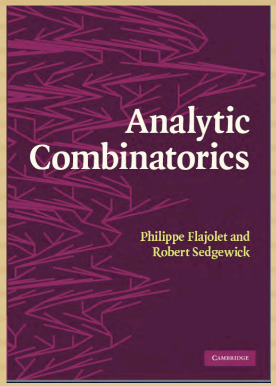


ANALYTIC COMBINATORICS

Philippe FLAJOLET

Bologna, June 2010

Counting...

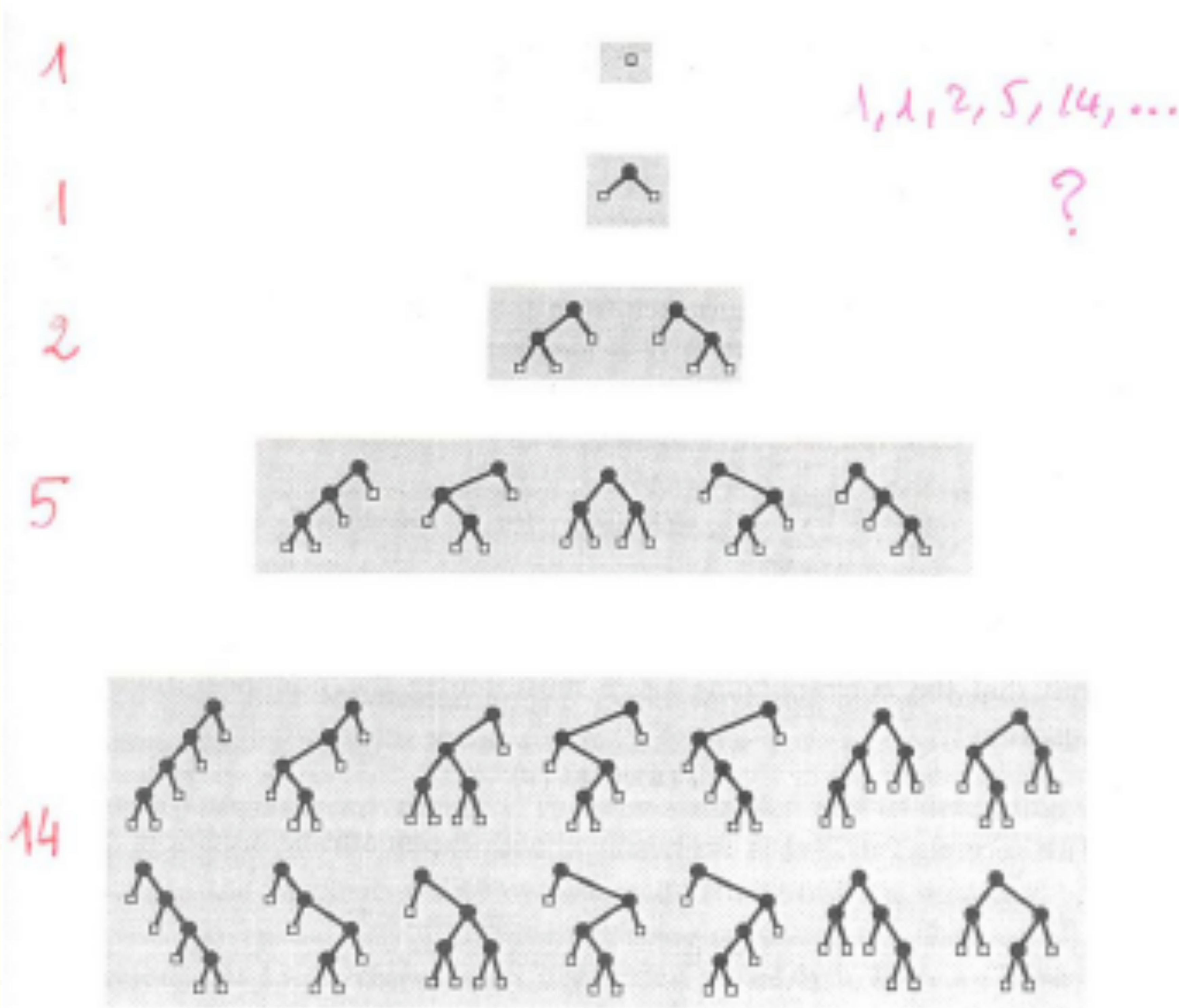


Figure 3.1 All binary trees with 1, 2, 3, 4, and 5 external nodes

Counting (and asymptotics)

- Binary trees => *Catalan numbers*
- **Formula is**

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!\,(n+1)!}$$

- **Growth rate is (asymptotics)**

$$C_n \sim \frac{4^n}{\sqrt{\pi n}}$$

Counting (and probabilities)

