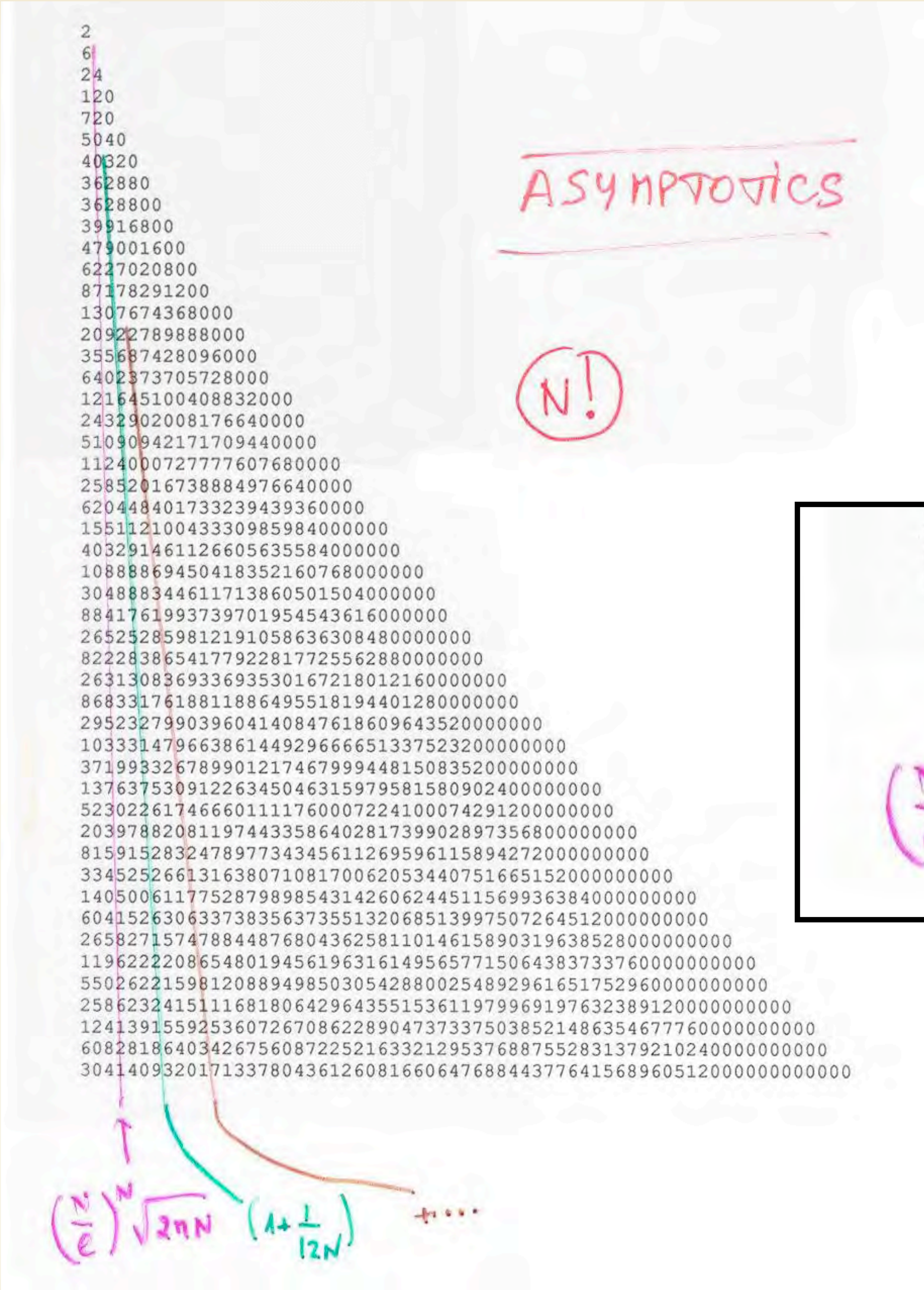
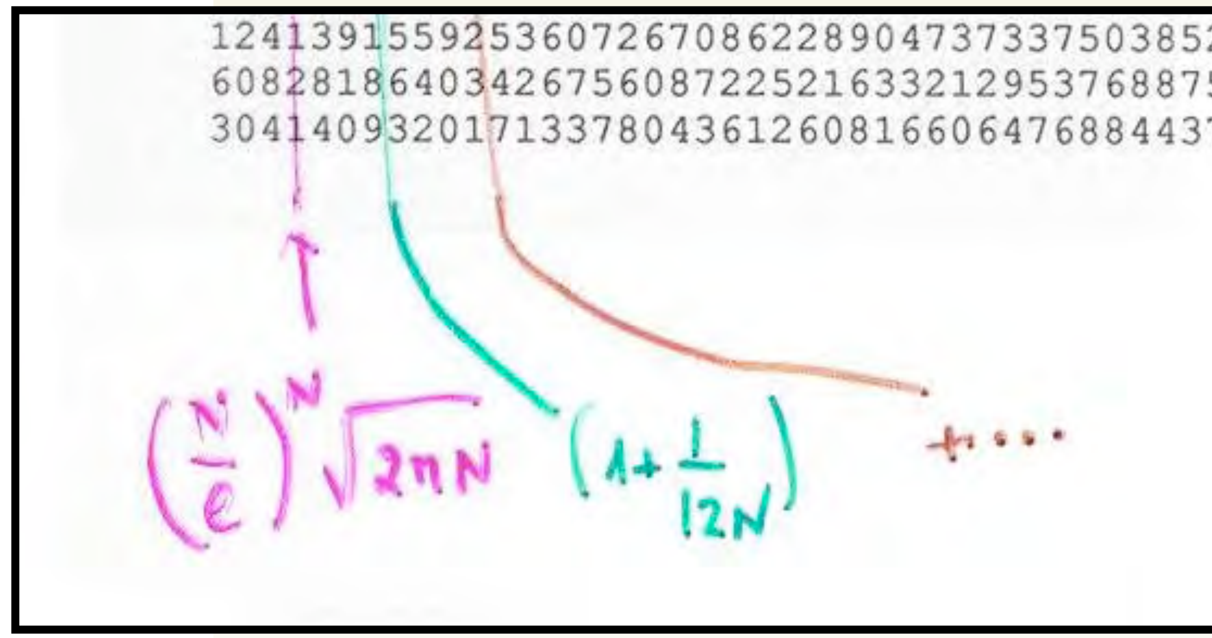
Part B
Complex Asymptotics

* Chapter 4: Complex Analysis
* Chapter 5: Rational and Meromorphic Asymptotics
* Chapter 6: Singularity Analysis of GFs
* Chapter 7: Applications of Singularity Analysis
* Chapter 8: Saddle-point Methods


ASYMPTOTICS
N!
2
6
24
120
720
5040
40320
362880
3628800
39916800
479001600
6227020800
87178291200
1307674368000
20922789888000
355687428096000
6402373705728000
121645100408832000
2432902008176640000
51090942171709440000
1124000727777607680000
25852016738884976640000
620448401733239439360000
15511210043330985984000000
403291461126605635584000000
10888869450418352160768000000
304888344611713860501504000000
8841761993739701954543616000000
265252859812191058636308480000000
8222838654177922817725562880000000
263130836933693530167218012160000000
8683317618811886495518194401280000000
295232799039604140847618609643520000000
10333147966386144929666651337523200000000
371993326789901217467999448150835200000000
13763753091226345046315979581580902400000000
523022617466601111760007224100074291200000000
20397882081197443358640281739902897356800000000
815915283247897734345611269596115894272000000000
33452526613163807108170062053440751665152000000000
1405006117752879898543142606244511569936384000000000
60415263063373835637355132068513997507264512000000000
2658271574788448768043625811014615890319638528000000000
119622220865480194561963161495657715064383733760000000000
5502622159812088949850305428800254892961651752960000000000
258623241511168180642964355153611979969197632389120000000000
12413915592536072670862289047373375038521486354677760000000000
608281864034267560872252163321295376887552831379210240000000000
30414093201713378043612608166064768844377641568960512000000000000
$\left(\frac{N}{e}\right)^N \sqrt{2\pi N} \left(1+\frac{1}{12N}\right) + \cdots$

N! for N=2,...,50


1241391559253607267086228904737337503852
6082818640342675608722521633212953768875
3041409320171337804361260816606476884437
$\left(\frac{N}{e}\right)^N \sqrt{2\pi N} \left(1+\frac{1}{12N}\right) + \cdots$

Real analysis?

Some functions are too complicated to be expanded

e.g. = TRAINS

Some GF's are not even explicit

e.g. = unlabelled, non-plane trees

$$U(z) = z \exp\left(U(z) + \frac{1}{2} U(z^2) + \frac{1}{3} U(z^3) + \cdots\right)$$

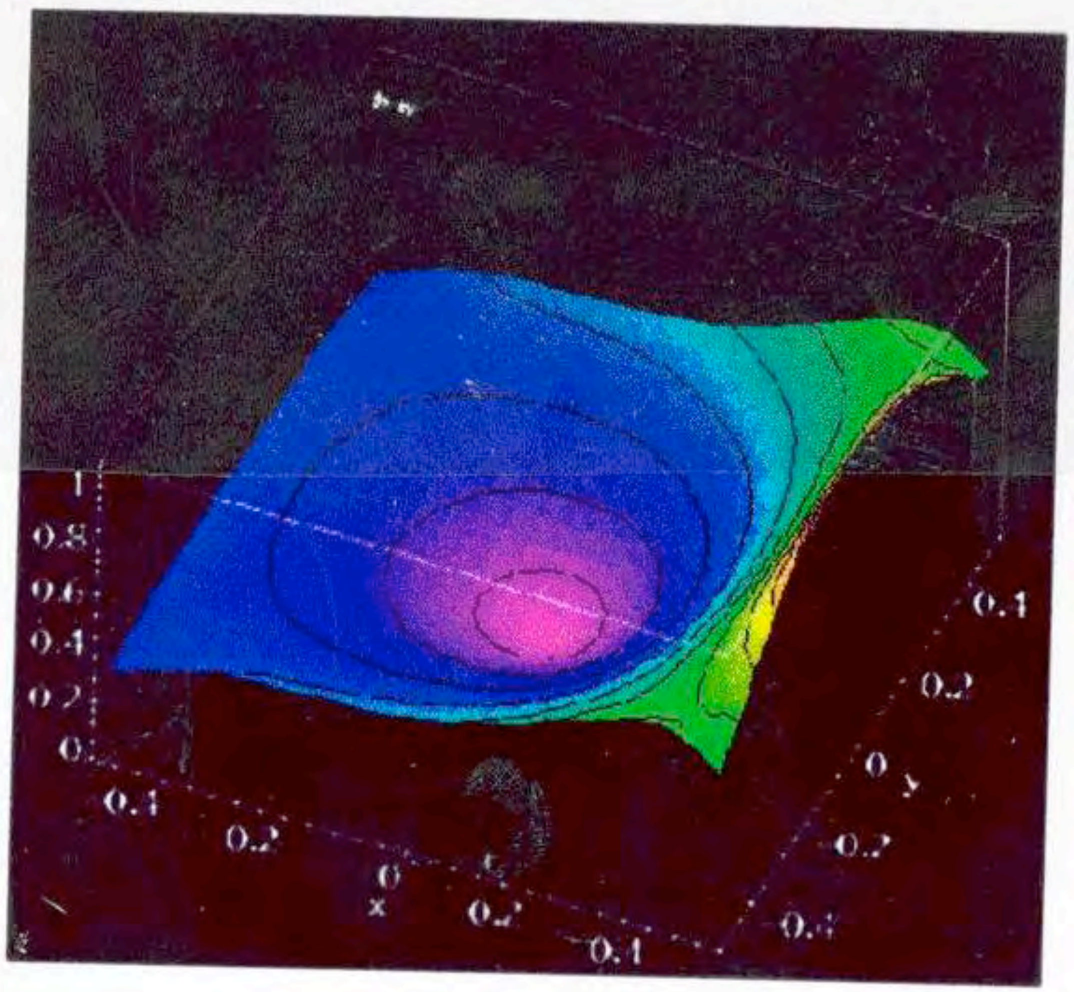
"Universality" phenomena are not apparent

- schemas applying to wide classes
- limit laws shared by — —

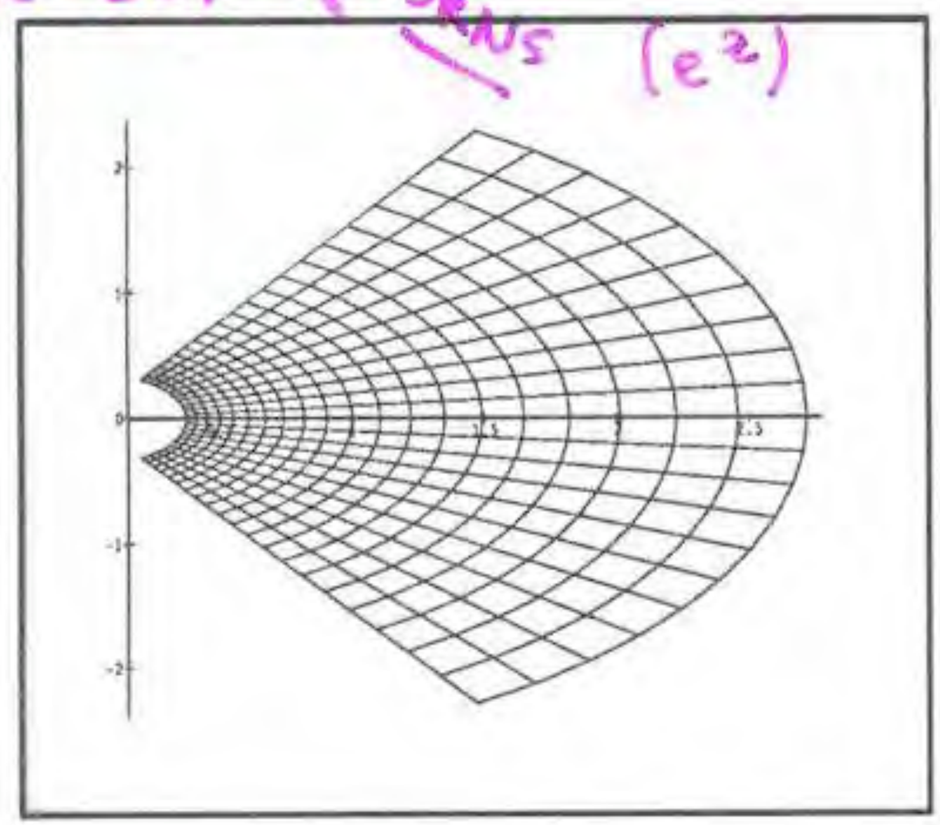
PRINCIPLE: Assign to the variable (z) complex values ⇒

View a GF as a geometric transformation from $\mathbb{C} \to \mathbb{C}$

modulus of GF of balanced trees



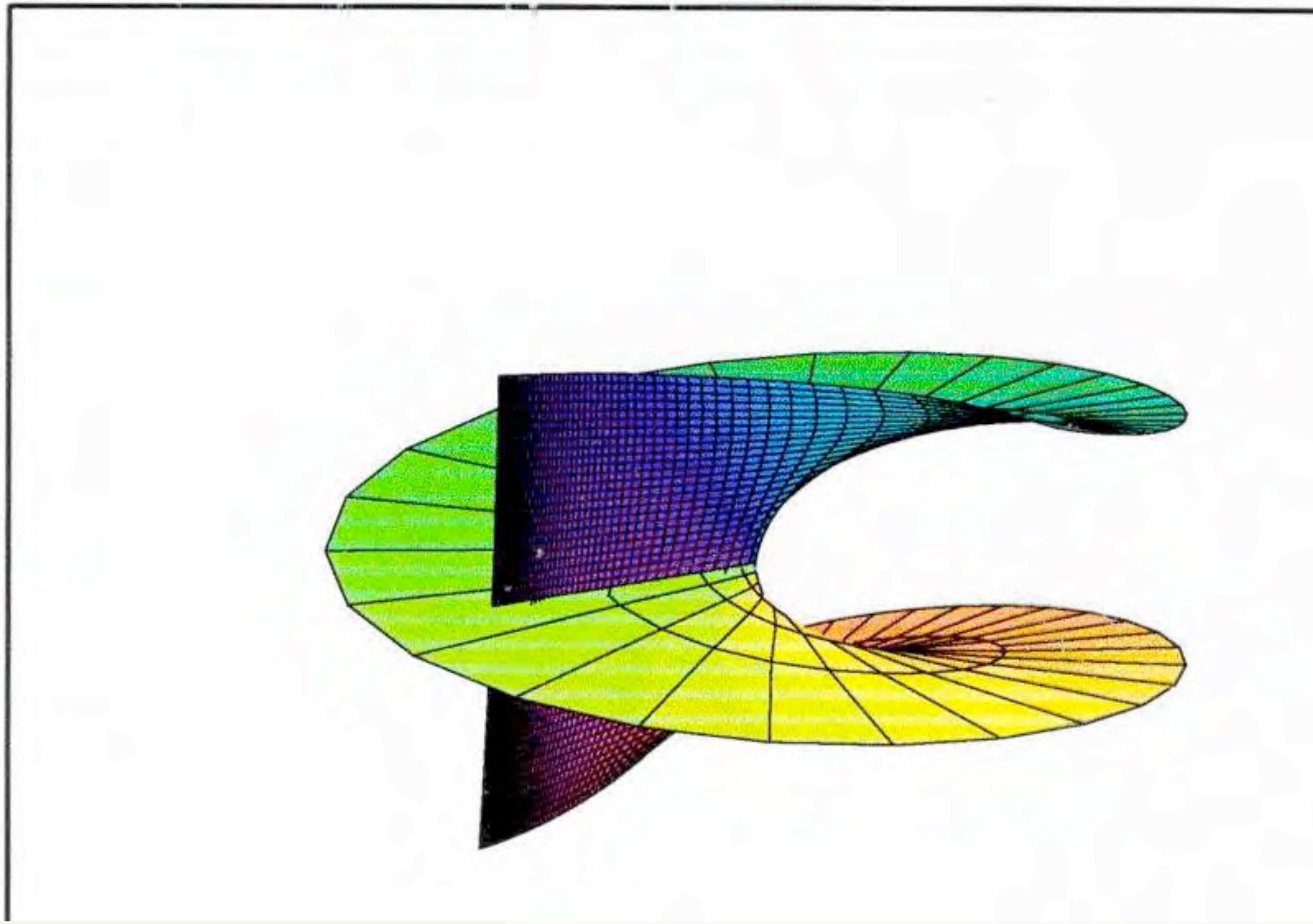
THE EGF of URNS (e^z)



(Catalan GF)

SINGULARITIES MATTER!

The √-singularity of the OGF of binary trees.



CHAPTER

4 ANALYTIC FUNCTIONS

GFs are (usually) analytic functions near 0.

- Analytic aka holomorphic functions
- Meromorphic functions
- Integrals and residues
- Singularities and exponential growth orders

Let $f(z)$ be defined from D (open connected set) to E:

Definition. • $f(z)$ is analytic at z_0 iff *locally*: $\boxed{f(z) = \sum_{n \geq 0} c_n (z - z_0)^n}$.

