell\mathbf{D}^\mathbf{m}.$$

Evolution equations

Definition

An equation for $F \equiv F(x,t)$, such as

$$\boxed{\frac{\partial}{\partial t} F = \Gamma \cdot F, \qquad F(x,0) = f(x),}$$

is known as an evolution equation (with initial value, or Cauchy, conditions), based on the "spatial" operator $\Gamma \in \mathbb{C}\left[x, \frac{\partial}{\partial x}\right]$.

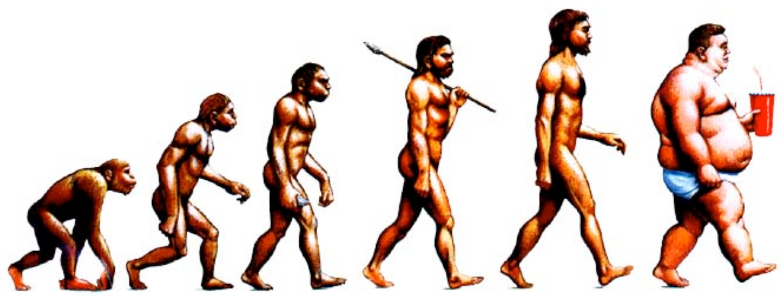
Proposition

Evolution equation is formally solved by $F(x,t) = e^{t\Gamma} \cdot f$

Solving $\frac{\partial}{\partial t} F = \frac{\partial}{\partial x} F + xF$

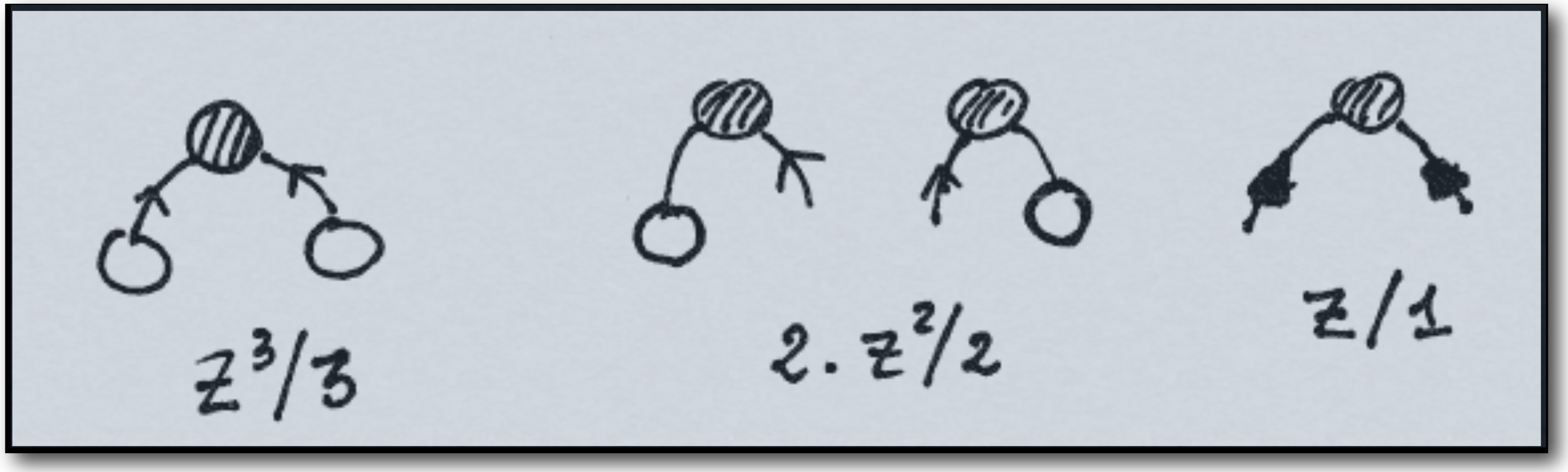
- Use **normal form** $F = e^{t(X+D)} = e^{t^2/2} \cdot e^{tx} \cdot e^{tD} \cdot f$.
- **Taylor** "means" $e^{tD} \cdot f(x) = f(x+t)$.
- Conclude:

$$F(x,t) = e^{t^2/2 + xt} f(x+t).$$



Extensions to $(X^2 + D)$, $(X + D^2)$, ...