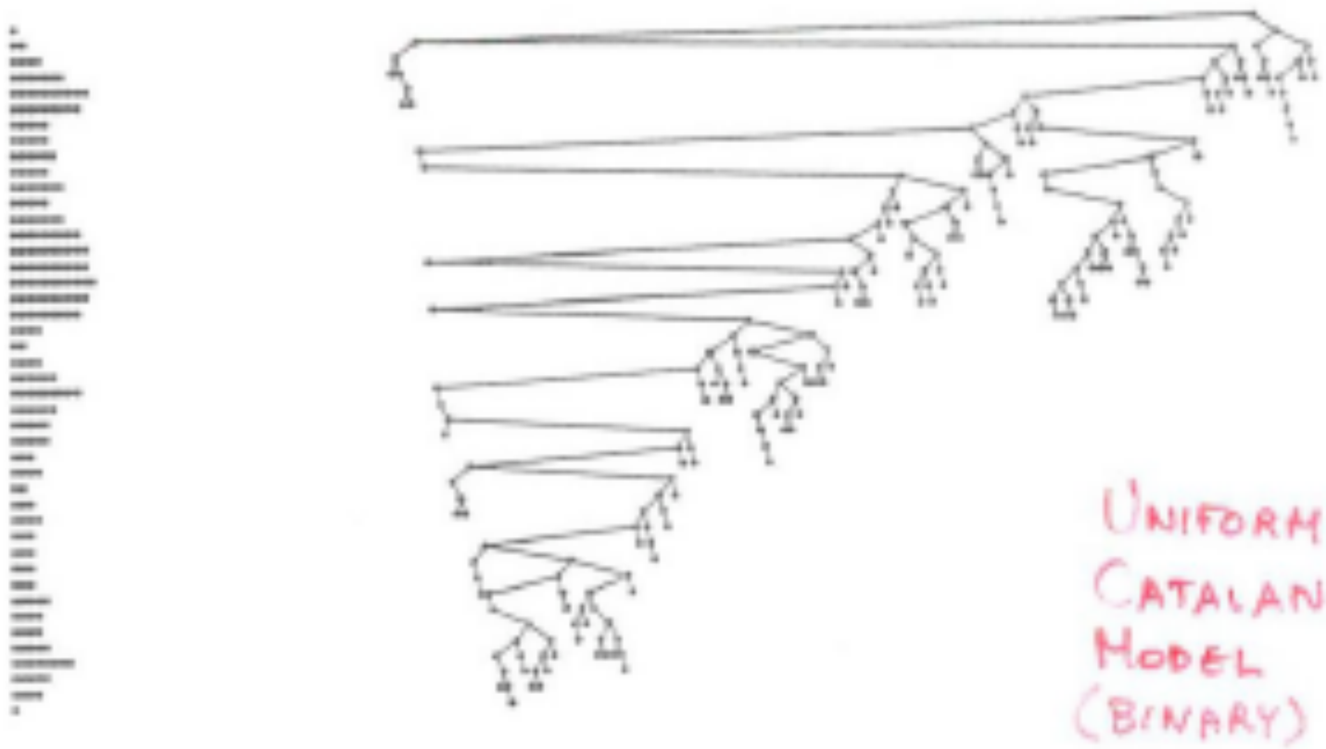


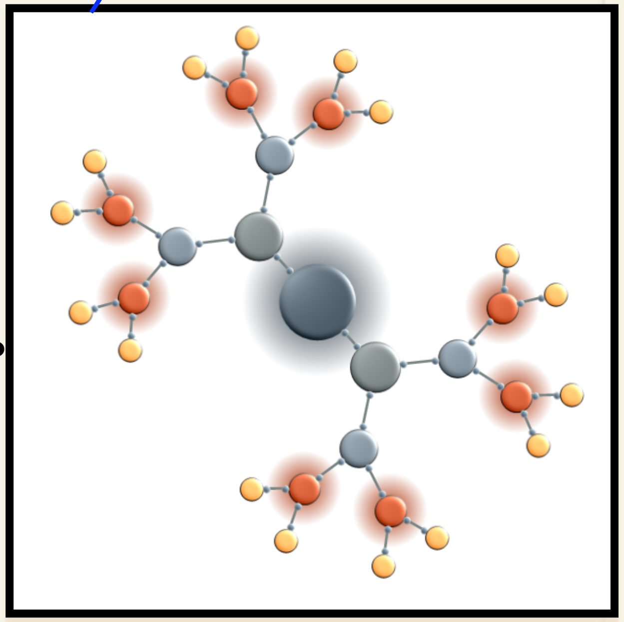
Figure 5.5 A random binary tree with 256 internal nodes