• $f(z)$ is *complex differentiable* iff

$$\exists \boxed{\lim_{h \to 0,\ h \in \mathbb{C}} \frac{f(z_0 + h) - f(z_0)}{h}} =: f'(z_0) \equiv \frac{d}{dz} f(z)\Big|_{z=z_0}.$$

$\leadsto$ f analytic/ differentiable in Ω , etc.

Theorem. Equivalence between the two notions!

Combinatorialists love power series; analysts love differentiability!

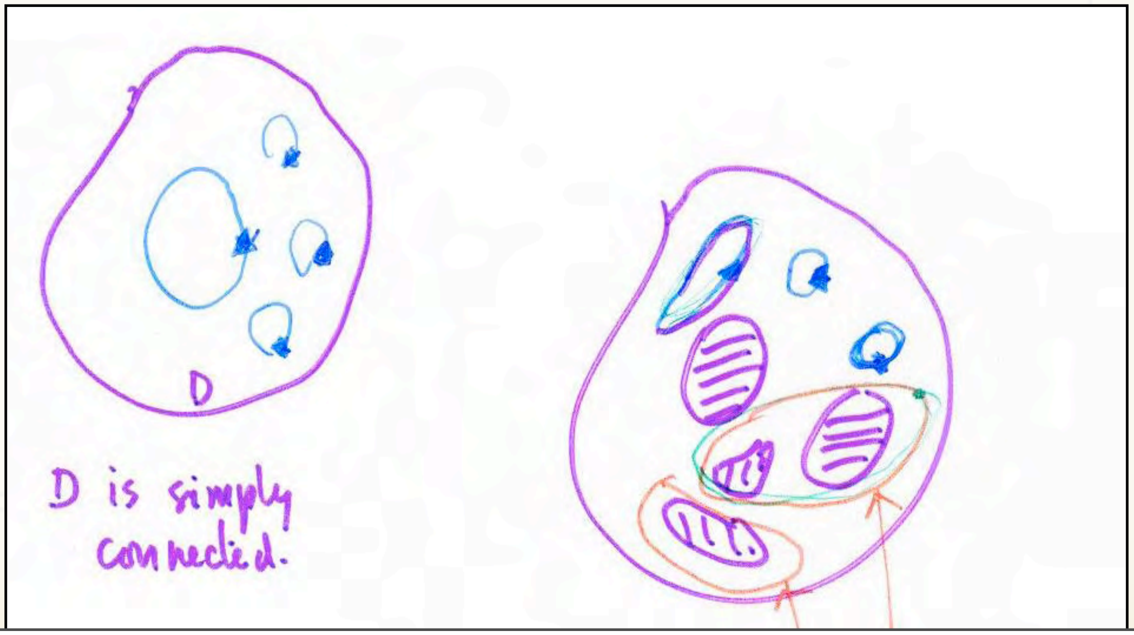
$\dfrac{\Delta f}{\Delta z}$ gives closure under $+, -, \times, \div$, composition, inversion, &c.

Integration and residues

Theorem. Let f be analytic in Ω and γ be contractible to a single point in Ω. Then

$$\int_\gamma f(z)\,dz = 0.$$

In particular $\int_A^B f(z)\,dz$ does not depend on path.



Definition. $g(z)$ is *meromorphic* in Ω iff near any z_0, one has $g(z) = \frac{A(z)}{B(z)}$, with A, B analytic.

A point z_0 such that $B(z_0) = 0$ is a *pole*. Its *order* is the multiplicity of z_0 as root of B (assume $A(z_0) \neq 0$).

Pole of order m: $g(z) = \dfrac{c_{-m}}{(z-z_0)^m} + \cdots + \dfrac{c_{-1}}{(z-z_0)} + c + 0 + \cdots$.

c_{-1} is called *residue* of $g(z)$ at z_0.

Cauchy's Residue Theorem. If $f(z)$ has poles, then

$$\frac{1}{2i\pi}\int_\gamma f(z)\,dz = \sum \text{Residues}\,.$$

Proof: local integration +

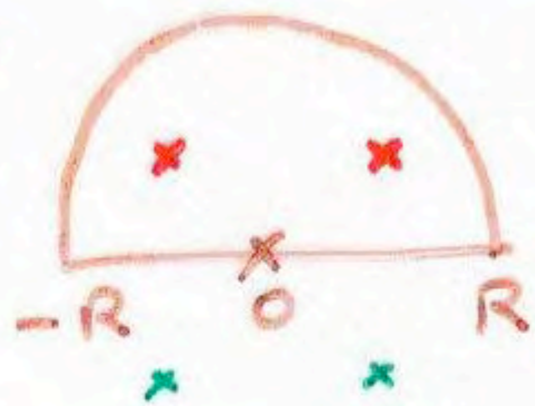
Cauchy's Coefficient Theorem. $$\text{coeff}[z^n]\,f(z) = \frac{1}{2i\pi}\int_\gamma f(z)\,\frac{dz}{z^{n+1}}$$

Proof: by residues:

Complex analysis: LOCAL vs GLOBAL

- **Computing integrals** global ⇝ local !!!

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^4} = \lim_{R\to\infty} \int$$



$$= 2i\pi \sum_{\zeta \in \{e^{i\pi/4},\, e^{3i\pi/4}\}} \operatorname{Res}\left(\frac{1}{1+x^4}; \zeta\right)$$

$$= \frac{\pi\sqrt{2}}{2}$$

Complex analysis: coeffs at 0 ~> elsewhere

- Estimating coefficients: $d_n := \mathbb{P}[\text{derangement}]$ over $\mathcal{P}_n$.

$$d_n = [z^n]\frac{e^{-z}}{1-z} = \frac{1}{2i\pi}\int_{|z|=1/2} \frac{e^{-z}}{1-z}\frac{dz}{z^{n+1}}.$$

Evaluate instead on $|z| = 2$:

$$\begin{aligned} J_n &= \frac{1}{2i\pi}\int_{|z|=2} \frac{e^{-z}}{1-z}\frac{dz}{z^{n+1}} = O(2^{-n}) \\ &= \text{Res}_{z=0} + \text{Res}_{z=1} = d_n - e^{-1}. \end{aligned}$$

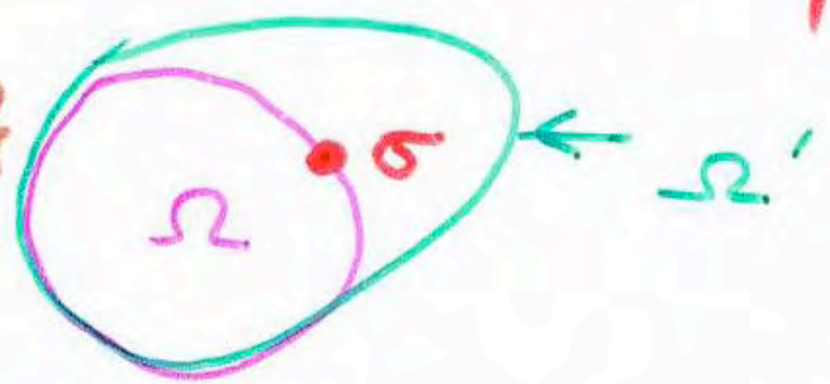
Thus: $\boxed{d_n = e^{-1} + O(2^{-n})}$.

Exercise: Double derangement: $[z^n]e^{-z-z^2/2}/(1-z)$. Generalize!

singularity

- $f(z)$ has a SINGULARITY at border point σ iff

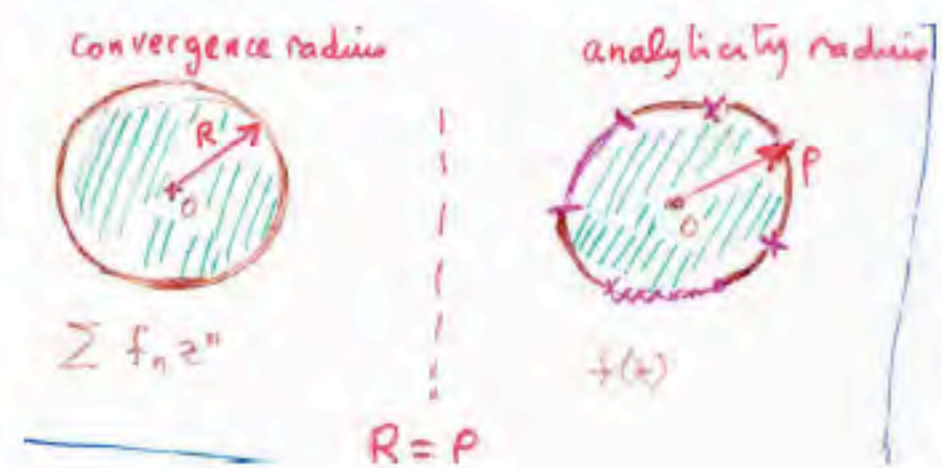
$\nexists\, \Omega' \supset \Omega \cup \{\sigma\}$



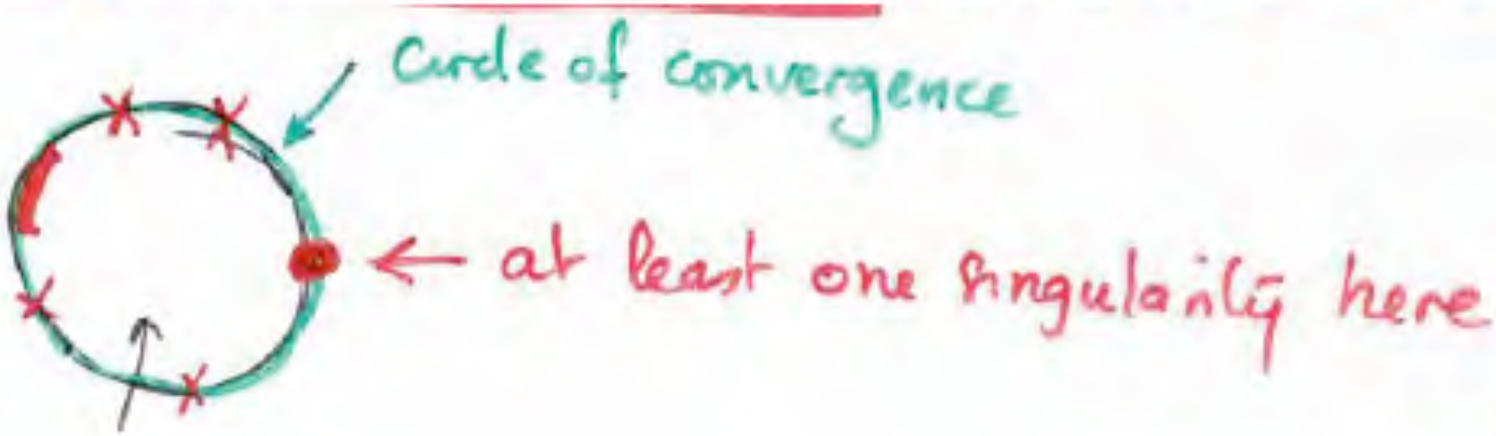
e.g.: $f(\sigma) = \infty$, $f'(\sigma) = \infty$, + other causes

Theorem. A series always has at least one singularity on its circle of convergence (but none inside).

Convergence radius ≡ Analyticity radius:



Pringsheim's Theorem. If $f_n \geq 0$, one such singulariy is positive.



Exponential growth of coefficients

$$f_n (R-\epsilon)^n \to 0; \qquad f_n(R+\epsilon)^n \text{ is unbounded.}$$

That is $\limsup |f_n|^{1/n} = \frac{1}{R}$, or

$$f_n = R^{-n}\vartheta(n), \quad \text{where } \vartheta(n) \text{ is "subexponental".}$$

Also write $\boxed{f_n \bowtie R^{-n}}$ with R := distance to nearest sing(s).

Find exponential growth by just "looking" at GF!!

Once you find the singularities nearest to the origin, you KNOW the exponential growth of the function's coefficients.

Exponential order of coeffs is computable:

$$\rho(f+g) = \min(\rho(f), \rho(g))$$

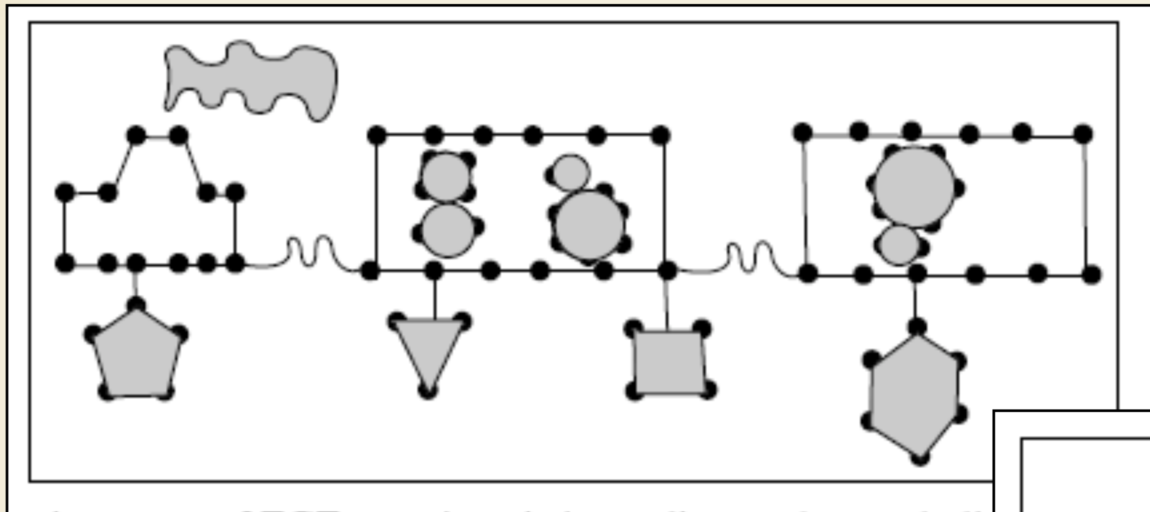
$$\rho(f \times g) = \min(\rho(f), \rho(g))$$

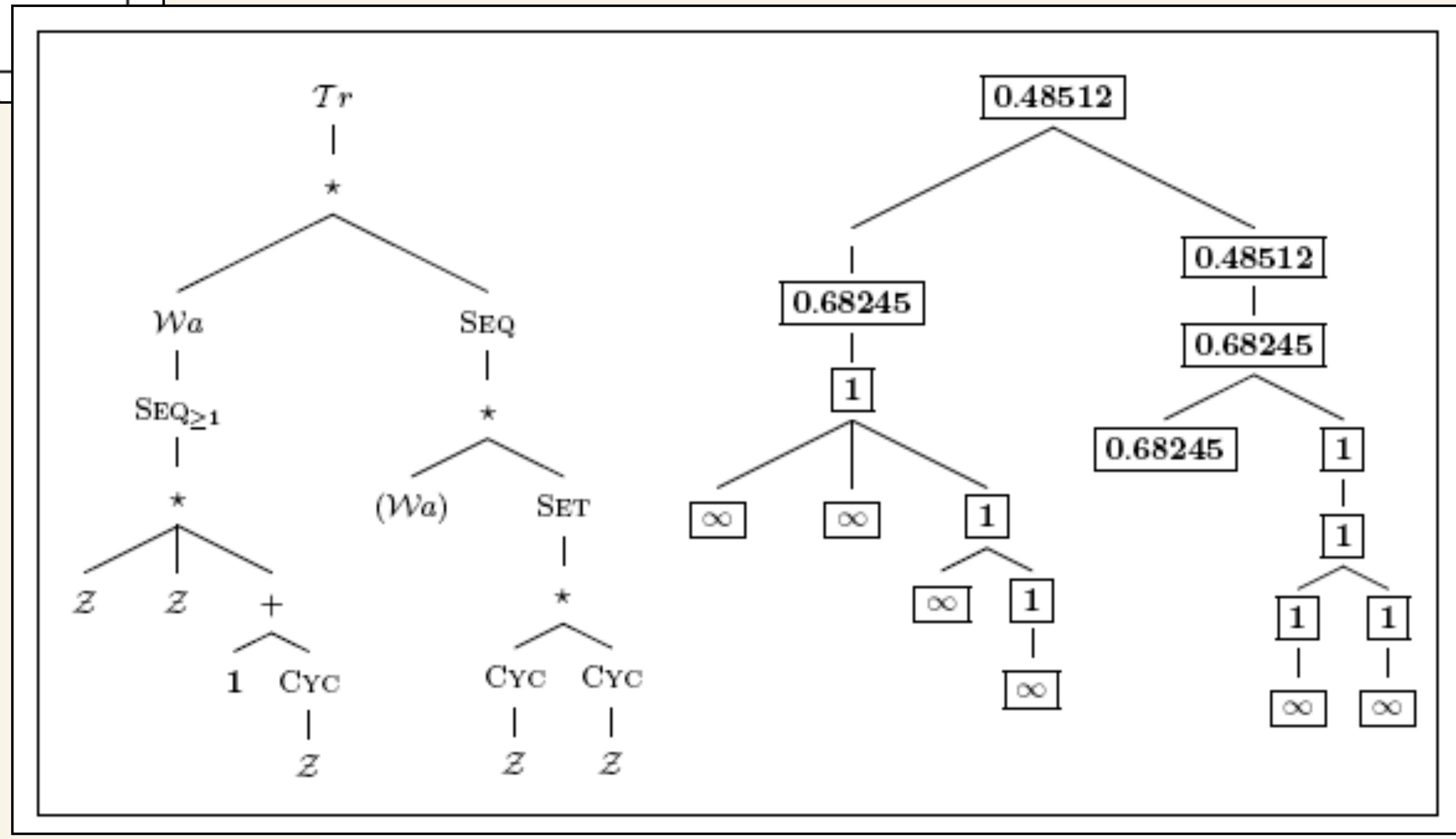
$$\rho\left(\frac{1}{1-f}\right) = \min(\rho(f), \{|z| \mid f(z) = 1\})$$

$$\rho(e^f) = \rho(f)$$

$$\rho\left(\log \frac{1}{1-f}\right) = \rho\; 1/(1-f)$$

+ Recursive structures can be approached via Implicit function theorem.





TRAINS

$$Tr(z) = \frac{z^2\left(1+\log((1-z)^{-1})\right)}{\left(1-z^2\left(1+\log((1-z)^{-1})\right)\right)}\left(1-\frac{z^2\left(1+\log((1-z)^{-1})\right)e^{\left(\log((1-z)^{-1})\right)^2}}{1-z^2\left(1+\log((1-z)^{-1})\right)}\right)^{-1},$$

Chapter 5

Rational and Meromorphic Asymptotics

Find subexponential factors in

$$f_n \bowtie R^{-n}, \qquad \text{meaning} \quad f_n = R^{-n}\vartheta(n),$$

where $\vartheta(n)$ is like n^α, $(\log n)^\beta$, $e^{\sqrt{n}}$, etc.

Here: simple case of Rat & Mero.

Coefficients of rational functions

Theorem. Each pole ζ with multiplicity r contributes to coefficients a term

$$\zeta^{-n} P(n),$$

where $P(n)$ is a polynomial of degree $r-1$.