The model is that of a **COMB**.



$$e^{z^3/3+z^2+z}$$

$$\mathfrak{N}\left[e^{z(D^2+X)}\right] = e^{z^3/3+zX} \cdot e^{zD^2+z^2D}.$$

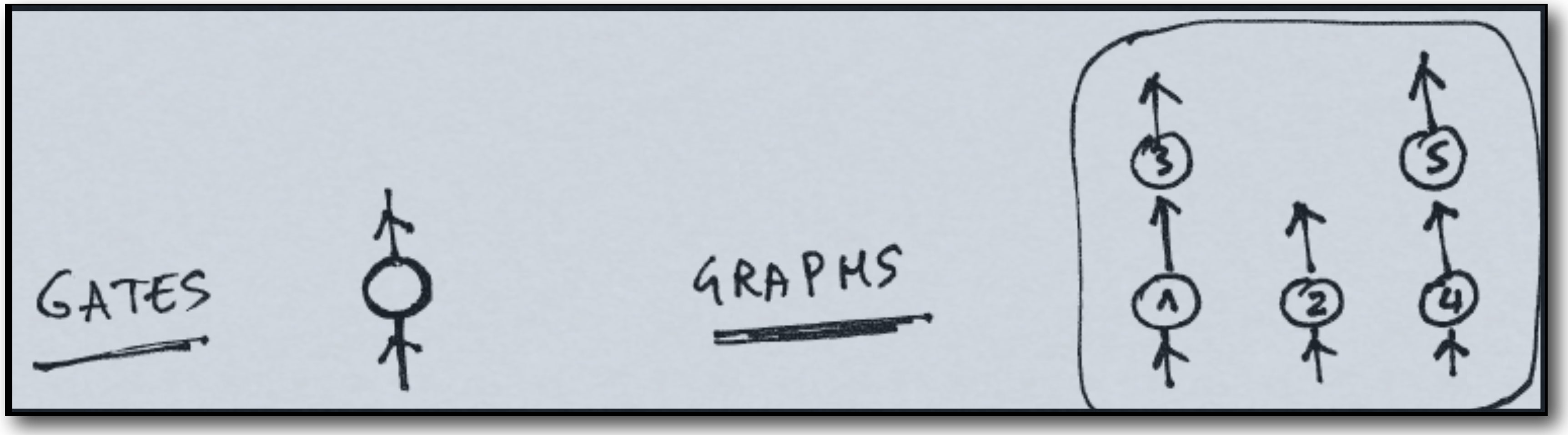
- Generalization to $(a(X) + D)$ or $(a(D) + X)$;
- **PDE: solve** $\frac{\partial}{\partial t}F = \frac{\partial}{\partial x}F + a(x)F$**.**
  [cf *method of characteristics; heat kernel*. . .]

3. The (XD) - forms

Set partitions & such…

The form (XD) and set partitions

- Well-known connections in analysis, difference calculus, and combinatorics(!)



$$\mathcal{G} = \textsc{Set}\left(u\,\textsc{Set}_{\geq 1}(\mathcal{Z})\right) \quad \Longrightarrow \quad G(z,u) = e^{u(e^z-1)}.$$

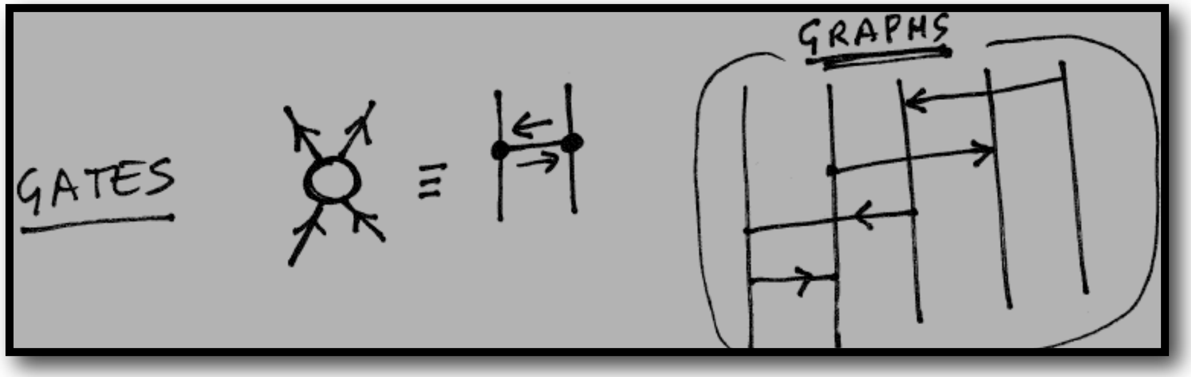
$$\mathfrak{N}\left[e^{z(XD)}\right] = \sum_{k\geq 0} \frac{1}{k!}(e^z-1)^k X^k D^k.$$