Figure 5.11 A binary search tree built from 256 randomly ordered keys

plane tree

increasing tree

Counting (methods)



- E.g. binary trees: `1, 1, 2, 5, 14, 42, ...`
- **Bijective combinatorics** = first principles
- **Generating function methods** ...
- **Algebraic methods** (e.g., symmetric fns, operator)

Generating Functions (GFs)

Combinatorial class $\mathcal{C}$; counting sequence (C_n):

$$\mathcal{C} \quad \Longrightarrow \quad \begin{cases} C(z) = \sum C_n z^n & \text{(OGF)} \\ \hat{C}(z) = \sum C_n \frac{z^n}{n!} & \text{(EGF)} \end{cases}$$

- Get GFs

  combinatorics $\rightsquigarrow$ algebra of special fns

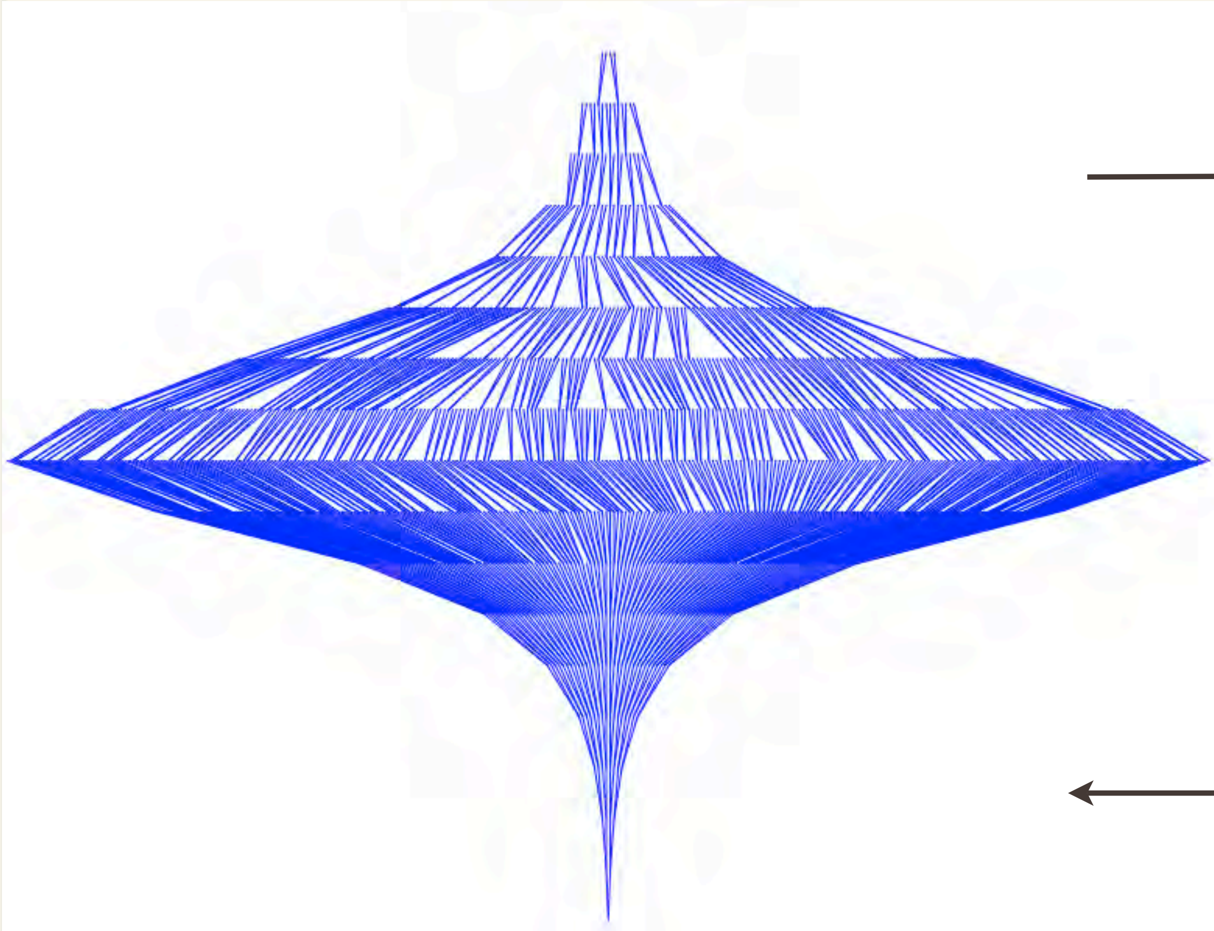
- Look at GFs as mappings of complex plane, $z \in \mathbb{C}$