Proof. $[z^n]\dfrac{1}{(z-\zeta)^r} = (-\zeta)^{-r}\dbinom{n+r-1}{r-1}\zeta^{-m}.$

Poles are arranged in order of increasing modulus. Dominant ones matter for exponential growth. Multiplicities give polynomial factors.

Example 1. Denumerants.

$$D(z) = \frac{1}{1-z^{s_1}} \times \frac{1}{1-z^{s_2}} \times \cdots \times \frac{1}{1-z^{s_m}}$$

Poles at various roots of unity w/order < m

at $z=1$ with order exactly m

$$D_n \equiv [z^n]\, D(z) \sim [z^n] \frac{1}{(1-z)^m} \frac{1}{\prod s_j}$$

Schur:

$$\sim \frac{n^{m-1}}{(m-1)!} \times \frac{1}{\prod s_j}$$

Example 2. Longest b-runs in strings. (cf Feller)

$$bbb\ \boxed{\mathbf{a}bb}\ \boxed{\mathbf{a}b}\ \boxed{\mathbf{a}}\ \boxed{\mathbf{a}bbbb}$$

$$\textsc{Seq}_{<m}(b) \times \textsc{Seq}(\mathbf{a}\ \textsc{Seq}_{<m}(b))$$

$$\frac{1-z^m}{1-z} \times \frac{1}{1-z\frac{1-z^m}{1-z}} = \frac{1-z}{1-2z+z^{m+1}}.$$

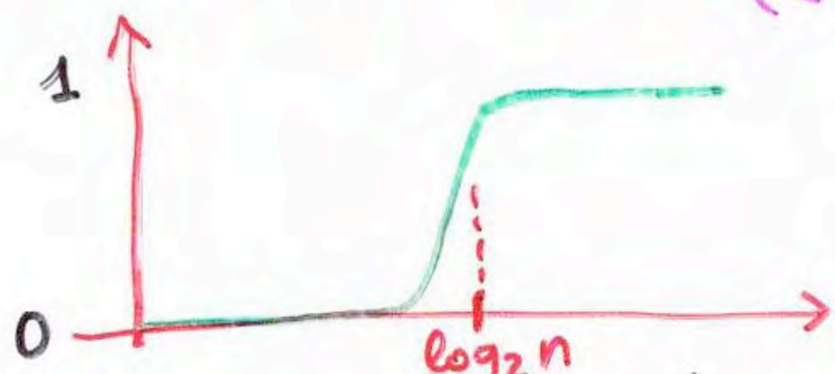
$$\frac{1-z^m}{1-2z+z^{m+1}}$$

- Dominant pole at $\rho_m \approx \frac{1}{2}$

$$\rho_m \approx \frac{1}{2}\left(1+\left(\frac{1}{2}\right)^{m+1}\right)$$

- Check error from dominant pole is good by $\int_{|z|=3/4}$

$$\Pr(\text{Longest run} < m) \approx \left(\frac{1}{2\rho_m}\right)^n \approx e^{-n/2^{m+1}}$$



Guibas-Odlyzko: Correlation Polynomials

Normality of strings; longest repeated substring ...

Coefficients of meromorphic functions

Theorem

EACH POLE ζ WITH MULTIPLICITY r CONTRIBUTES A TERM

$$\zeta^{-n} P(n) \text{ with } \deg(P) = r-1$$

AND ERROR TERM $O(R^{-n})$

Proof: 1) Let $h(z)$ gather contributions of poles.

Then $[g(z) - h(z)]$ is analytic in $|z| \leq R$

Cauchy coeff. formula + trivial bounds

2) Estimate $\frac{1}{2i\pi} \int g$ on $|z| = R$ by residues.

Worked out Example: *derangements*

$$\mathcal{D} = \text{Set}(\text{Cycle}(\mathcal{Z}, \text{card} \geq 2))$$

$$D(z) = \exp\left(\log \frac{1}{1-z} - z\right)$$

$$D(z) = \frac{e^{-z}}{1-z}$$

$$D(z) \sim \frac{e^{-1}}{1-z} \quad \text{at singularity } z = 1$$

$$\Rightarrow \quad \frac{D_n}{m!} \sim [z^n] \frac{e^{-1}}{1-z} = e^{-1}$$

<u>Prop.</u> A perm is a derangement with probability $e^{-1} = 0.345\ldots$

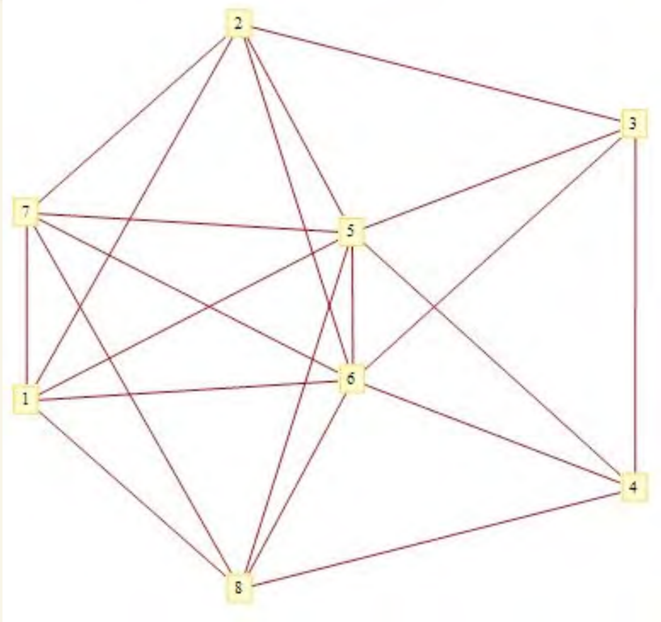
Generalized derangements $\mathcal{D}^*$

$$D^*(z) = \frac{e^{-z-\frac{z^2}{2}}}{1-z}$$

$$\frac{D^*_n}{n!} \sim e^{-3/2}$$

in general, get

proba $\sim e^{-H_k}$ of all cycles of length $> k$.



Example 4. Paths-in-graphs models.

Encapsulates finite automata and finite Markov chains. GFs are rational.

If the graph Γ is *strongly connected* and *aperiodic*, then there is unicity and simplicity of dominant pole ($\ll$ Perron-Frobenius): $f_n \sim c\rho^{-n}$.

Generalized patterns in random strings (F, Nicodème, Régnier, Salvy, Szpankowski, Vallée, &c).

Example 5. Surjections and Supercritical $\textsc{Seq}$ Schema.

Random surjection $\equiv$ ordered partition (pref. arrangement)

$$\mathcal{R} = \textsc{Seq}(\textsc{Set}_{\geq 1}(\mathcal{Z})) \qquad \Longrightarrow R(z) = \frac{1}{2 - e^z}.$$

Pole at $\zeta = \log 2$; subdominant ones at $\zeta = \log 2 \pm 2ik\pi$, etc.

$$\frac{R_n}{n!} \sim c(\log 2)^{-n}.$$

Also, mean number of blocks via $\dfrac{1}{1 - u(e^z - 1)}$ is $O(n)$. There is concentration, etc.

Any supercritical sequence should similarly behave $\leadsto$ schema.

APPLICATION : Supercritical schema (sequences)

Assume $\mathcal{F} = \mathrm{Seq}(\mathcal{G}) \Rightarrow F(z) = \frac{1}{1-G(z)}$

$\underline{H_1}$: $G(z)$ reaches 1 before it becomes singular

($\underline{H_2}$: the schema is aperiodic : $F_n > 0$ for all $n \geq n_0$.)

Then $F_n \sim C \cdot \rho^{-n}$ $\rho = G^{(-1)}(1)$

Also : number of $\mathcal{G}$-components in random $\mathcal{F}$-structure has

mean $\sim \mu n$; variance $\sim \sigma^2 n \Rightarrow$ concentration

SCHEMA

Chapter 6

Singularity Analysis

- Singularities more general than poles.
- Subexponential factors more general than polynomials:

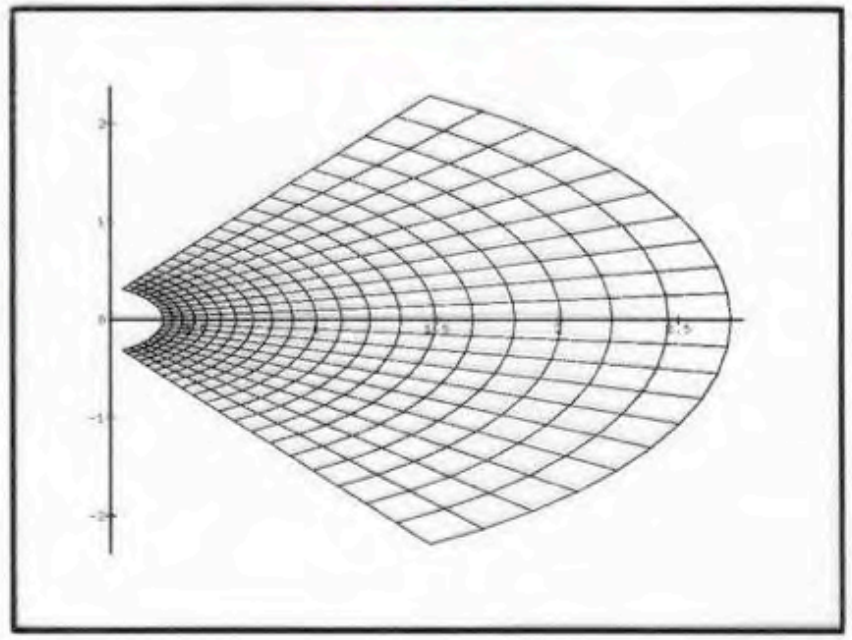
$$f_n \sim R^{-n}\vartheta(n),$$

with $\vartheta(n)$ of the form $n^{\alpha}(\log n)^{\beta}$.

Note: May assume singularity at 1 by scaling $[z^n]f(\lambda z) = \lambda^n [z^n] f(z)$.

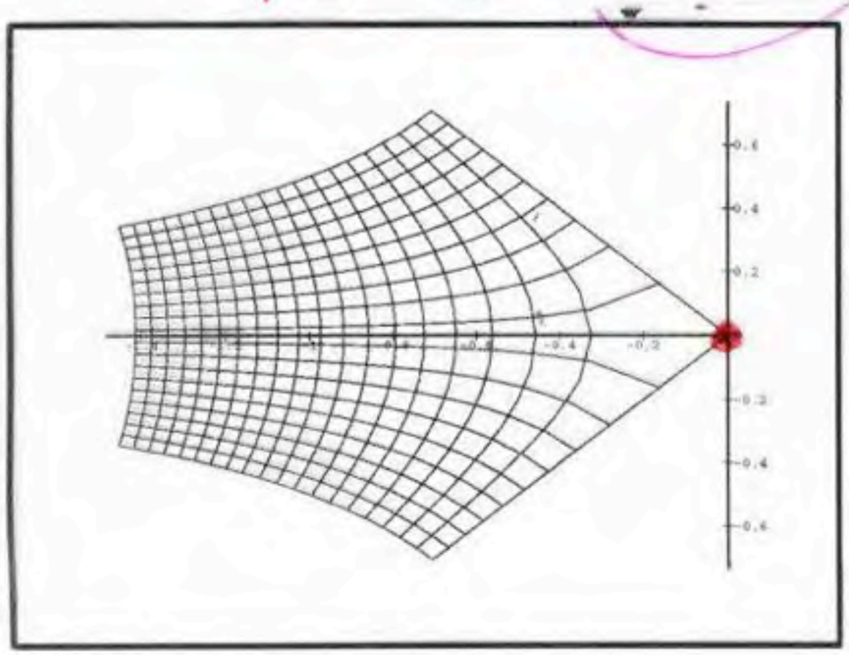
Regular point $f(z) \approx f(z_0) + f'(z_0)(z - z_0)$

$\exp(z)$

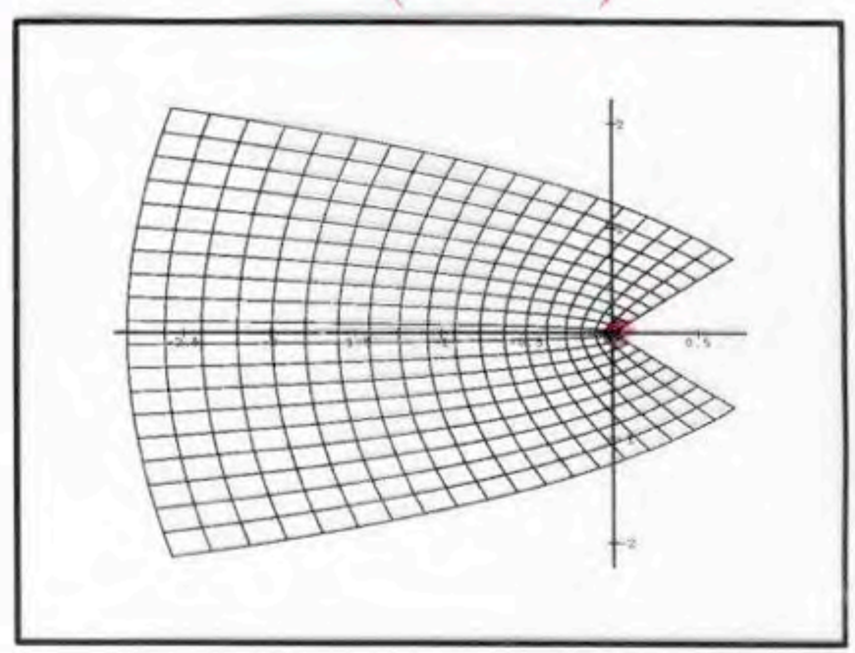


singular point

$-\sqrt{1-z}$



$-(1-z)^{3/2}$



(Singularity analysis)

[F + Odlyzko, 90]

PRINCIPLE

$$f(z) \underset{[z\to 1]}{\sim} C \cdot (1-z)^{-\alpha} \log^k \frac{1}{1-z}$$

ANALYTIC CONTINUATION

$$\Longrightarrow \quad [z^n]f(z) \underset{[n\to\infty]}{\sim} C \cdot \frac{n^{\alpha-1}}{\Gamma(\alpha)} \log^k n$$

(earlier: Darboux-Pólya; Tauberian thms)

From functions to coefficients:

$\dfrac{1}{(1-z)^2}$	$\longrightarrow$	$n+1$	$\sim$	n
$\dfrac{1}{1-z}\log\dfrac{1}{1-z}$	$\longrightarrow$	$H_n \equiv \frac{1}{1} + \ldots + \frac{1}{n}$	$\sim$	$\log n$
$\dfrac{1}{1-z}$	$\longrightarrow$	1	$\sim$	1
$\dfrac{1}{\sqrt{1-z}}$	$\longrightarrow$	$\dfrac{1}{2^{2n}}\dbinom{2n}{n}$	$\sim$	$\dfrac{1}{\sqrt{\pi n}}$

$$\begin{cases} \text{Location of sing's : Exponential factor } & \rho^{-n} \\ \text{Nature of sing's : "Polynomial" factor } & \vartheta(n) \end{cases}$$

Principles of Singularity Analysis

Larger functions tend to have larger coefficients.

— Establish this for basic scales $(1-z)^{-\alpha}$. Enrich with $\log$'s, $\log\log$'s, etc.

— Prove transfer theorems. If f "resembles " g via $O(\cdot)$, $o(\cdot)$, then f_n resembles g_n.

Theorem 1. Coefficients of basic scale:

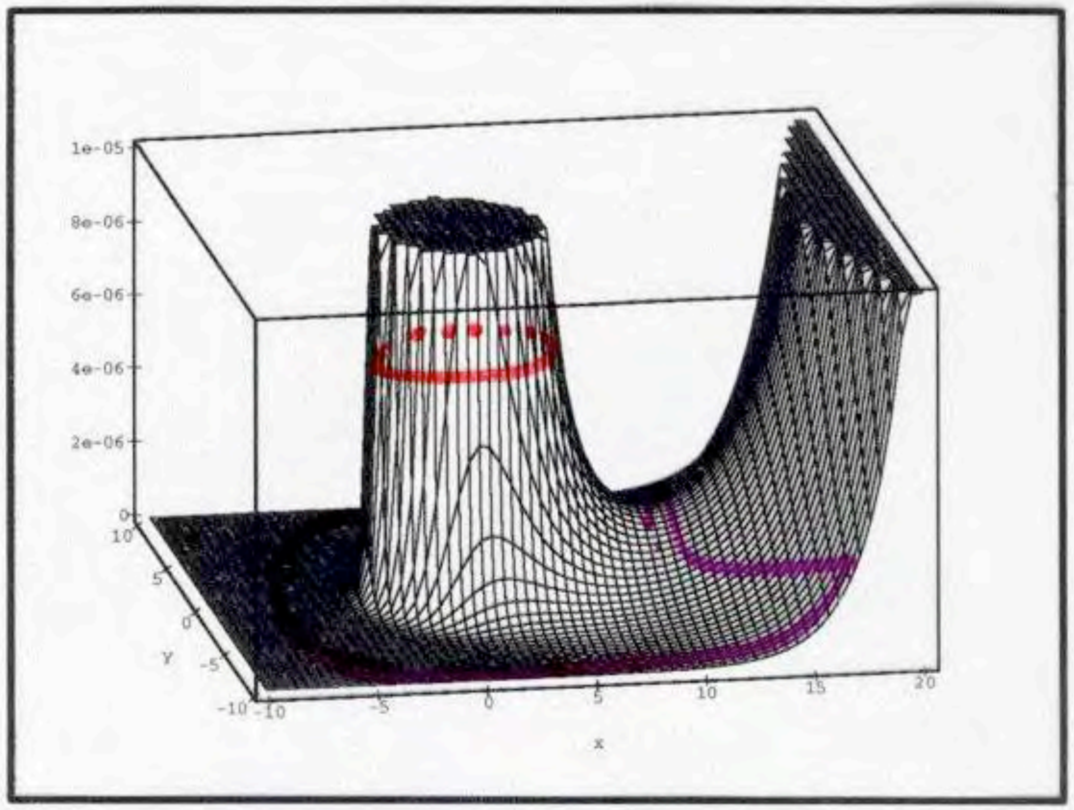
$$[z^n](1-z)^{-\alpha} \sim \frac{1}{\Gamma(\alpha)} n^{\alpha-1}.$$