$\rightsquigarrow$ **PDE** $\dfrac{\partial}{\partial t}F = x\dfrac{\partial}{\partial x}F$ is solved by $F(x,t) = f(xe^t)$.

The forms (X^2D^2) and such, after Blasiak, Penson *et al.*

$$
\begin{array}{lcll}
(X^2D^2) & = & X^2D^2 & \mathbf{1} \\
(X^2D^2)^2 & = & 2X^2D^2 + 4X^3D^3 + X^4D^4 & \mathbf{7} \\
(X^2D^2)^3 & = & \cdots \text{ 87 terms } \cdots & \mathbf{87}
\end{array}
$$

[OEIS A20556]



bi-labelled structures : $e^u G(z,x) = \sum_k (k(k-1))^n \frac{x^k}{k!}\frac{z^n}{n!}$

$$\left\{ {n \atop k} \right\}_{2,2} = \frac{1}{k!}\sum_j (-1)^{k-j}\binom{k}{j}(k(k-1))^n; \quad \omega_n^{(2,2)} = e^{-1}\sum_\ell \frac{(l(l-1))^n}{\ell!}.$$

- Count **matrices** with two ones per line, no null column.
- **Coupon collector** with group drawings [DuFlRoTa]:

$$\mathbb{E}_m(T) = \frac{m(m-1)}{2m-1}\left[H_m + 12m - 1 - \frac{(-1)^m}{(m+1)\binom{2m-1}{m+1}}\right] \sim \frac{1}{2}\log m.$$

- Also **set partitions** with constrained contiguities.

4. The (X^2+D^2)-form

Permutations & such

The "circle" form $(X^2 + D^2)$

- Get two types of gates: CUPS (X^2) and CAPS (D^2)

- These assemble into chains that are either open or closed.
- As we go along a chain, **label values alternate**.
- There are symmetry factors since we enter from left or right.

Alternating (zigzag) perms have EGF $\tan(z)$, $\sec(z)$ [André 1881]
Alternating cycles have EGF $\log\cos(z)$.
We must take $\textsc{Set}$s of these.

$$G(x,y,z) = \frac{1}{\sqrt{\cos(2z)}} \exp\left(\left(\frac{x^2}{2} + \frac{y^2}{2}\right)\tan(2z) + xy(\sec(2z) - 1)\right).$$

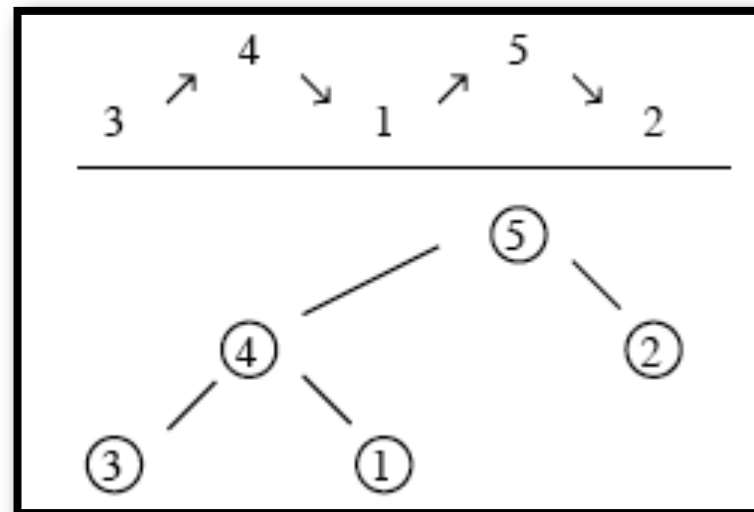
The general quadratic form $(\alpha X^2 + \beta D^2 + \gamma XD + \delta DX)$

Principle: similarly follow the spaghetti!