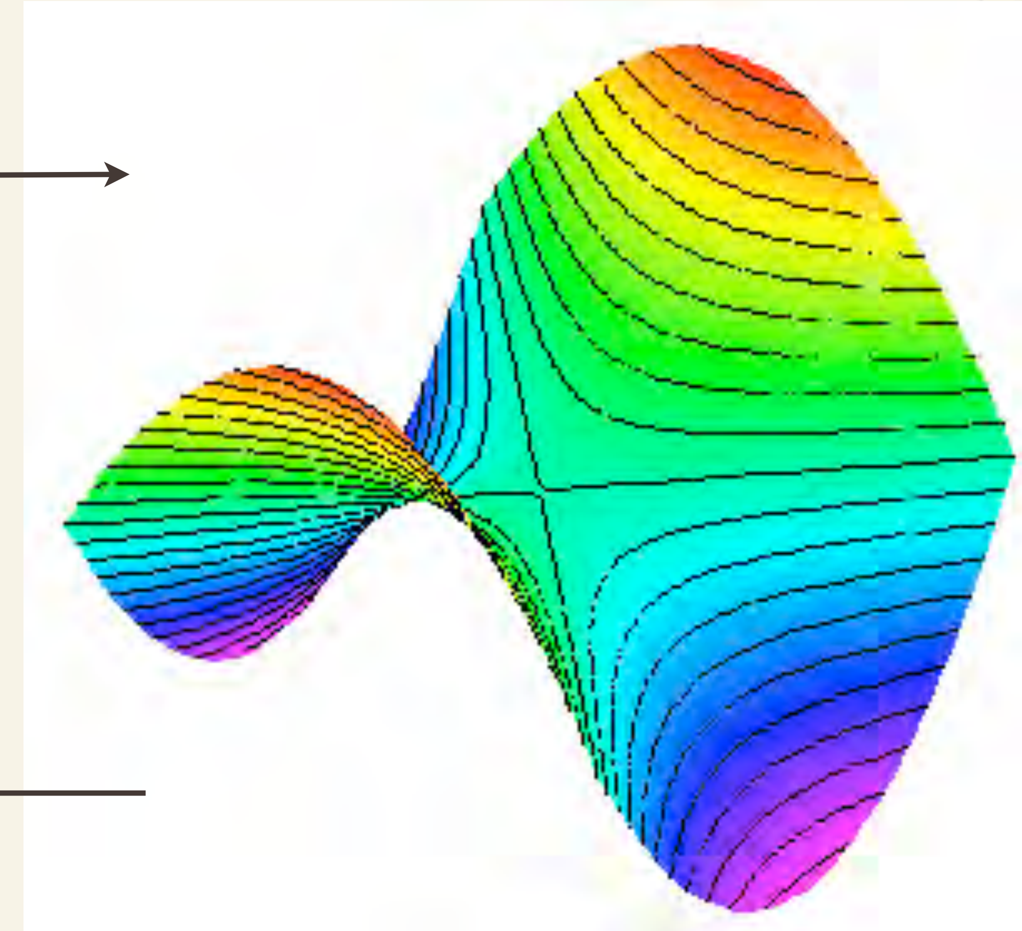
  algebra of special fns $\rightsquigarrow$ complex analysis