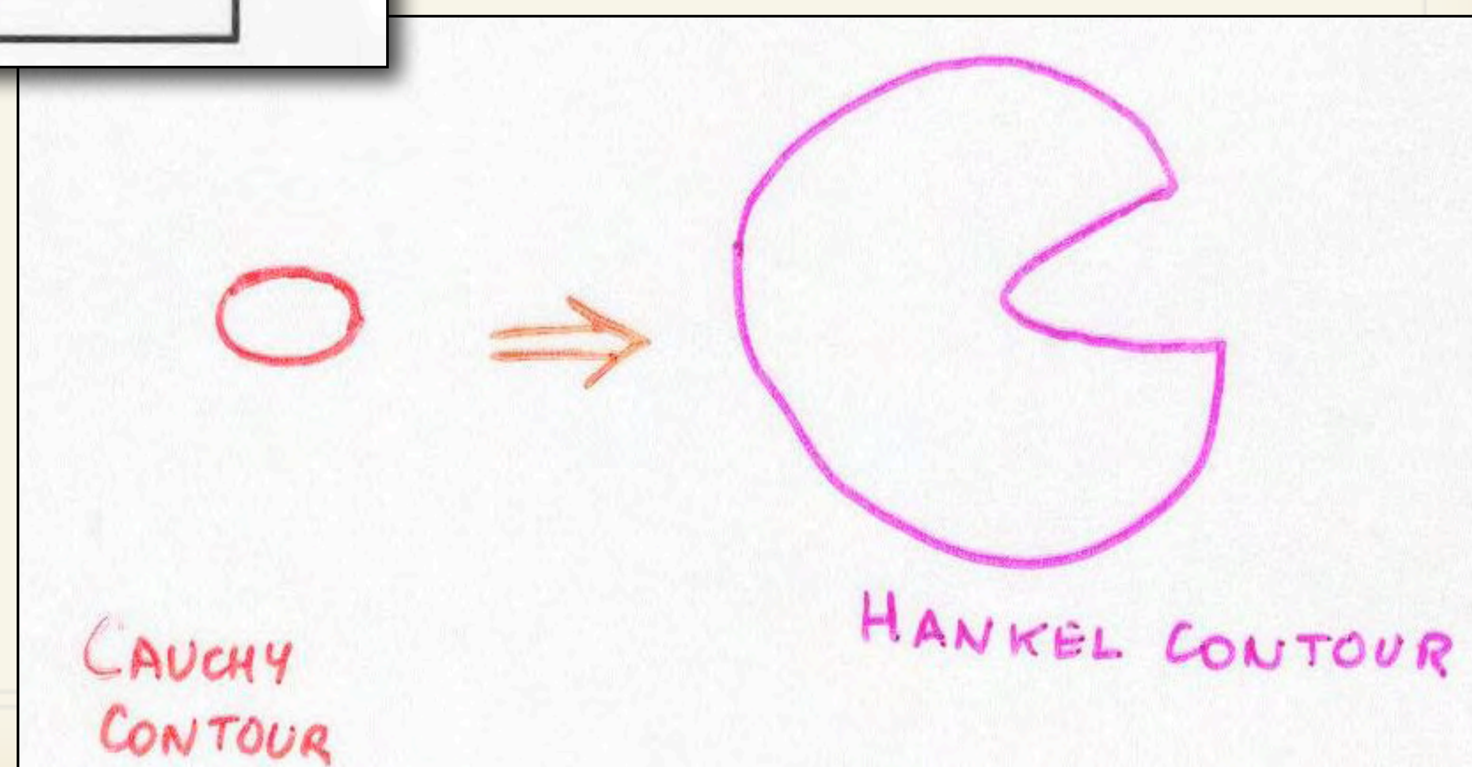
Also: full expansion, log's log-log's, etc.

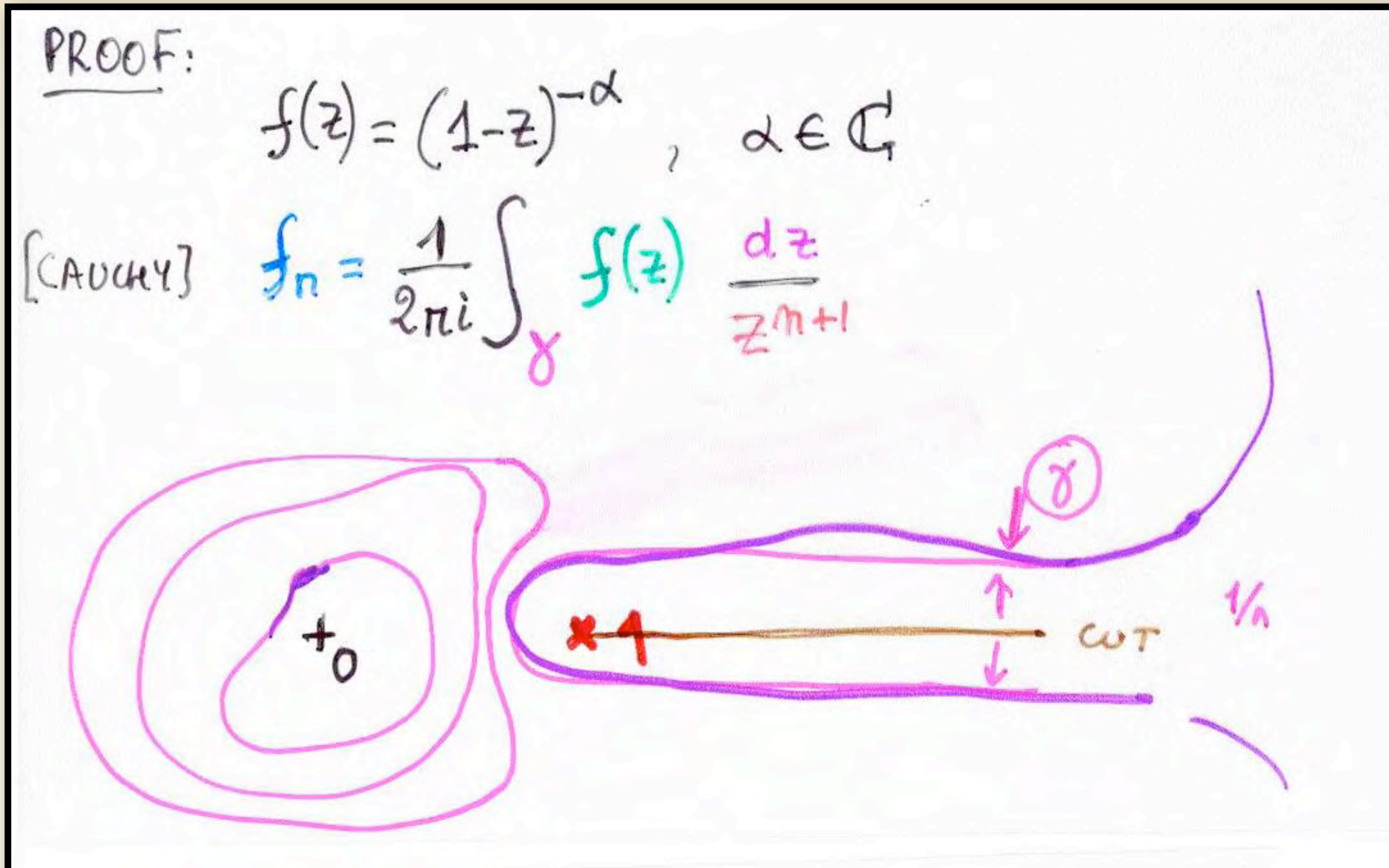
Gamma function: $\Gamma(s) := \int_0^\infty e^{-t} t^{s-1}\,dt$, with analytic continuation by $\Gamma(s+1) = s\Gamma(s)$.

$(1-z)^{-\alpha}$

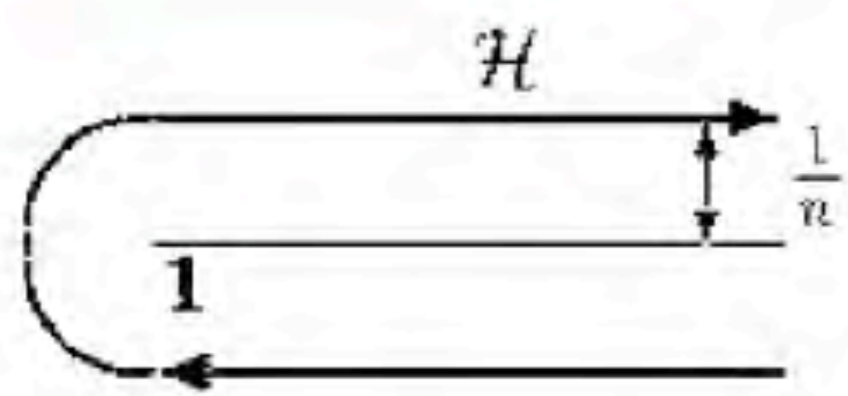
$$\mathrm{Coeff}\,[z^n]\,f(z) = \frac{1}{2i\pi}\oint f(z)\,\frac{dz}{z^{n+1}}.$$







$$f(z) = (1-z)^{-\alpha}$$



$$[z^n]f(z) = \frac{1}{2i\pi}\int_\gamma f(z)\,\frac{dz}{z^{n+1}}$$

$$\Downarrow \qquad \left(z = 1 + \frac{t}{n}\right) \qquad \Downarrow$$

$$\frac{1}{2i\pi}\int_{\mathcal{H}} \left(-\frac{t}{n}\right)^{-\alpha} e^{-t}\,\frac{dt}{n}$$

$$n^{\alpha-1} \times \frac{1}{\Gamma(\alpha)}.$$

Theorem 2. Transfer of asymptotic properties.

If $f(z) = O((1-z)^{-\alpha})$ as $z \to 1$ in a Camembert region

Then

$$\text{coeff}\,[z^n]\, f(z) = O(n^{\alpha-1})$$

* same for $\Theta(-)$; * same for log's, etc..

Proof: similarly by Hankel contours.

"ALGORITHM": ⊛ Make sure function exists in larger region than disc.

⊛ Expand as $z \to 1$

⊛ Translate formally (and safely) by

THEOREMS 1+2 .

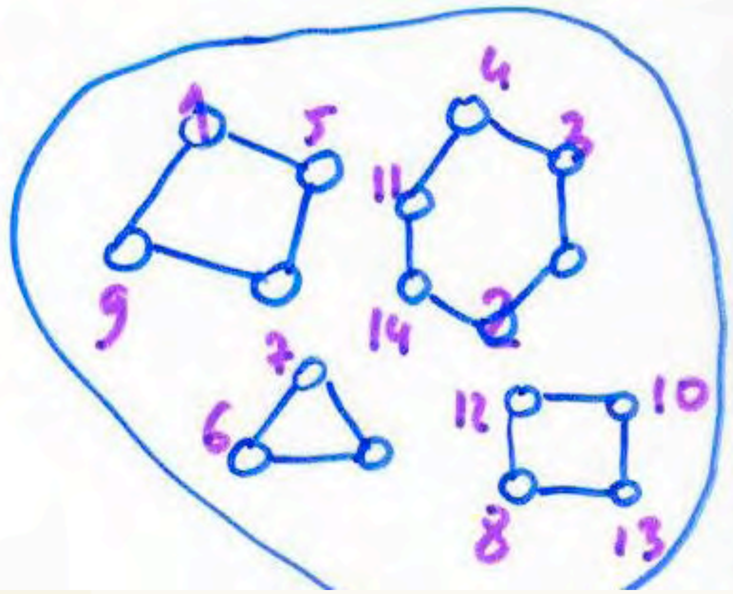
2-regular graphs

[Each node has exact degree = 2]

$$\mathcal{R} = \text{Set}(\text{UnorderedCycle}(\mathcal{Z}, \text{card} \geq 3))$$

$$R(z) = \exp\left(\frac{1}{2}\log\frac{1}{1-z} - \frac{z}{2} - \frac{z^2}{4}\right)$$

$$R(z) = \frac{e^{-\frac{z}{2}-\frac{z^2}{4}}}{\sqrt{1-z}}$$



By singularity analysis,

$$R(z) \sim \frac{e^{-3/4}}{\sqrt{1-z}}$$

$$\frac{R_n}{n!} \sim e^{-3/4}\binom{2n}{n}4^{-n}$$

$$\sim \frac{e^{-3/4}}{\sqrt{\pi n}}$$

TREES (Catalan model, binary variety)

$\mathcal{B} = \square +$ 

$$B(z) = \frac{1-\sqrt{1-4z}}{2z}$$

$\mathrm{Sing}(B) = \frac{1}{4}$; exponent: $\alpha = -\frac{1}{2}$ in form $(1-z)^{-\alpha}$

$$B_n \sim \frac{1}{4\sqrt{\pi n^3}} \cdot 4^n .$$

- Unary binary trees.

$$T = z + zT + zT^2$$

$$\Rightarrow \quad T = \frac{1 - z - \sqrt{1 - 2z - 3z^2}}{2z}$$

$$1 - 2z - 3z^2 = (1 - 3z)(1 + z)$$

$\Rightarrow$ $\sqrt{-}$ singularity @ $\frac{1}{3}$,

$$T_n \sim c \cdot 3^n n^{-3/2}$$

In fact: *universality* of $n^{-3/2}$ law (later).

Example 3. Cycles in Perms.

Mean number of cycles in a random perm is coeff$[z^n]$ in

$$M(z) = \frac{\partial}{\partial u} \exp\left(u \log \frac{1}{1-z} \right)\Bigg|_{u \to 1} = \frac{1}{1-z} \log \frac{1}{1-z}.$$

Thus $\boxed{[z^n]M(z) \sim \log n}$.

Exercise: Holds for perms with finitely many excluded cycle lengths.

In fact: *universality* for the "exp-log" schema.

Generating Function $\rightsquigarrow$ Coefficients

Solving a "Tauberian" problem

Real–Tauberian	Darboux-Pólya	Singularity An.

(large $\Longrightarrow$ large)	(smooth $\Longrightarrow$ small)	(Full mappings)

+ Singularity analysis preserves uniformity $\rightsquigarrow$ distributions.

Closures

Theorem 3. *Generalized polylogarithms*

$$\mathrm{Li}_{\alpha,k} := \sum (\log n)^k n^{-\alpha} z^n$$

are of S.A.-type.

Theorem 4. *Functions of S.A.-type are closed under integration and differentiation.*

Theorem 5. *Functions of S.A.-type are closed under Hadamard product*

$$f(z) \odot g(z) := \sum_n (f_n g_n) z^n.$$

Chapter 7

Applications of Singularity Analysis

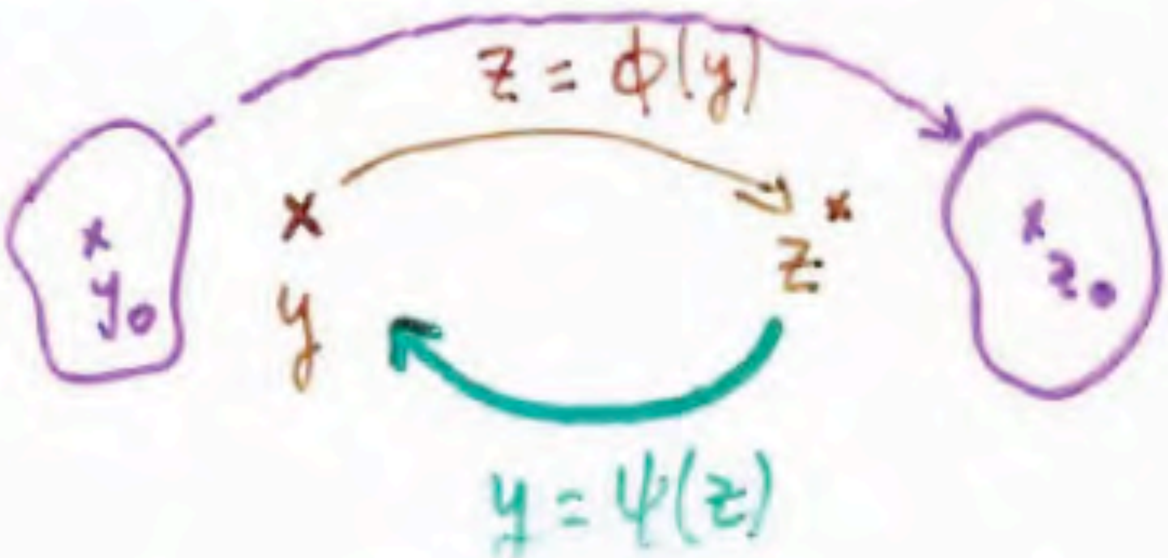
Recursive structures

— TREES

- Universality of the $\sqrt{\ }$- law for functions

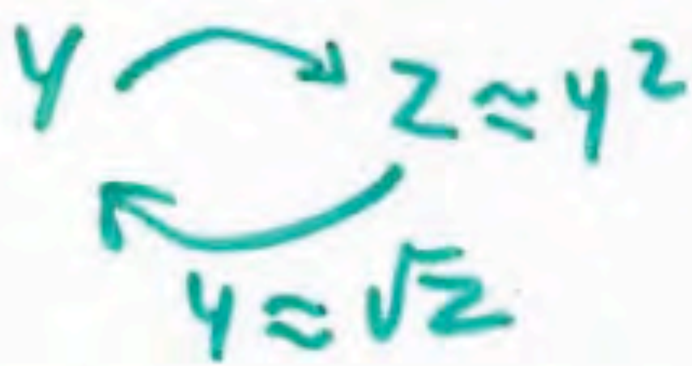
- Universality of $\rho^{-n} n^{-3/2}$ estimate for counts

Inversion:



Inversion Theorem: ϕ is analytically invertible iff $\phi'(y_0) \neq 0$.

If not invertible
$\phi''(y_0) \neq 0$:



Square-root singularity is expected for inverse functions.

Cayley trees : $T = z e^{T}$ or $z = T e^{-T}$

not invertible if $\frac{d}{dT}(Te^{-T}) = (1-T)e^{-T} = 0$,

that is $T = 1$; $z = e^{-1}$

square-root singularity

$$T(z) \underset{z \to e^{-1}}{\sim} 1 - \sqrt{2}\sqrt{1-ez} + O((1-ez))$$

$$[z^n]\, T(z) \sim \frac{e^n}{\sqrt{2\pi n^3}}$$

$$\left(= \frac{n^{n-1}}{n!} \right)$$

Theorem 1. Let ϕ have nonnegative coeffs and be entire. Then the function that solves

$$Y(z) = z\phi(Y(z))$$

has a square-root singularity, so that

$$[z^n]Y(z) \sim C\rho^{-n}n^{-3/2}.$$

— Characteristic equation (singular value of Y) is $\tau : \frac{d}{dy}\frac{y}{\phi(y)} = 0$, i.e., $\tau\phi'(\tau) - \phi(\tau) = 0$. Then $\rho = \frac{\tau}{\phi(\tau)}$. All is computable.

— $\sqrt{\ }$–singularity propagates via suitable compositions, so that parameters can be analysed.

— Phenomena are robust.

functional equation

Example 2. Unlabelled trees. Recall

$$U(z) = ze^{U(z)+\frac{1}{2}U(z^2)+\cdots}.$$

Express as T composed with an analytic function and get SQRT sing:
$U = \zeta e^U$, where $\zeta := z\exp(\frac{1}{2}U(z^2) + \cdots)$.

- Using BGF's and singularities
  - Meir & Moon: Path length is on average $\sim c\, n^{3/2}$
  - F. Odlyzko: Height is on average $\sim c' n^{1/2}$
  - Marckert et al.: Width is on average $\sim c'' n^{1/2}$

Height is universally $O(\sqrt{n})$ with local and integral limit laws (of theta type). Similarly for width (Marckert et al.). Leaves are universally normally distributed, etc.