— Get the famous peaks, troughs, double rises, double falls.

— Cf: Carlitz; Françon–Viennot, ...

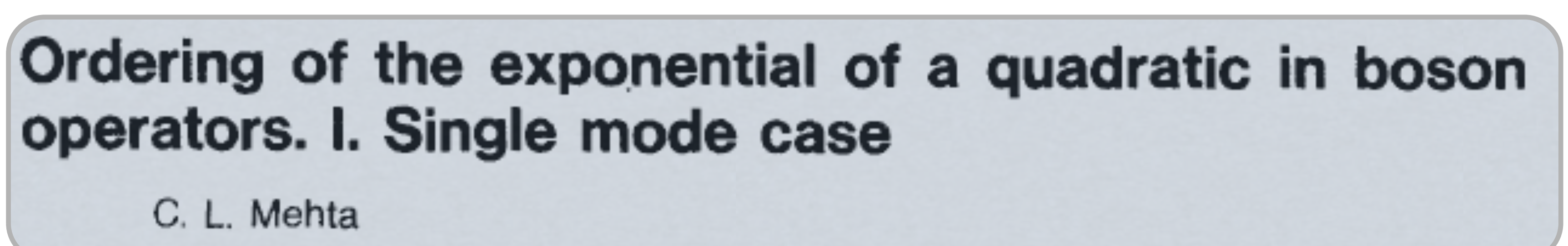
- Use **tree decomposition** of perms:



- Make use of "max-rooting $\rightsquigarrow \int$".

$$\mathcal{A} = (\mathcal{B}^{\square} \star \mathcal{C}) \quad \Longrightarrow \quad A(z) = \int_0^z (\partial_t B(t)) \cdot C(t)\, dt$$

Contrast with Lie algebra or *ad hoc* computations.


Ordering of the exponential of a quadratic in boson operators. I. Single mode case

C. L. Mehta

5. The tree form (X^2D)

... and trees

The special form (X^2D)

- The unique **gate is a '$\mathbb{Y}$'**.

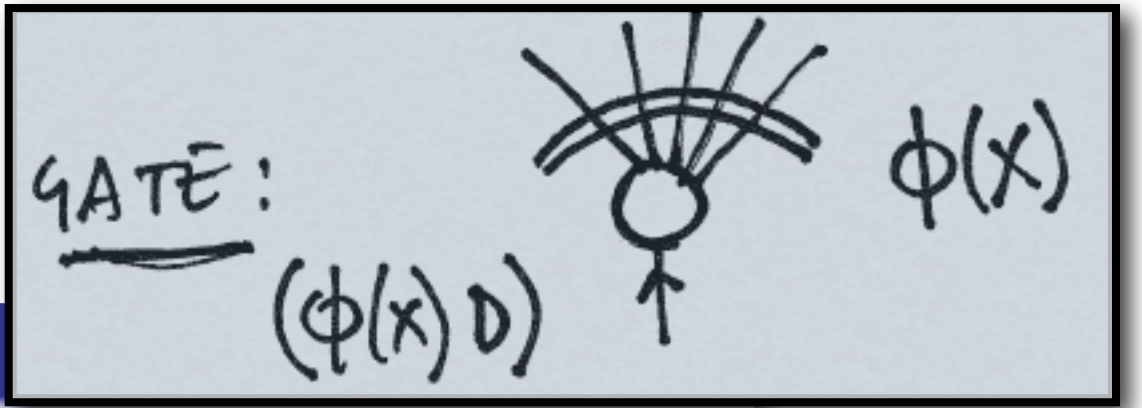
- Get **all permutations** as connected components.
- Take SETs of these.
- GFs are variants of $\boxed{\exp\left(\frac{z}{1-z}\right)}$.

$$\mathfrak{N}\left[e^{zXD^2}\right] = \exp\left(z\frac{X^2}{1-zX}D\right).$$

The semilinear form $\phi(X)D$

Gives rise to **increasing trees**, with $\phi(X)$ **the basic constructor.**

- Case (XD): threads (unary trees) $\rightsquigarrow$ set partitions.
- Case (X^2D): binary trees.
- Case (X^rD): r–ary trees, . . .