- For parameters, add extra variables

  complex analysis $\rightsquigarrow$ perturbation theory

A Calculus of Discrete Structures



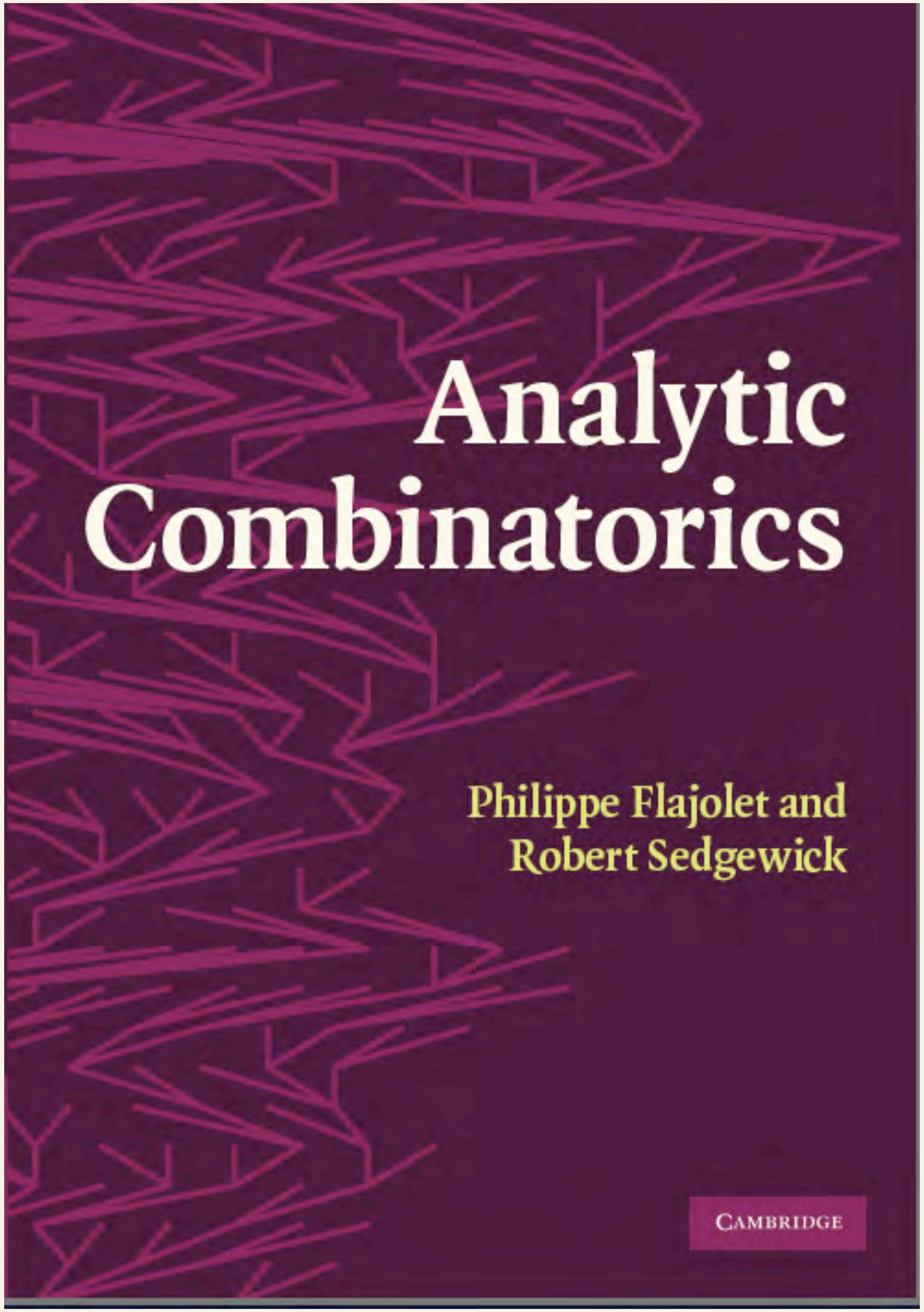


Discrete

Continuous

(a digital tree aka trie of size 500)

(a generating function in the complex plane)



FREE

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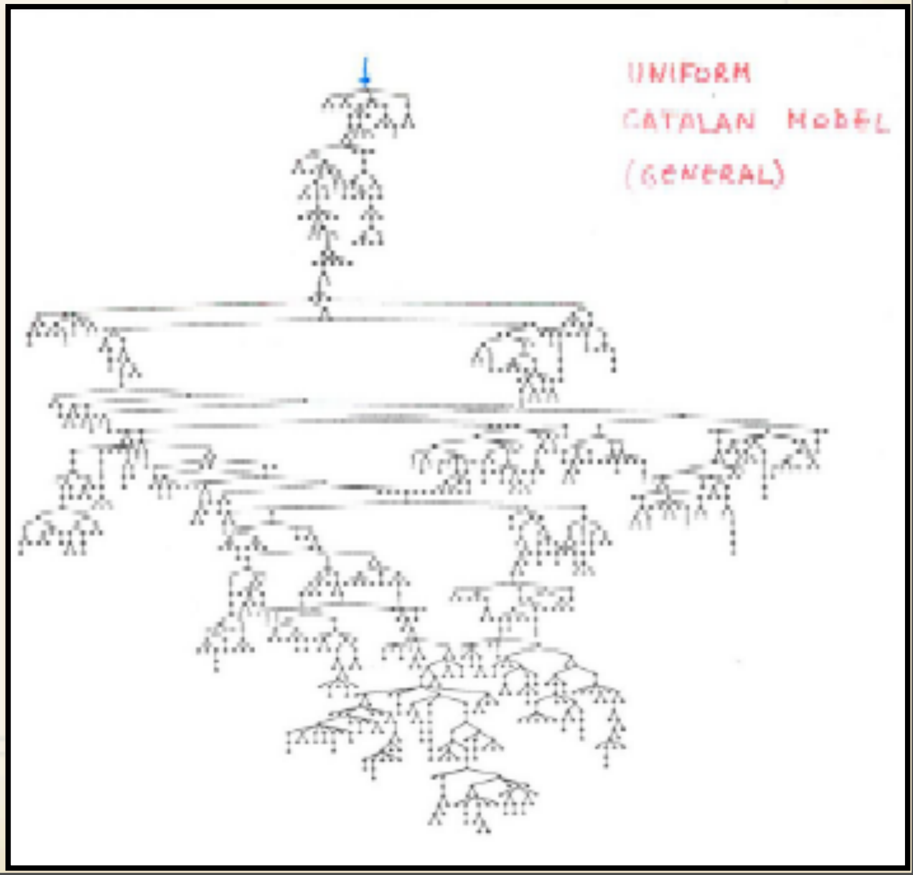
Analytic Combinatorics