Example 3. Mappings (cyclic points).

$$\begin{cases} \text{graph:} & \mathcal{G} = \mathrm{Set}(\mathcal{K}) \\ \text{connected:} & \mathcal{K} = \mathrm{Cyc}(u\,\mathcal{T}) \\ \text{tree:} & \mathcal{T} = \circ \star \mathrm{Set}(\mathcal{T}) \end{cases} \qquad \begin{cases} G = e^{K} \\ K = \log \dfrac{1}{1-uT} \\ T = z e^{T} \end{cases}$$

Mean number of cyclic points is

$$\mu_n = \frac{[z^n]\ \partial/\partial u\, G\,|_{u=1}}{[z^n]\ G\,|_{u=1}}, \leftarrow \quad G = \frac{1}{1-uT}$$

Develop a theory of degree-constrained mappings: (Arney-Bender), (F.-Odlyzko).

Mean number of cyclic points is

$$\mu_n = \frac{[z^n]\ \partial/\partial u\, G\,|_{u=1}}{[z^n]\quad G\,|_{u=1}}, \longleftarrow G = \frac{1}{1-uT}$$

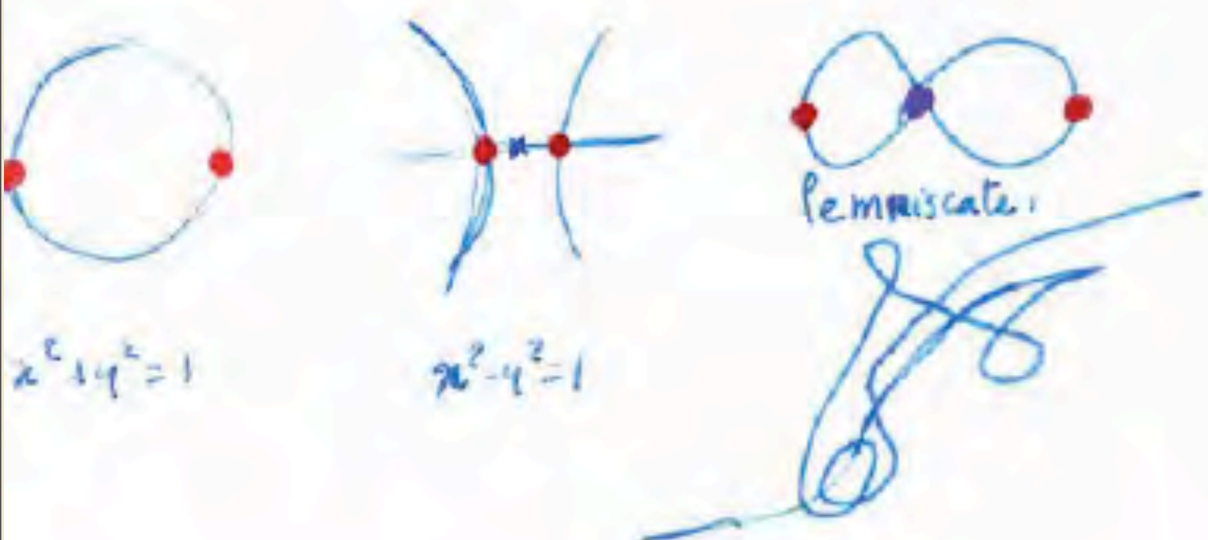
$$= \frac{[z^n]\ T/(1-T)^2}{[z^n]\ 1/(1-T)}$$

$$\sim \frac{[z^n]\, 2(1-ez)^{-1}}{[z^n]\, \sqrt{2}\,(1-ez)^{-1/2}} \quad \begin{matrix} \longleftarrow e^n n^0 \\ \longleftarrow e^n n^{-1/2} \end{matrix}$$

$$\mu_n \equiv \text{Mean \# cyclic points} \sim \sqrt{\frac{\pi n}{2}}$$

Algebraic functions

Singularity analysis applies to any algebraic function



NEWTON-PUISEUX THEOREM

Around any point ζ, $y(z)$ admits a fractional power expansion

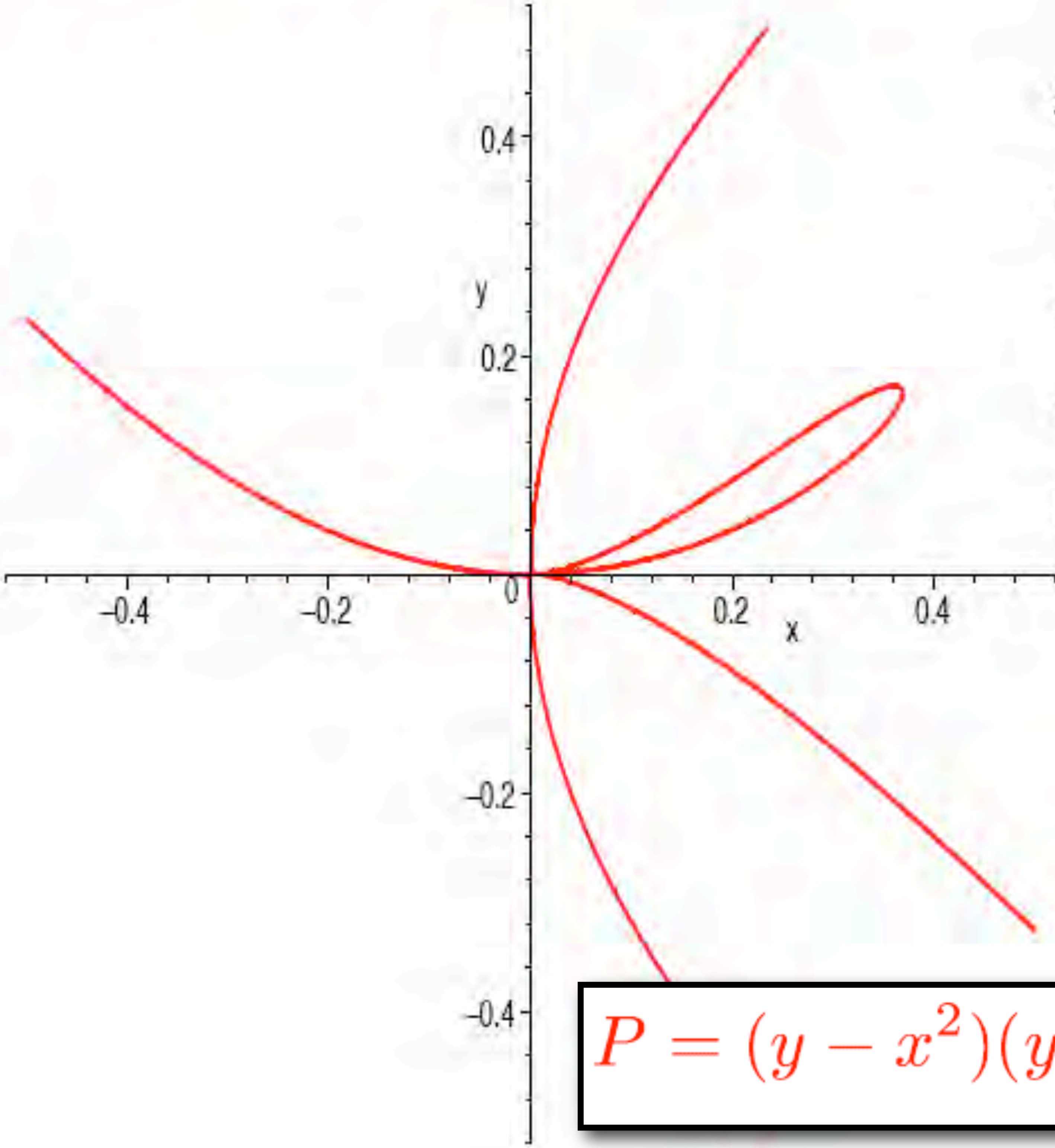
$$y(z) = \sum_{j \geq -m} c_j (z-\zeta)^{\alpha j} \qquad \alpha = p/q \in \mathbb{Q}$$

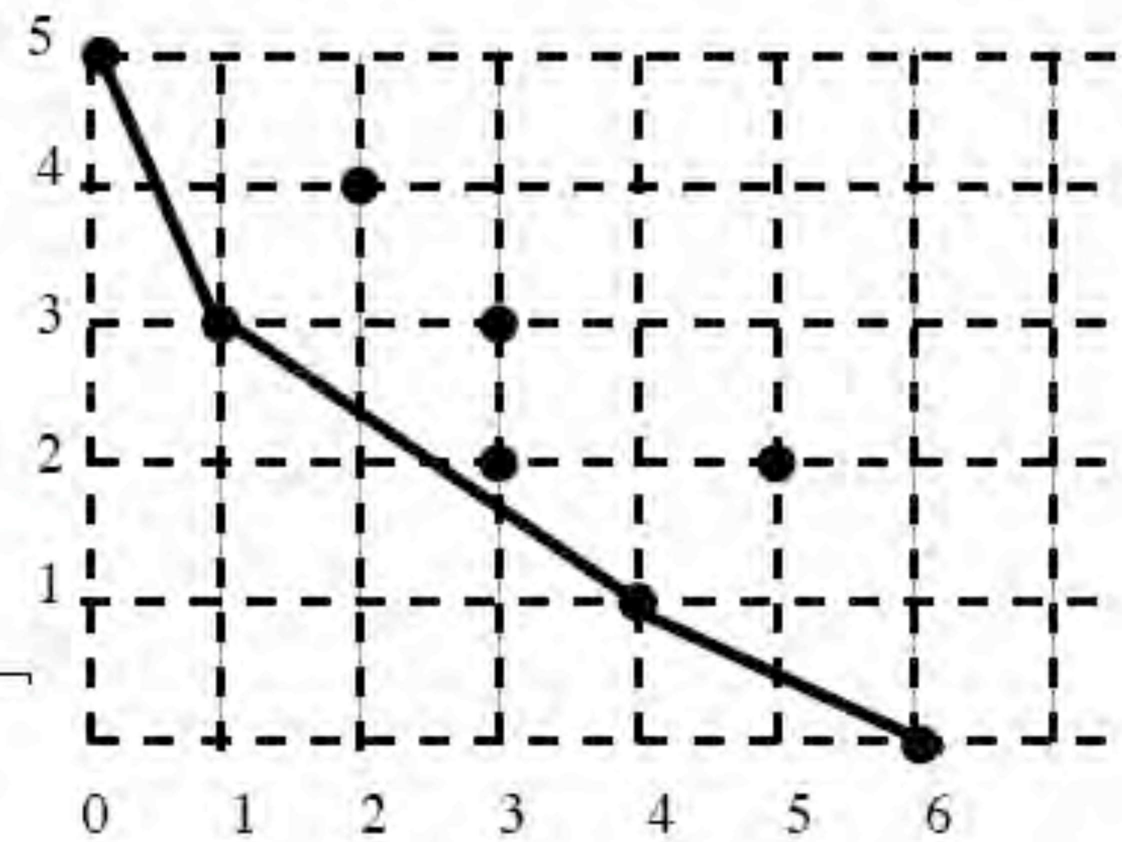
Algebraic function $\Longrightarrow$ Fractional exponents @ singularities.

Theorem [Newton-Puiseux] At a singularity solutions group themselves into "cycles" that can be expanded into fractional power series (Puiseux series):

$$y(z) = H((z - z_0)^{1/r}), \qquad H(w) = \sum_{j=-m}^{\infty} h_j w^j.$$

Newton diagram





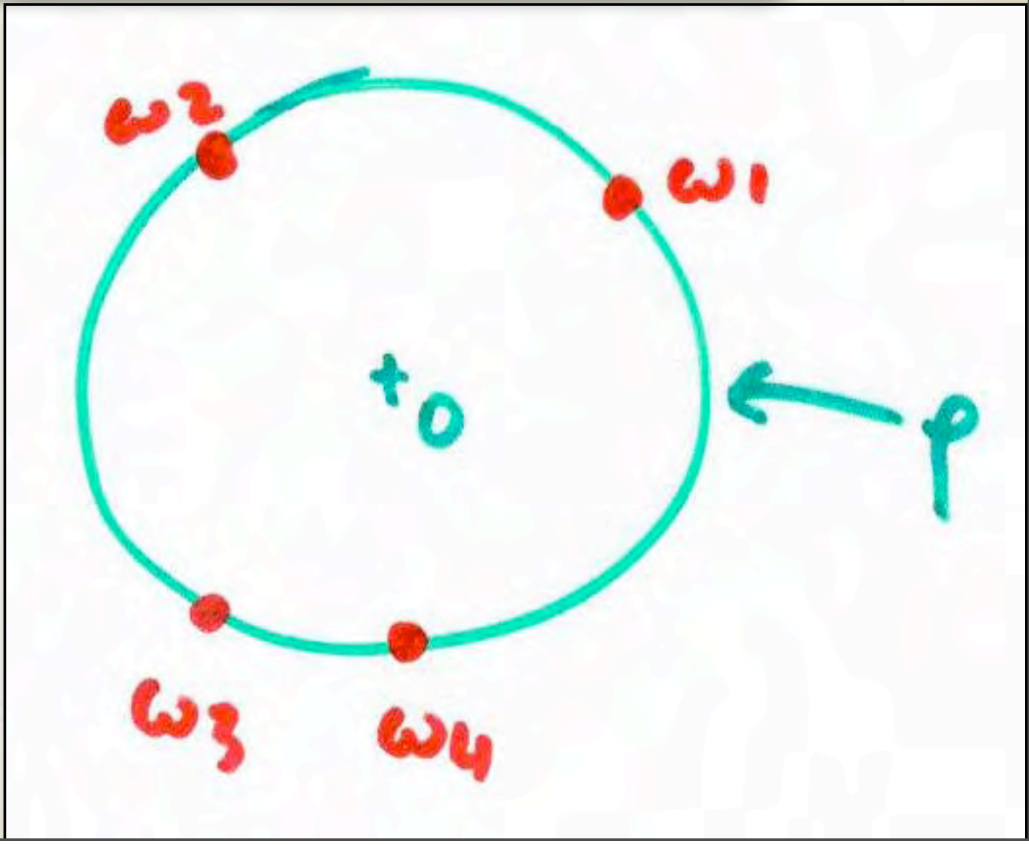
$$P = (y - x^2)(y^2 - x)(y^2 - x^3) - x^3y^3$$

Theorem Let $f(z)$ be an alg. fun. (branch).
— One dominant singularity at α_1:

$$f_n \sim \alpha_1^{-n} \left(\sum_{k \geq k_0} d_k n^{-1-k/\kappa} \right),$$

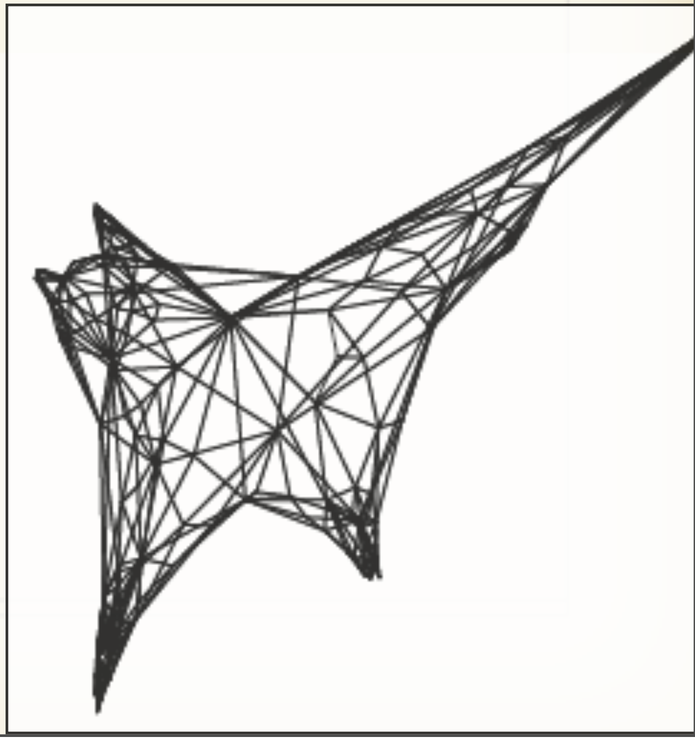
where $k_0 \in \mathbb{Z}$ and κ is an integer ≥ 2. — Several: a finite linear combination of such plus exponentially smaller error terms.