Proposition

The EGF of increasing trees with "rule" ϕ is

$$T(z) = Inv \int_0^t \frac{dw}{\phi(w)}.$$

Some exactly solvable models

[Bergeron-Fl-Salvy, 1992] →

Plane d–ary	$y' = (1+y)^d$	$y(z) = -1 + [1-(d-1)z]^{-1/(d-1)}$
Plane Strict d–ary	$y' = 1 + y^d$	$d=2$ $y(z) = \tan z$ $d>2$ —
Non plane Strict d–ary	$y' = 1 + \frac{y^d}{d!}$	$d=2$ $y(z) = \sqrt{2}\tan\frac{z}{\sqrt{2}}$ $d>2$ —
Plane unary-binary	$y' = 1 + y + y^2$	$y(z) = \frac{\sqrt{3}}{2}\tan\left(\frac{\sqrt{3}}{2}z + \frac{\pi}{6}\right) - \frac{1}{2}$
Non plane unary-binary	$y' = 1 + y + y^2/2$	$y(z) = \tan\left(\frac{z}{2} + \frac{\pi}{4}\right) - 1$
Plane "Recursive"	$y' = \frac{1}{1-y}$	$y(z) = 1 - \sqrt{1-2z}$
(Non plane) "Recursive"	$y' = \exp(y)$	$y = \log\frac{1}{1-z}$

Example: XD^r, after Blasiak, Penson, Solomon

$$G(x, y, z) = \exp\left(\frac{xy}{(1 - \rho x^{\rho} z)^{1/\rho}} - xy\right), \qquad \rho := r - 1.$$

- For $r = 3$, get $\exp\left(\frac{1}{\sqrt{1-2z}} - 1\right)$, which is evocative of binary trees(?).
- Explicit binomial sums are available for $\mathfrak{N}$ of powers of XD^r.

6. The (X^a+D^a) forms

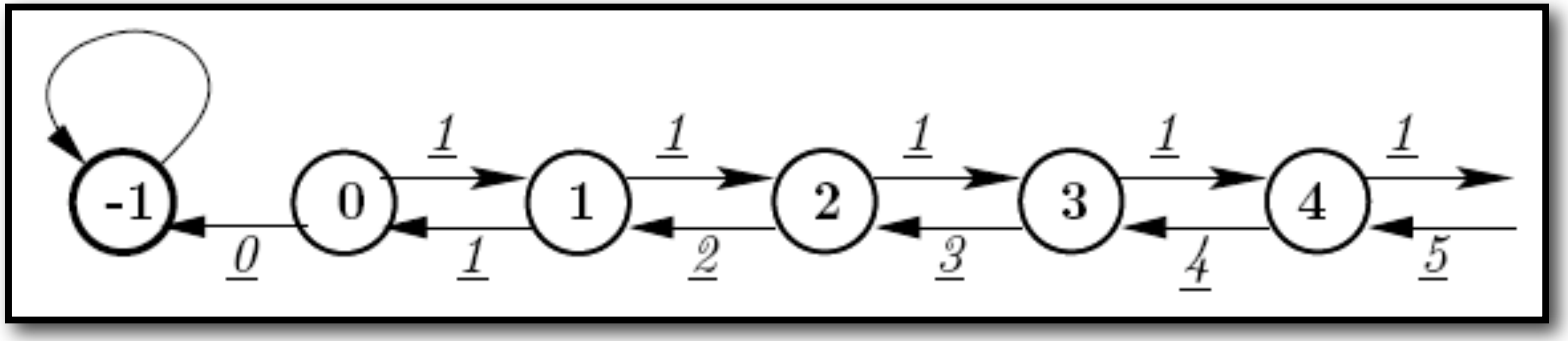
Dyck path, histories, & such

Paths and operators

In the **canonical basis** (x^k), X and D become matrices:

$$\mathbf{D} = \begin{bmatrix} 0 & 1 & \cdot & \cdot & \cdot & \cdots \\ \cdot & 0 & 2 & \cdot & \cdot & \cdots \\ \cdot & \cdot & 0 & 3 & \cdot & \cdots \\ \cdot & \cdot & \cdot & 0 & 4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \qquad \mathbf{X} = \begin{bmatrix} 0 & \cdot & \cdot & \cdot & \cdot & \cdots \\ 1 & 0 & \cdot & \cdot & \cdot & \cdots \\ \cdot & 1 & 0 & \cdot & \cdot & \cdots \\ \cdot & \cdot & 1 & 0 & \cdot & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Schematically, effect on basis (x^k) is described by the "Weyl graph":



Proposition

Consider a monomial $\mathfrak{f}$ in X, D. The constant term (of its normal form) is nonzero if and only if the associated path $\pi(\mathfrak{f})$ in the Weyl graph, starting from vertex 0, returns to vertex 0. In that case, this constant term is equal to the multiplicative weight of the path $\pi(\mathfrak{f})$.