- **A.** *Combinatorial structures*
- **B.** *Analytic structures*
- **C.** *Randomness properties*

for objects given by constructions



Quotations (1)

- Laplace discovered the remarkable correspondence between set theoretic operations and operations on formal power series and put it to great use to solve a variety of combinatorial problems.— G.–C. ROTA
- La méthode des fonctions génératrices, qui a exercé ses ravages pendant un siècle, est tombée en désuétude... — Claude BERGE

Quotations (2)

- Despite all appearances they [generating functions] belong to algebra and not to analysis.
- Combinatorialists use recurrence, generating functions, and such transformations as the Vandermonde convolution;
  others to my horror, use contour integrals, differential equations, and other resources of mathematical analysis. — John RIORDAN

PART I

Symbolic Methods

*1. Unlabelled structures & OGFs

* 2. Labelled structures and EGFs

* 3. Parameters and multivariate GFs

Embed a fragment of set theory into a language of constructions; map to algebra(s) of special functions.

Chapter 1

Unlabelled structures and OGFs

Symbolic Methods

Embed a fragment of set theory into a language of constructions;

map combinatorics to algebra(s) of special functions.

1 UNLABELLED STRUCTURES AND OGFS

Ordinary Generating Function (OGF)

$$(f_n) \longrightarrow f(z) := \sum_{n=0}^{\infty} f_n z^n.$$

(f_n) is number sequence, e.g., counting sequence.

Later: Exponential Generating function (EGF): $(f_n) \longrightarrow f(z) := \sum_{n=0}^{\infty} f_n \frac{z^n}{n!}.$

ΣΥ∫ Symbolic approach

- An object of size n is viewed as composed of *n atoms* (with additional structure): words, trees, graphs, permutations, etc.
- Replace each atom by symbolic weight z:

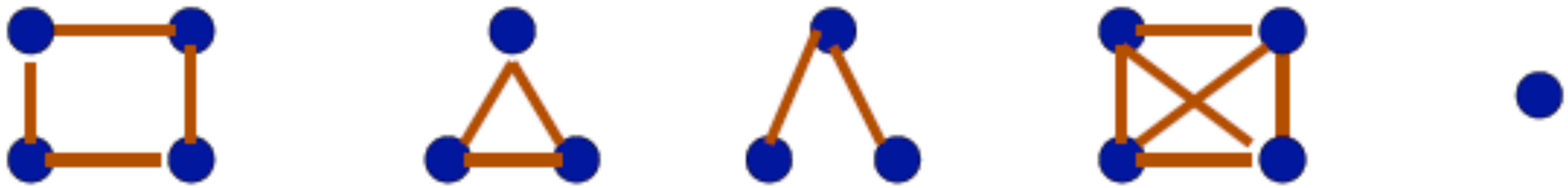
— Class: $\sum$ objects. Object: $\gamma \rightsquigarrow z^{|\gamma|}$.

Gives the **Ordinary Generating Function (OGF)**:

$$\mathcal{C} \qquad \rightsquigarrow \qquad C(z) := \sum_{\gamma \in \mathcal{C}} z^{|\gamma|} \equiv \sum_{n} C_n z^n.$$

E.g.: a class of graphs enumerated by # vertices

$\mathcal{C} =$



$$C(z) = z\,z\,z\,z + z\,z\,z + z\,z\,z + z\,z\,z\,z + z$$

$$= \mathbf{1} \cdot z + \mathbf{2} \cdot z^3 + \mathbf{2} \cdot z^4$$

$$(C_n) = (0, 1, 0, 2, 2).$$

Principle (Symbolic method)

The OGF of a class: (i) encodes the counting sequence; (ii) is nothing but a reduced form of the class itself.