Exponential $* \; n^{rational}$

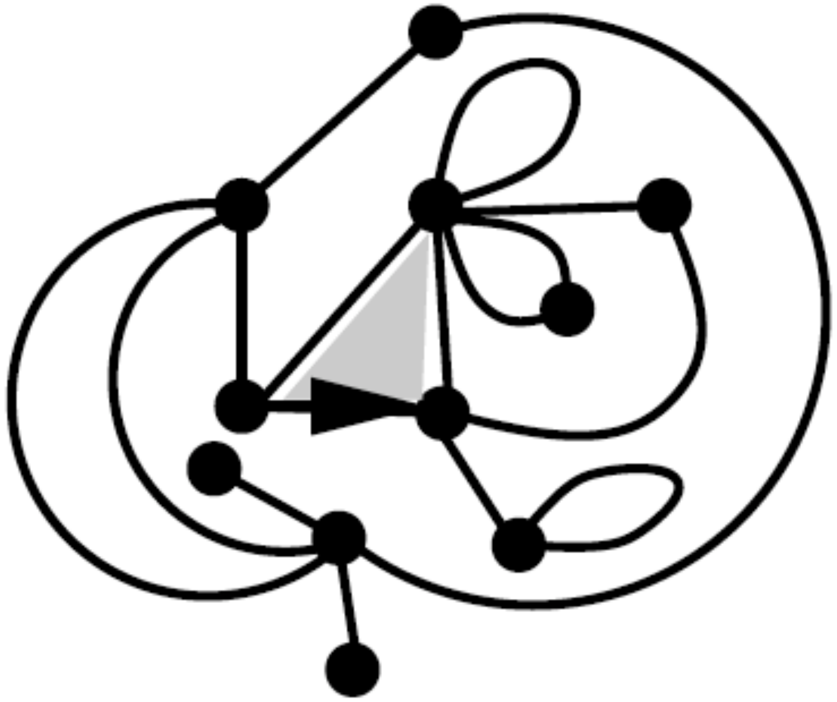


APPLICATIONS of ALGEBRAIC FUNCTIONS

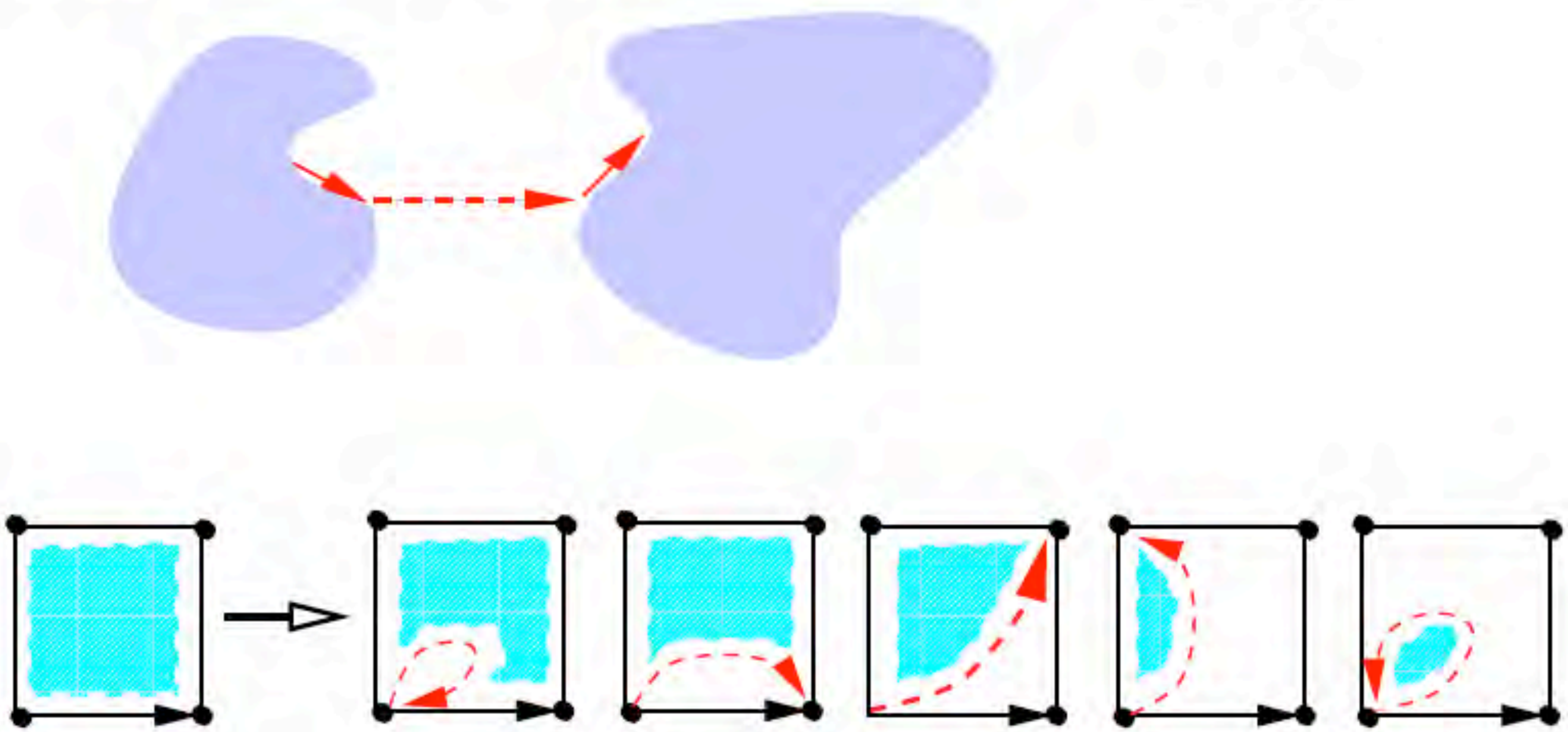
- Trees with a finite set of node degrees
- Excursions with finite set of steps [Lalley, BaFl]
- Maps embedded into the plane [Tutte,...]
  Gimenez-Noy: Planar graphs
- Context-free structures =
  Drmota-Lalley-Woods Thm.



Maps



$$M(z,u) = 1 + u^2 z M(z,u)^2 + uz \frac{M(z,1) - uM(z,u)}{1-z}.$$



$$u^4 \quad \mapsto \quad zu^5 \; + \; zu^4 \; + \; zu^3 \; + \; zu^2 \; + \; zu^1.$$

Theorem [DLW]. Assume positive irreducible system.

All y_j have same dominant singularity ρ.
$\exists$ functions h_j analytic at the origin such that

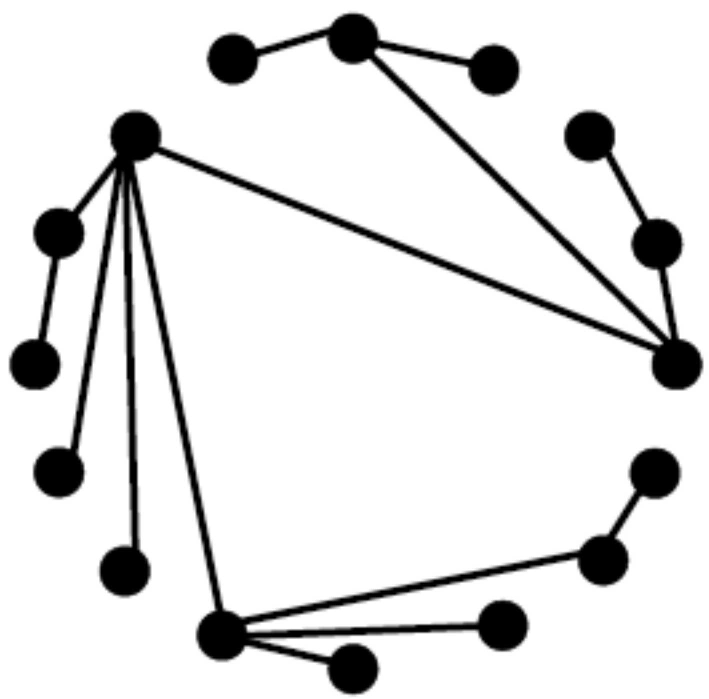
$$y_j = h_j\left(\sqrt{1 - z/\rho}\right) \qquad (z \to \rho^-).$$

All other dominant sing. of the form $\rho\omega^j$, with ω root of unity—this, iff strongly periodic.

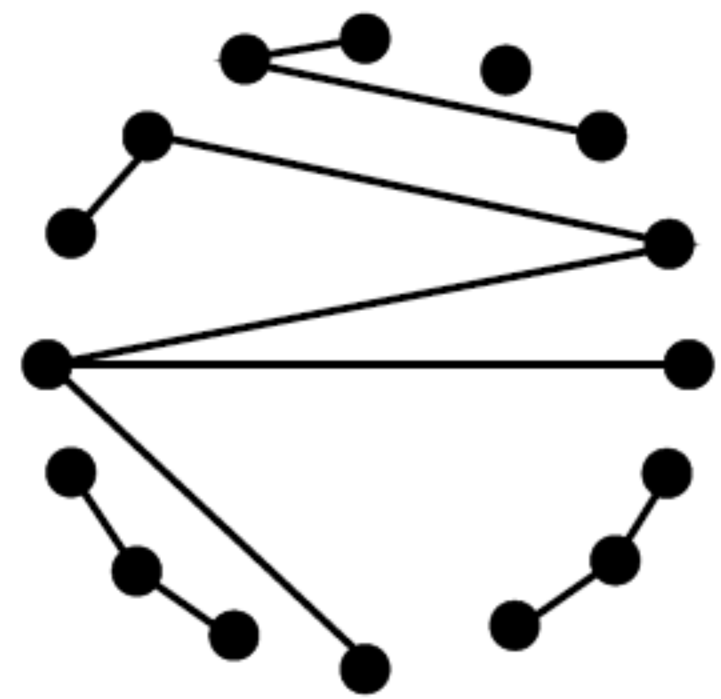
Asymptotics of the form (single sing.)

$$[z^n]y_j(z) \sim \rho^{-n}\left(\sum_{k\geq 1} d_k n^{-1-k/2}\right).$$

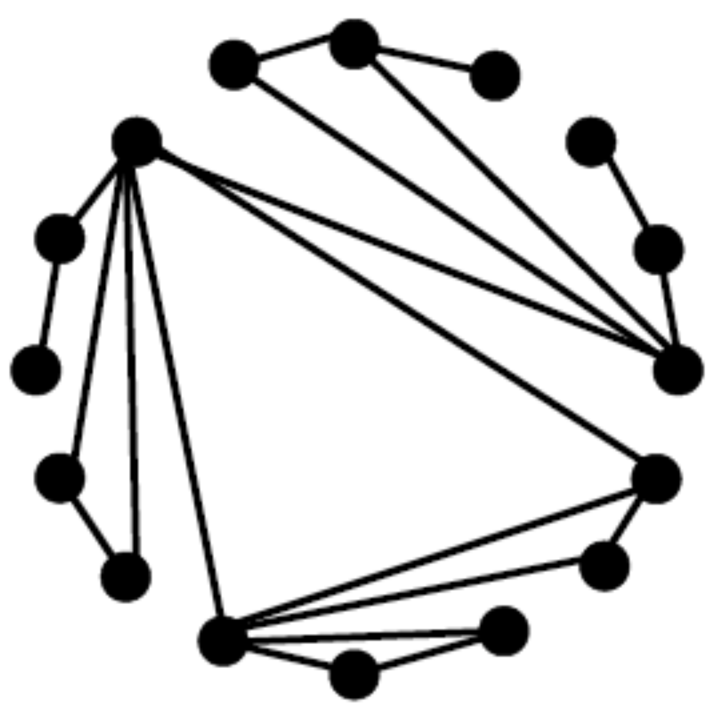
EXAMPLE. **Noncrossing graphs** [F.Noy, Disc. Math'99]



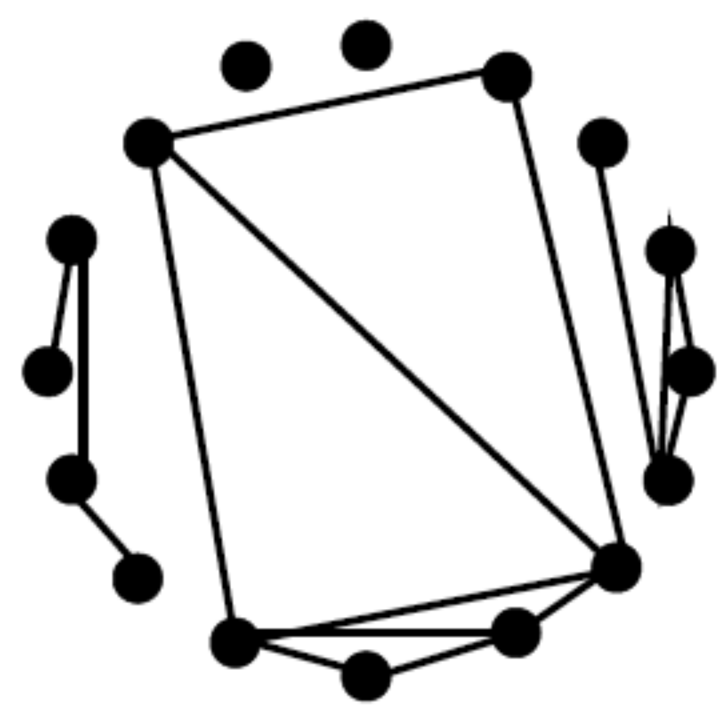
(tree)



(forest)



(connected graph)



(graph)

Configuration / OGF	Coefficients (exact / asymptotic)
Trees (*EIS*: **A001764**)	$z + z^2 + 3z^3 + 12z^4 + 55z^5 + \cdots$
$T^3 - zT + z^2 = 0$	$\frac{1}{2n-1}\binom{3n-3}{n-1}$
	$\sim \frac{\sqrt{3}}{27\sqrt{\pi n^3}}(\frac{27}{4})^n$
Forests (*EIS*: **A054727**)	$1 + z + 2z^2 + 7z^3 + 33z^4 + 181z^5 \cdots$
$F^3 + (z^2 - z - 3)F^2 + (z+3)F - 1 = 0$	$\sum_{j=1}^{n} \frac{1}{2n-j}\binom{n}{j-1}\binom{3n-2j-1}{n-j}$
	$\sim \frac{0.07465}{\sqrt{\pi n^3}}(8.22469)^n$
Connected graphs (*EIS*: **A007297**)	$z + z^2 + 4z^3 + 23z^4 + 156z^5 + \cdots$
$C^3 + C^2 - 3zC + 2z^2 = 0$	$\frac{1}{n-1}\sum_{j=n-1}^{2n-3}\binom{3n-3}{n+j}\binom{j-1}{j-n+1}$
	$\sim \frac{2\sqrt{6}-3\sqrt{2}}{18\sqrt{\pi n^3}}\left(6\sqrt{3}\right)^n$
Graphs (*EIS*: **A054726**)	$1 + z + 2z^2 + 8z^3 + 48z^4 + 352z^5 + \cdot$
$G^2 + (2z^2 - 3z - 2)G + 3z + 1 = 0$	$\frac{1}{n}\sum_{j=0}^{n-1}(-1)^j\binom{n}{j}\binom{2n-2-j}{n-1-j}2^{n-1-}$
	$\sim \frac{\sqrt{140-99\sqrt{2}}}{4\sqrt{\pi n^3}}\left(6+4\sqrt{2}\right)^n$

Singularity Analysis applies to

- non-linear ODEs = models of "logarithmic trees"
- the holonomic framework = solutions of linear ODEs with rational coefficients.

• "Holonomic" functions. Defined as solutions of linear ODE's with coeffs in $\mathbb{C}(z)$ [Zeilberger] $\equiv$ $\mathcal{D}$-finite.

$$\mathcal{L}[f(z)] = 0, \qquad \mathcal{L} \in \mathbb{C}(z)[\partial_z].$$

• Stanley, Zeilberger, Gessel: Young tableaux and permutation statistics; regular graphs, constrained matrices, etc.

Fuchsian case (or "regular" singularity) $(Z^\beta \log^k Z)$:

$$[z^n]f(z) \approx \sum \omega^n n^\beta (\log n)^k, \qquad \omega, \beta \in \overline{\mathbb{Q}}, \quad k \in \mathbb{Z}_{\geq 0}.$$

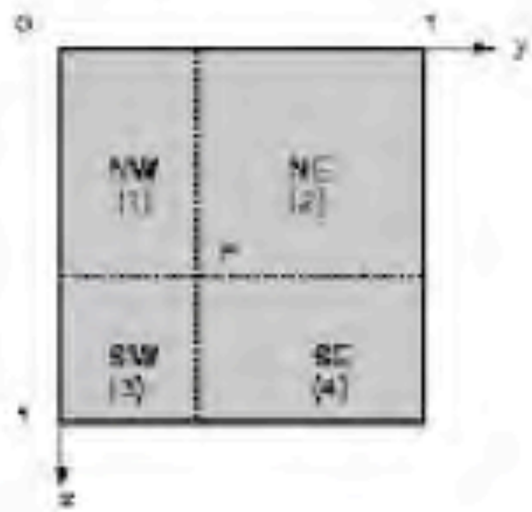
S.A. applies automatically to classical classification.

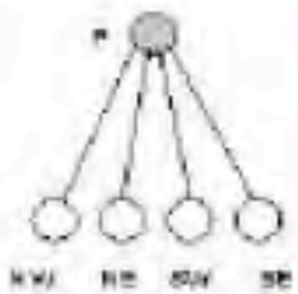
Asymptotics of coeff is decidable

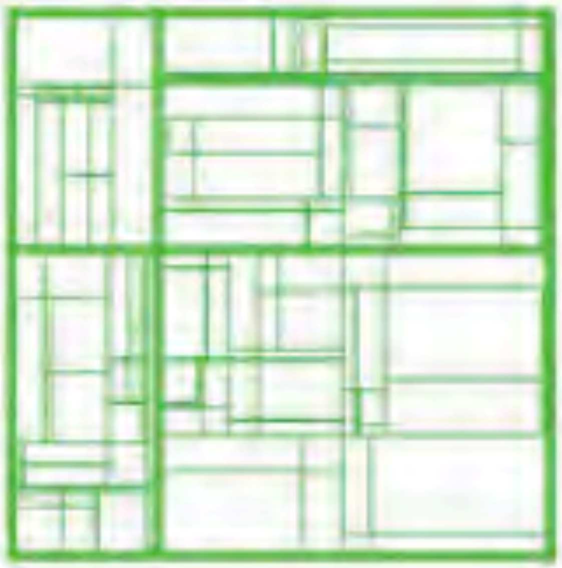
— general case: modulo oracle for connection problem;

— strictly positive case: "usually" OKay.

QTrees:







EXAMPLE 6. *Quadtrees—Partial Match* [FGPR'92]

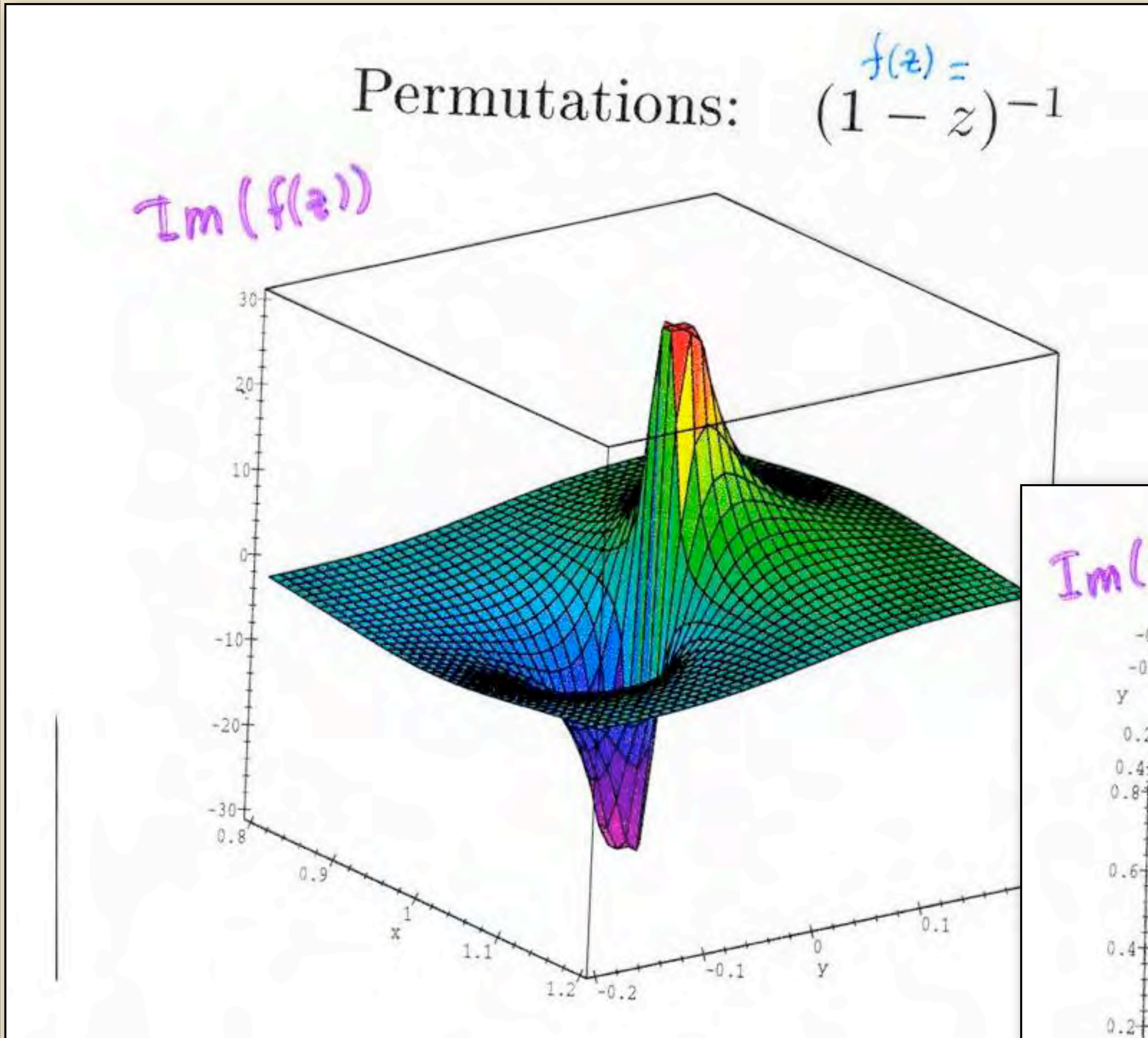
Divide-and-conquer recurrence with coeff. in $\mathbb{Q}(n)$
Fuchsian equation of order d (dimension) for GF