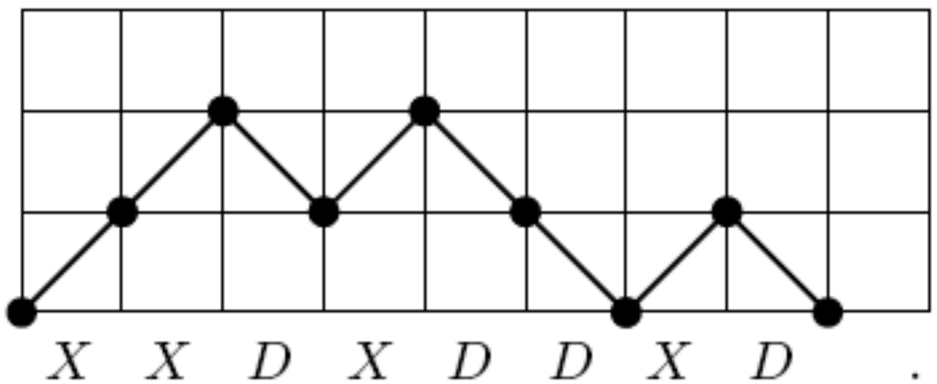
Constant Terms (C.T.) = weighted Dyck paths

Theorem F-1980]:

$$F(\mathbf{a}, \mathbf{d}, \boldsymbol{\ell}) = \cfrac{1}{1 - \ell_0 - \cfrac{a_0 d_1}{1 - \ell_1 - \cfrac{a_1 d_2}{1 - \ell_2 - \cfrac{a_2 d_3}{\ddots}}}}.$$

Dyck paths

level steps *ascents* *descents*



$$\mathfrak{h} = DXDDXDXX$$

$$\text{C.T.}(\mathfrak{h}) = 1 \times 1 \times 2 \times 1 \times 2 \times 1 \times 1 \times 1 = 4.$$

Proposition 2. *The normal ordering of $(X + D)$ corresponds to the continued fraction expansion*

$$\text{C.T.}\left(\frac{1}{1 - z(X+D)}\right) \equiv \text{C.T.}\left(\mathcal{L}\left[e^{z(X+D)}\right]\right) \equiv$$

$$\sum_{n \geq 0} [1 \cdot 3 \cdots (2n-1)]\, z^{2n} = \cfrac{1}{1 - \cfrac{1 \cdot z^2}{1 - \cfrac{2 \cdot z^2}{1 - \cfrac{3 \cdot z^2}{\ddots}}}}.$$

(X+D)

Laplace transform (L)

(X^2+D^2)

$$\mathcal{L}\left[\frac{1}{\sqrt{\cos(2z)}}\right] = \cfrac{1}{1 - \cfrac{1\cdot 2\cdot z^2}{1 - \cfrac{3\cdot 4\cdot z^2}{1 - \cfrac{5\cdot 6\cdot z^2}{\ddots}}}}.$$

(X^3+D^3)

$$\Phi_3(z) = \cfrac{1}{1 - \cfrac{1\cdot 2\cdot 3\cdot z^2}{1 - \cfrac{4\cdot 5\cdot 6\cdot z^2}{1 - \cfrac{7\cdot 8\cdot 9\cdot z^2}{\ddots}}}}$$

$$= 1 + 6\,z^2 + 756\,z^4 + 458136\,z^6 + 76534113$$

??

Identify? Cf Dixonian elliptic functions; Apéry's $z(3)$

Openings

Partial difference equations, q-analogues,...

MISC.

- Relation with PDEs?
  Eg. *Duchon's Clubs* and the *cubic oscillator*.
- *Difference equations* and q-analogues
- Relation with *Rook Polynomials* [Varvak]...
- Relation with various *tableaux*? exclusion...?
  Cf Viennot, Corteel, Josuat-Verges, ...