How many binary trees B_n with n external nodes?

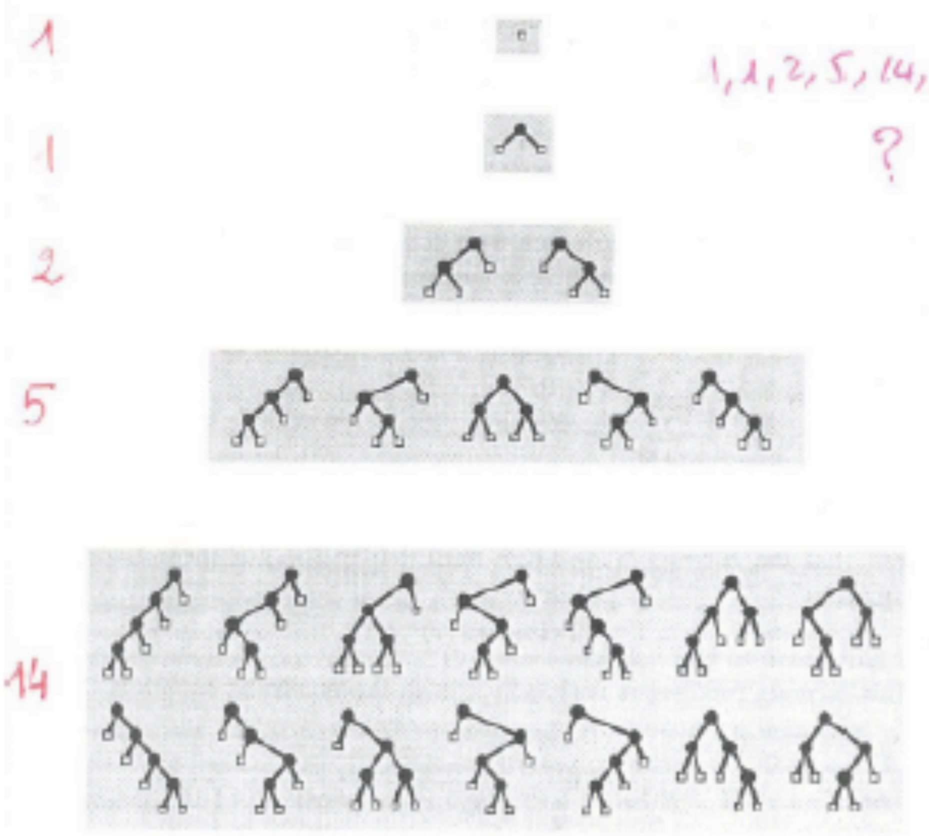


Figure 3.1 All binary trees with 1, 2, 3, 4, and 5 external nodes

$\mathcal{B} = \square + \bullet, (\mathcal{B} \times \mathcal{B})$.

Euler-Segner (1743): Recurrence

$$B_n = \sum_{k=1}^{n-1} B_k B_{n-k}.$$

Form OGF: $B(z) = z + (B(z) \times B(z))$.

Solve equation (quadratic):

$$B(z) = \frac{1}{2}(1 - \sqrt{1-4z}) = \frac{1}{2} - \frac{1}{2}(1-4z)^{1/2}.$$

Expand:

$$B_n = \frac{1}{n}\binom{2n-2}{n-1}$$ (Catalan numbers)

Analogy: $\mathcal{B} = \square + (\bullet \mathcal{B} \times \mathcal{B}) \rightsquigarrow B(z) = z + (B(z) \times B(z))$

Outline

Define a collection of constructions

union, product, sequence, set, cycle, . . .

allowing for *recursive definitions*.

meta-THM1: *OGFs are automatically computable (equations!)*

meta-THM2: *Counting sequences are automatically computable in time $O(n^2)$, and even $O(n^{1+\epsilon})$.*

meta-THM3: *Random generation is fast in $O(n \log n)$ arithmetic op´ns.*

Several set-theoretic constructions translate into GFs.

disjoint union	$\sum_{\mathcal{A}\oplus\mathcal{B}} = \sum_{\mathcal{A}} + \sum_{\mathcal{B}}$
cartesian product	$\sum_{\mathcal{A}\times\mathcal{B}} = \sum_{\mathcal{A}} \cdot \sum_{\mathcal{B}}$

There is a micro-dictionary:

disjoint union	$\mathcal{C} = \mathcal{A} \cup \mathcal{B}$	$\Longrightarrow$	$C(z) = A(z) + B(z)$
cartesian product	$\mathcal{C} = \mathcal{A} \times \mathcal{B}$	$\Longrightarrow$	$C(z) = A(z) \cdot B(z)$