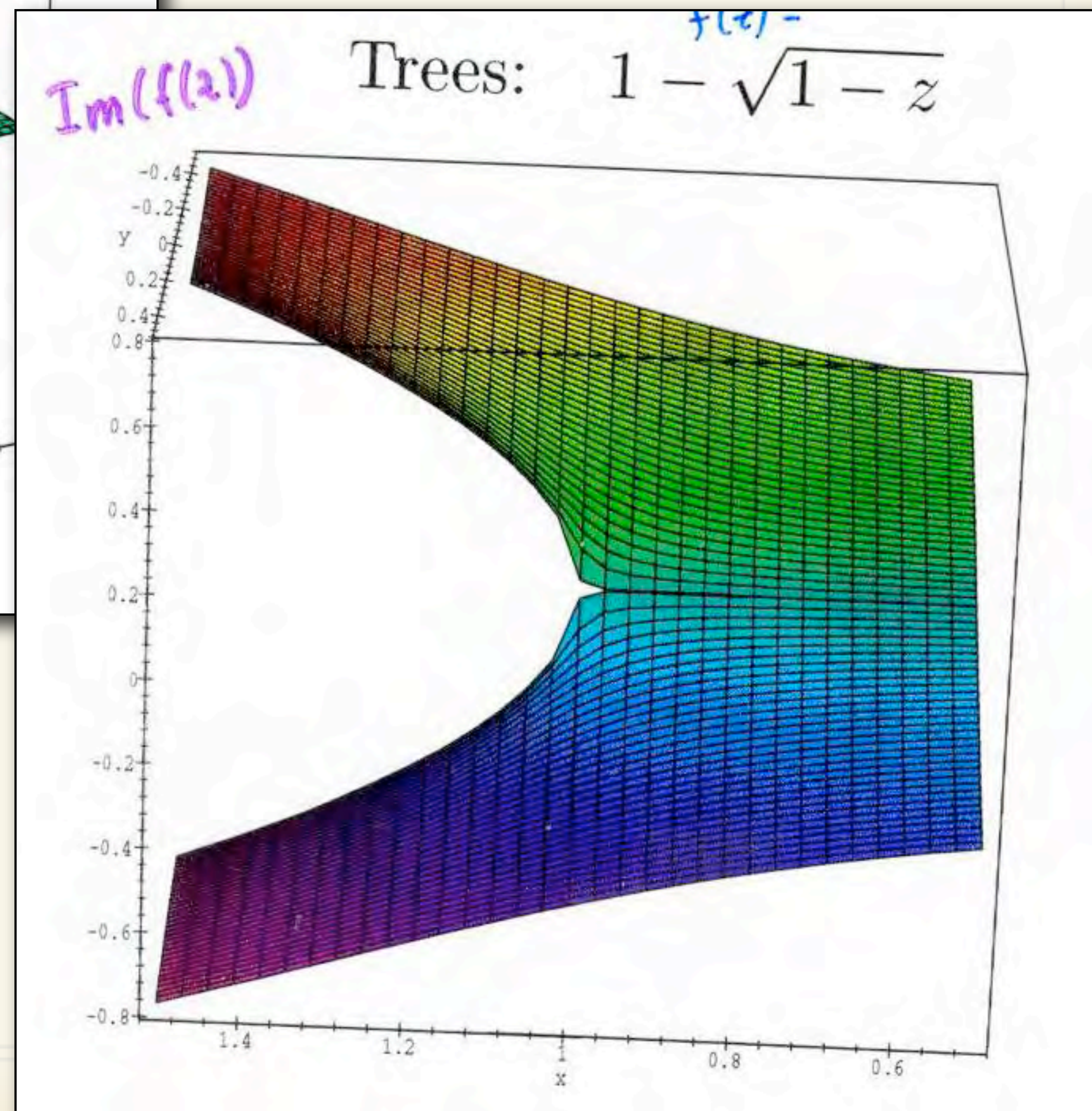
$$Q_n^{(d=2)} \asymp n^{(\sqrt{17}-3)/2}.$$

E.g., $d = 2$: Hypergeom ${}_2F_1$ with algebraic arguments.

Extended by Hwang et al. Cf also Hwang's *Cauchy ODE* cases.
Panholzer-Prodinger+Martinez, ...

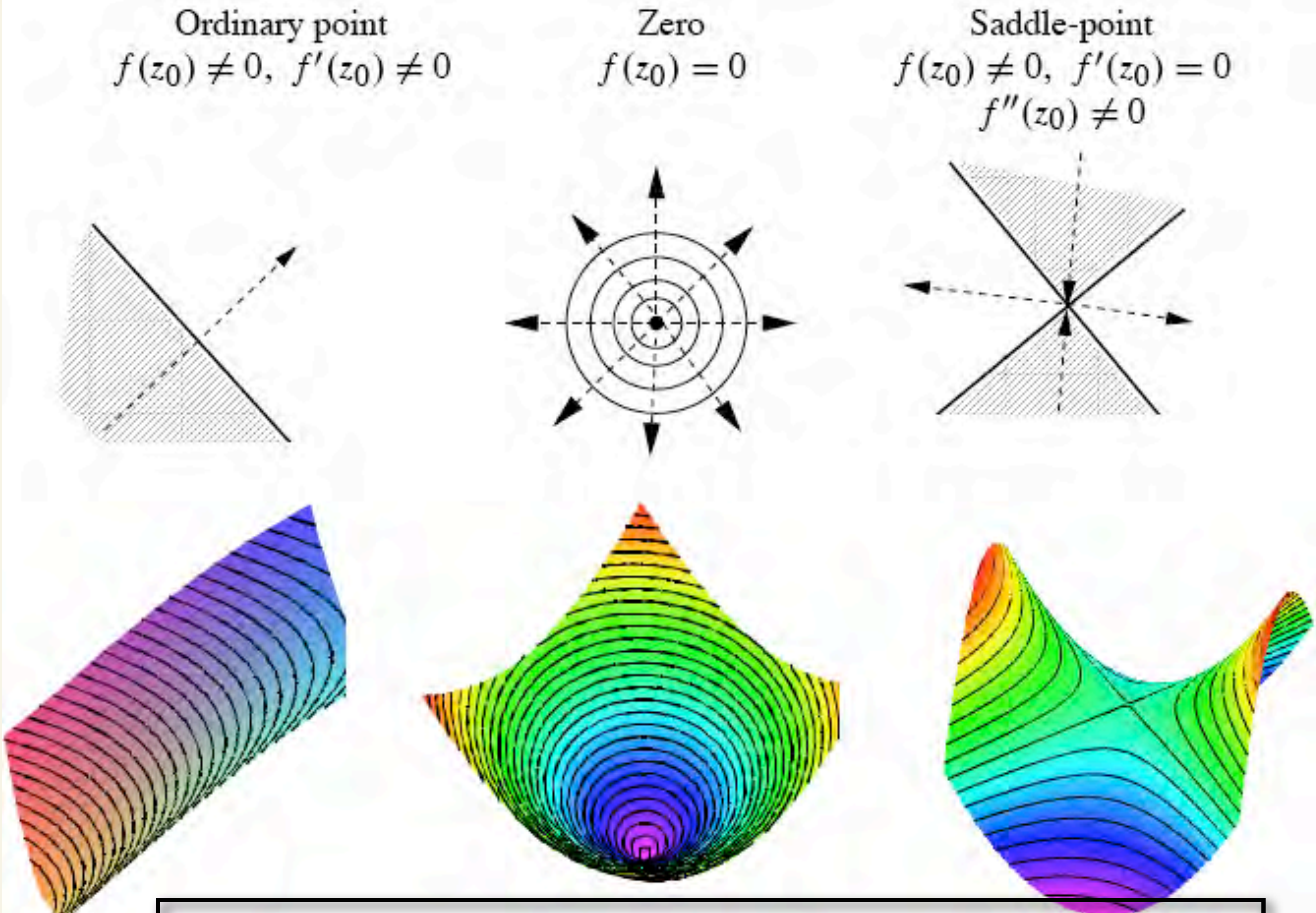
Conclusion: SCHEMAS





Chapter 8

Saddle-point Asymptotics



Modulus of an analytic function

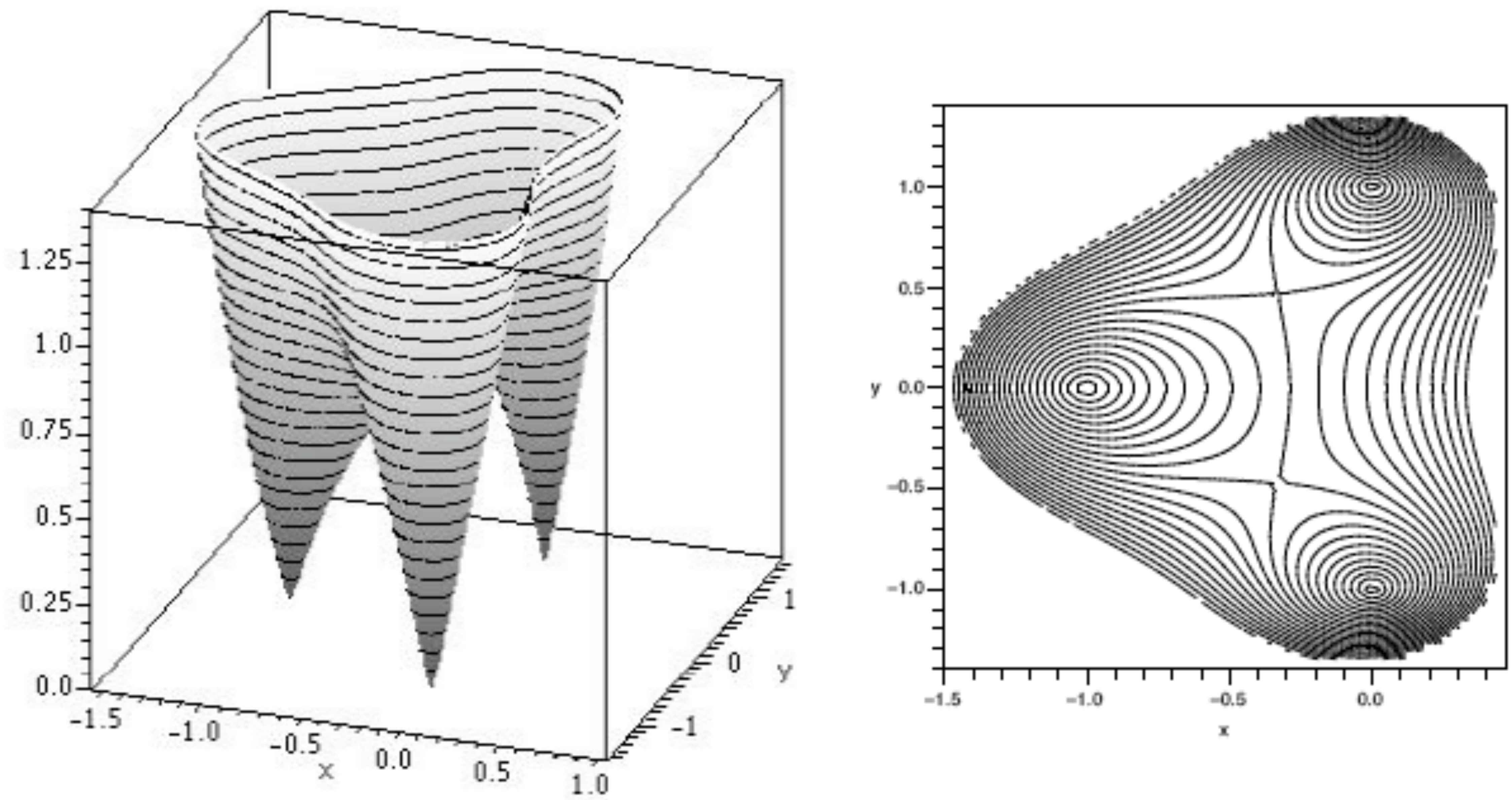
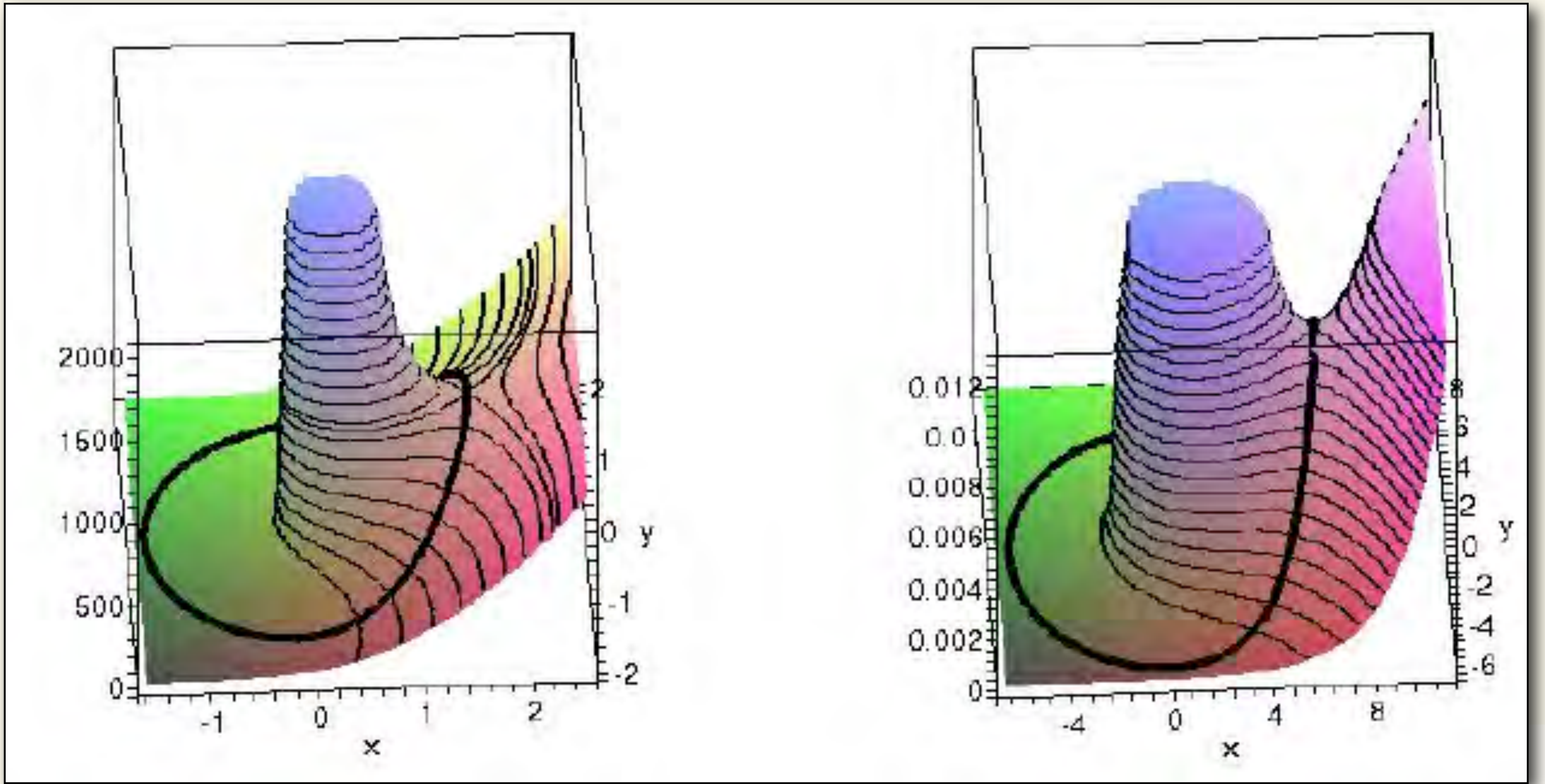


Figure VIII.2. The "tripod": two views of $|1+z+z^2+z^3|$ as function of $x = \Re(z)$, $y = \Im(z)$: (left) the modulus as a surface in $\mathbb{R}^3$; (right) the projection of level lines

Cauchy coefficient integrals



$$J_n = \frac{1}{2i\pi} \oint (1+z)^{2n} \frac{dz}{z^{n+1}}, \qquad K_n = \frac{1}{2i\pi} \oint e^z \frac{dz}{z^{n+1}},$$

Saddle-point method:

= concentration + local quadratic approximation.

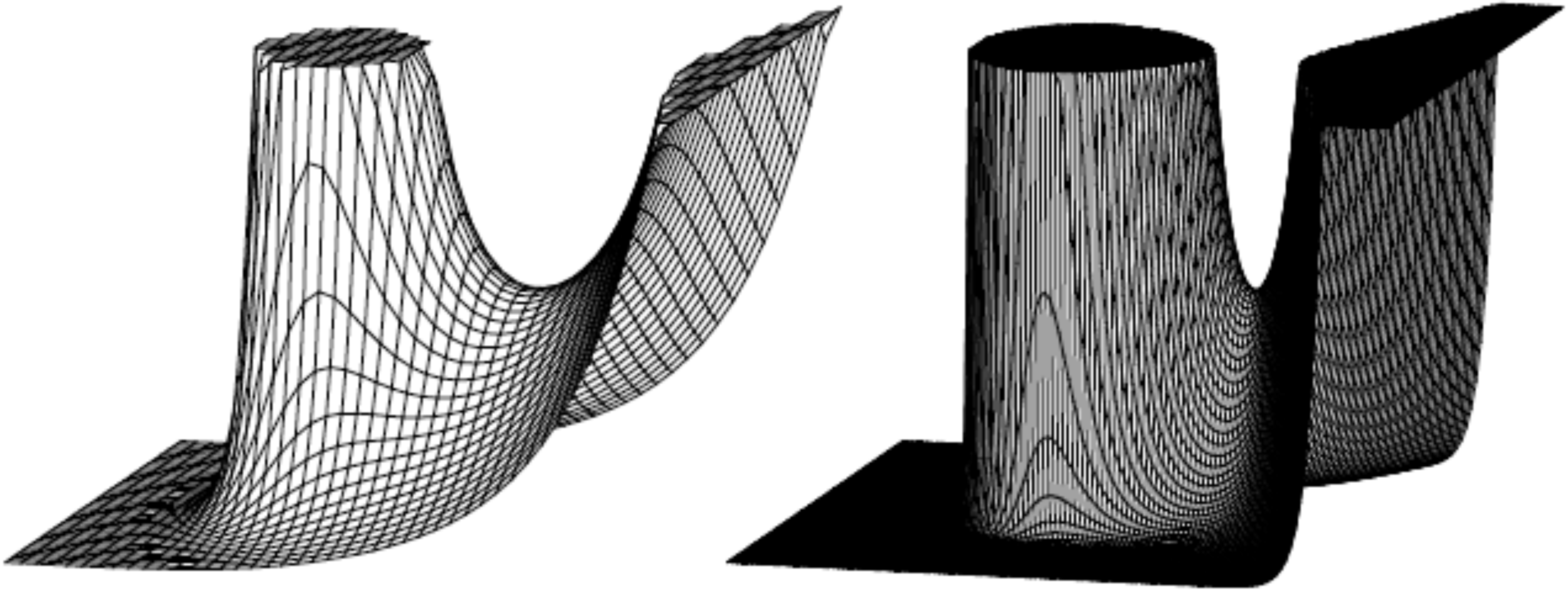


Figure VIII.5. Plots of $|e^z z^{-n-1}|$ for $n = 3$ and $n = 30$ (scaled according to the value of the saddle-point) illustrate the essential concentration condition as higher

The simple saddle point (review)

$$I(n) = \frac{1}{2i\pi} \int e^{h_n(z)}\, dz$$

saddle point at $\zeta \equiv \zeta_n \ : \ h'_n(\zeta) = 0$

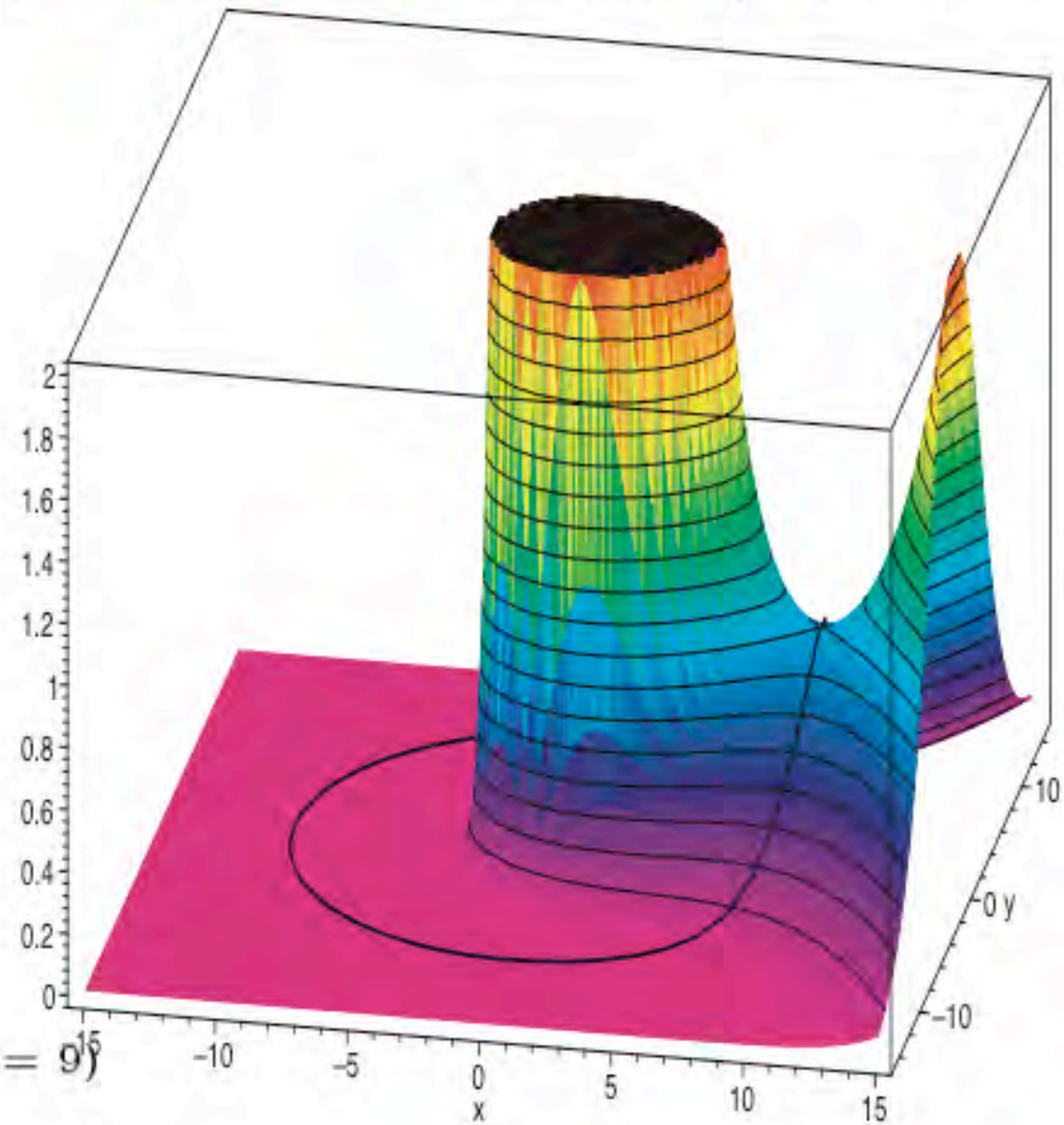
via: $e^{h_n(\zeta)} \int \exp(\frac{t^2}{2} h''_n(\zeta))\, dt$

$+$ concentration as $n \to \infty$

$$I(n) \sim \frac{1}{2\pi} \left((2!)^{1/2} \Gamma(1/2) \right) \frac{1}{\sqrt{h''_n(\zeta)}} e^{h_n(\zeta)}.$$

Normalization to:

Exp $\circ$ Quadratic; scales $\sqrt{n}$, ...



$\int e^z z^{-n-1}\, dz$ $(n = 9)$

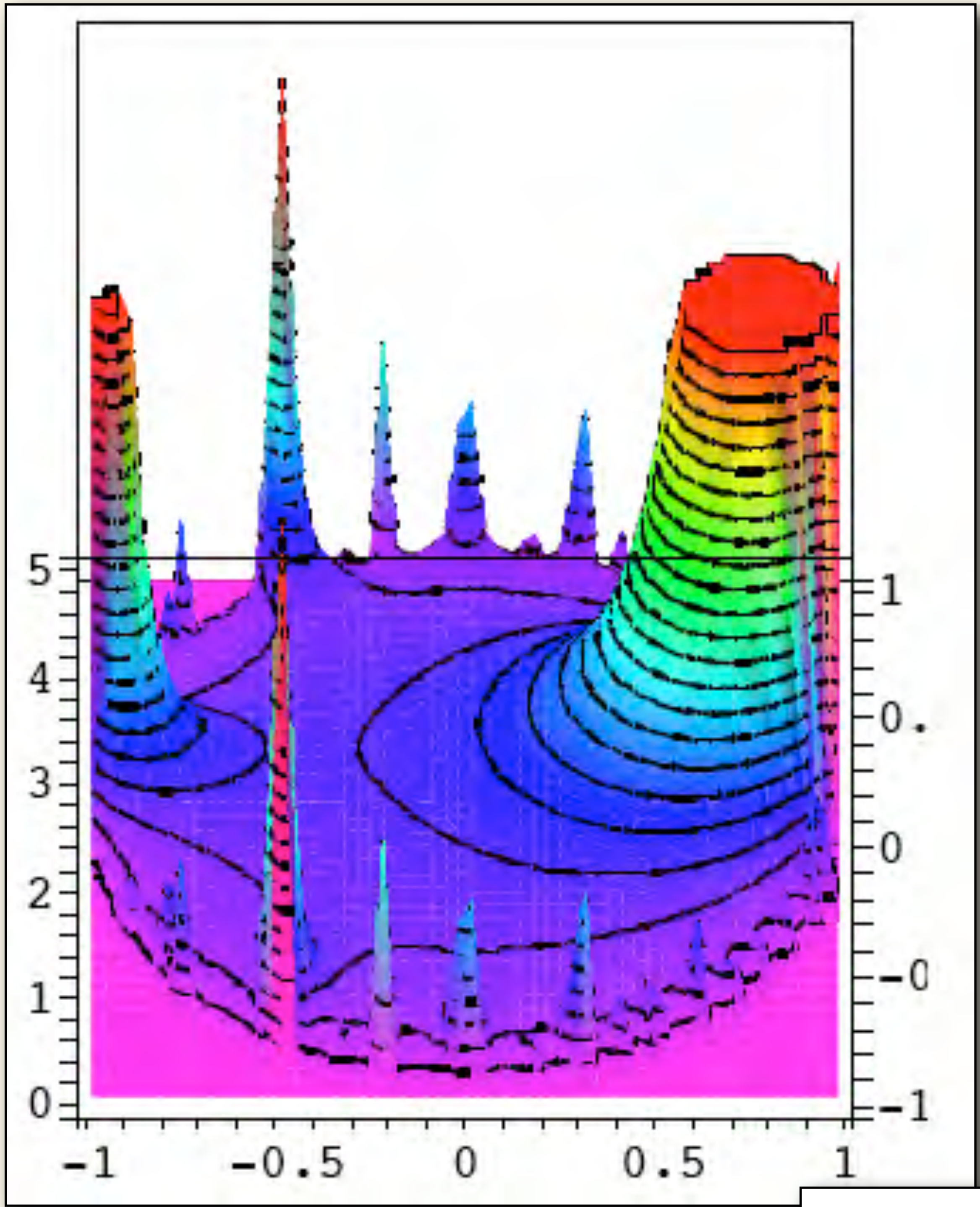
The saddle-point theorem:

$$\frac{1}{2i\pi}\int_A^B e^{f(z)}\,dz \sim \varepsilon e^{-i\phi/2}\frac{e^{f(\zeta)}}{\sqrt{2\pi|f''(\zeta)|}} = \pm\frac{e^{f(\zeta)}}{\sqrt{2\pi f''(\zeta)}}.$$

under conditions: concentration + local quadratic approximation

+ Hayman: *admissible functions*
= **closure properties**

Class	*EGF*	*radius* (r)	*angle* (θ_0)	*coeff* $[z^n]$ *in EGF*
urns $\text{SET}(\mathcal{Z})$ (Ex. VIII.3, p. 555)	e^z	n	$n^{-2/5}$	$\sim \frac{e^n n^{-n}}{\sqrt{2\pi n}}$
involutions $\text{SET}(\text{CYC}_{1,2}(\mathcal{Z}))$ (Ex. VIII.5, p. 558)	$e^{z+z^2/2}$	$\sim \sqrt{n} - \frac{1}{2}$	$n^{-2/5}$	$\sim \frac{e^{n/2-1/4} n^{-n/2}}{2\sqrt{\pi n}} e^{\sqrt{n}}$
set partitions $\text{SET}(\text{SET}_{\geq 1}(\mathcal{Z}))$ (Ex. VIII.6, p. 560)	e^{e^z-1}	$\sim \log n - \log\log n$	$e^{-2r/5}/r$	$\sim \frac{e^{e^r-1}}{r^n \sqrt{2\pi r(r+1)e^r}}$
fragmented perms $\text{SET}(\text{SEQ}_{\geq 1}(\mathcal{Z}))$ (Ex. VIII.7, p. 562)	$e^{z/(1-z)}$	$\sim 1 - \frac{1}{\sqrt{n}}$	$n^{-7/10}$	$\sim \frac{e^{-1/2+2\sqrt{n}}}{2\sqrt{\pi} n^{3/4}}$

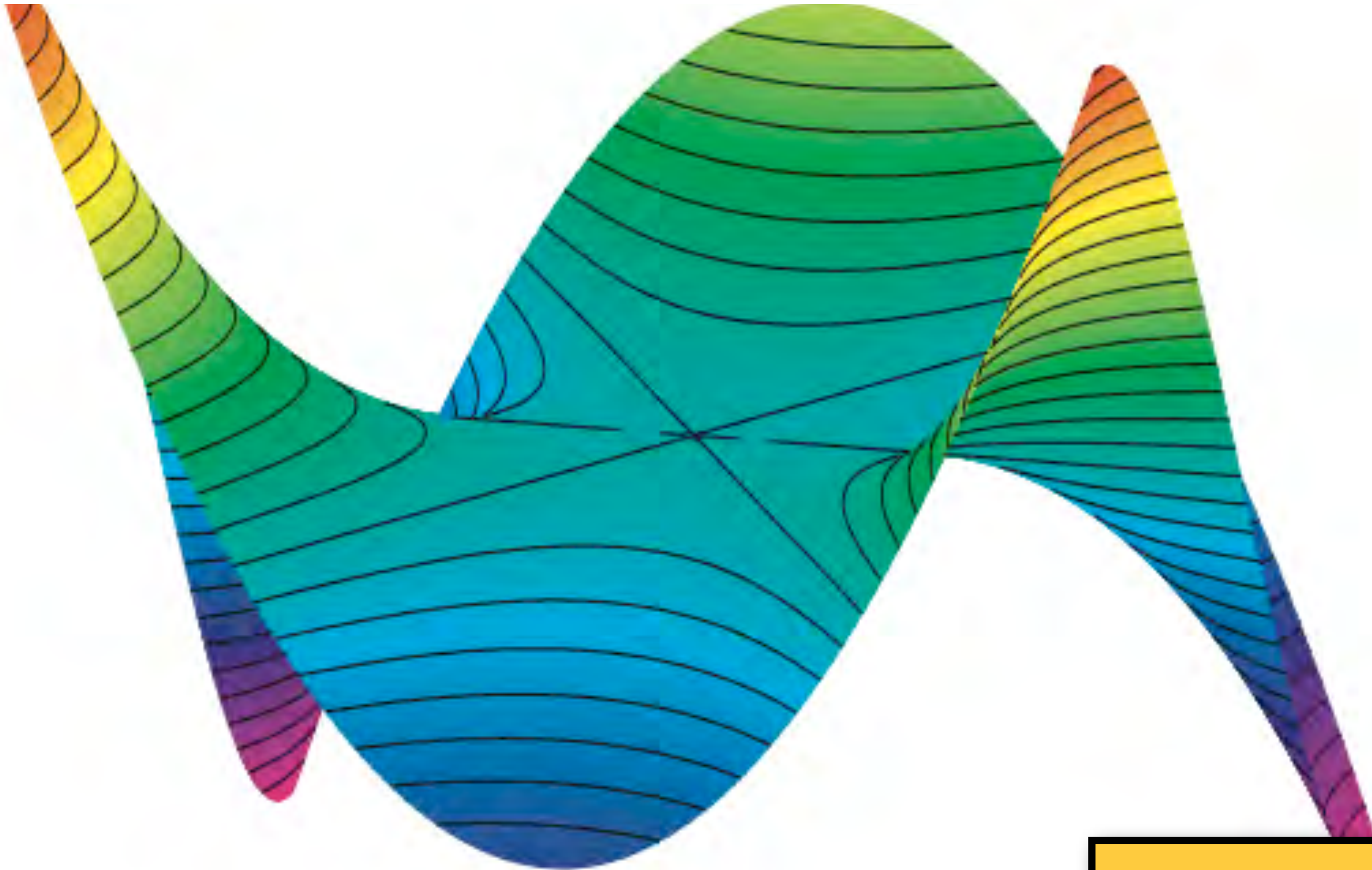


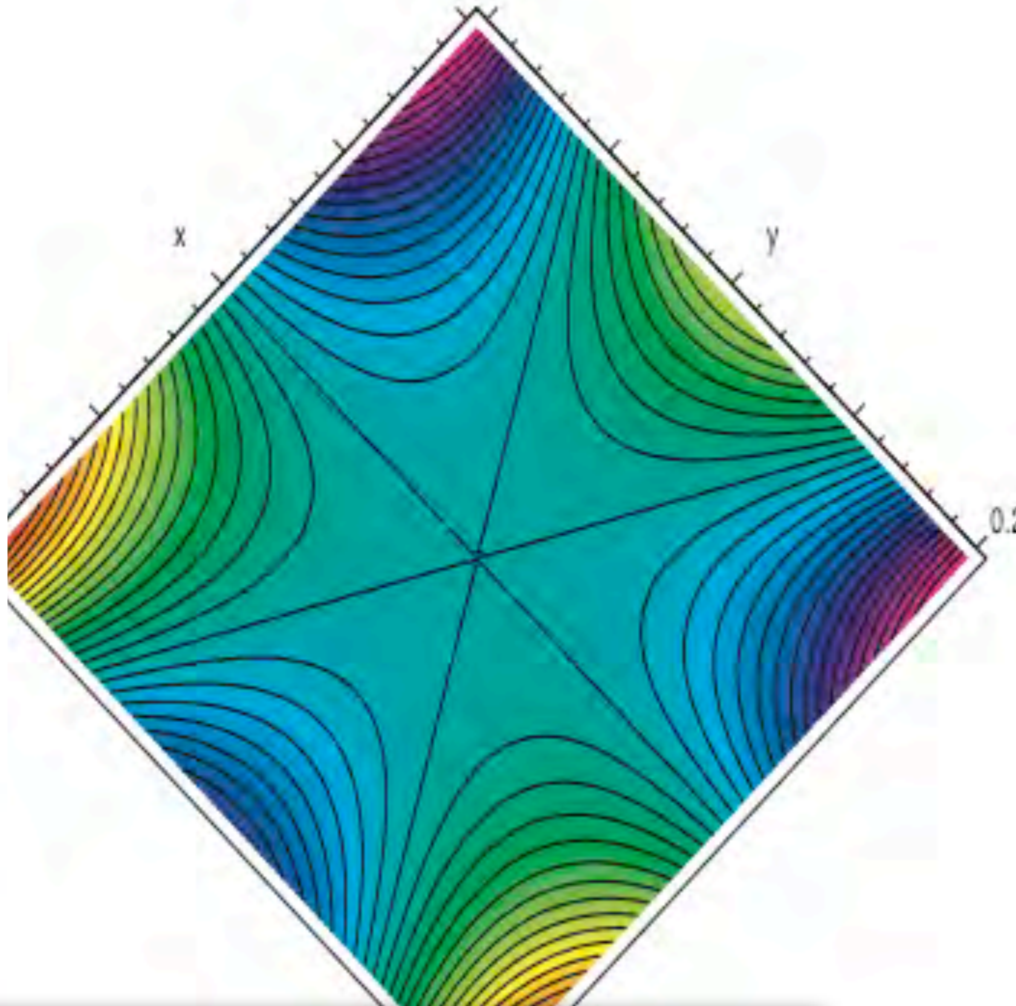
Hardy & Ramanujan

The number p_n *of partitions of integer* n *satisfies*

$$p_n \equiv [z^n] \prod_{k=1}^{\infty} \frac{1}{1-z^k} \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}}$$

(double saddle-point)





Coalescence; e.g., maps

Gauss => Airy

Conclusions: Saddle-point method

- Applies to many entire functions: involutions, set partitions, etc.
- Applies to function with violent growth at singularity(-ies): integer partitions
- Applies to coefficients of large order in large powers