Theorem. *There exists a dictionary:*

Construction	OGF
$\mathcal{C} = \mathcal{A} + \mathcal{B}$	$C(z) = A(z) + B(z)$
$\mathcal{C} = \mathcal{A} \times \mathcal{B}$	$C(z) = A(z) \cdot B(z)$
$\mathcal{C} = \textsc{Seq}(\mathcal{A})$	$C(z) = \dfrac{1}{1 - A(z)}$
$\mathcal{C} = \textsc{MSet}(\mathcal{A})$	$C(z) = \mathrm{Exp}(A(z))$
$\mathcal{C} = \textsc{PSet}(\mathcal{A})$	$C(z) = \widehat{\mathrm{Exp}}(A(z))$
$\mathcal{C} = \textsc{Cyc}(\mathcal{A})$	$C(z) = \mathrm{Log}\,\dfrac{1}{1 - A(z)}$

$\mathcal{E}$ or $\mathbf{1}$: "neutral class" formed with element of size $0 \mapsto E(z) = 1$.

$\mathcal{Z}$: "atomic class" formed with element of size $1 \mapsto E(z) = 1$.

$$\mathrm{Exp}(g(z)) = \exp\left(\sum_{k\geq 1} \frac{1}{k} g(z^k)\right); \ \widehat{\mathrm{Exp}}(g(z)) = \exp\left(\sum_{k\geq 1} \frac{(-1)^k}{k} g(z^k)\right);$$

$\mathrm{Log}(g(z)) = \sum_{k\geq 1} \frac{\varphi(k)}{k} g(z^k)$ with $\varphi(k)$= Euler totient.

SEQUENCE

$$\mathcal{C} = \mathrm{Seq}(\mathcal{A})$$

$$C(z) = \frac{1}{1-A(z)}$$

(i) $\mathcal{C} = \mathbf{1} + (\mathcal{A} \times \mathcal{C})$ (ii) $\frac{1}{1-f} = 1 + f + f^2 + \dots$

SET

$$\mathcal{C} = \mathrm{Set}(\mathcal{A}) \cong \prod_{\alpha \in \mathcal{A}} (1 + \{\alpha\})$$

$$C(z) = \exp\left(A(z) - \frac{A(z^2)}{2} + \dots\right)$$

PÓLYA OPERATOR

$$C(z) = \prod_{n=1}^{\infty} (1 + z^n)^{A_n} = \exp\left(\sum_n A_n \log(1 + z^n)\right) = \dots$$

MULTISET

$$\mathcal{C} = \mathrm{MSet}(\mathcal{A}) \cong \prod_{\alpha \in \mathcal{A}} \mathrm{Seq}(\{\alpha\})$$

$$C(z) = \prod_{n=1}^{\infty} (1 - z^n)^{-A_n} = \exp\left(A(z) + \frac{1}{2} A(z^2) + \cdots\right).$$

End of Proof ∎

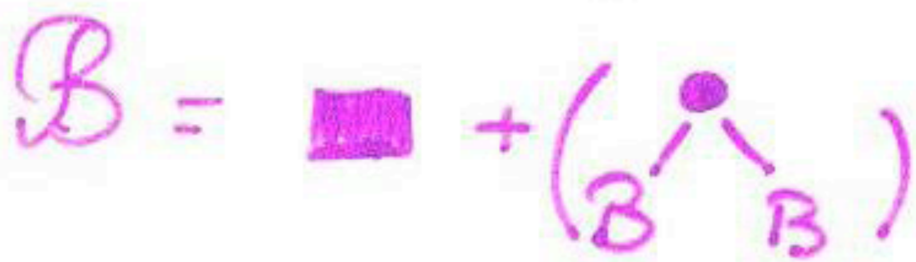
Summary:

Theorem (Symbolic method)

A dictionary translates constructions into generating functions:

Union	$+$
Product	$\times$
Sequence	$\frac{1}{1-\cdots}$
Set	Exp
Cycle	Log

This theorem permits us to write AUTOMATICALLY for binary trees



$\Downarrow$

$$B(z) = z + B(z)^2$$

Roots...

- A modicum of Pólya theory (1937)
- Schützenberger: languages and GFs (~1960)
- Rota-Stanley = MIT School (1970s)
- Goulden-Jackson = constructions (~1980)
- Joyal's theory of species +BLL (1980s)

Example 1. Binary words

$$\mathcal{W} = \textsc{Seq}(\{a,b\}) \quad \Longrightarrow \quad W(z) = \frac{1}{1-2z}.$$

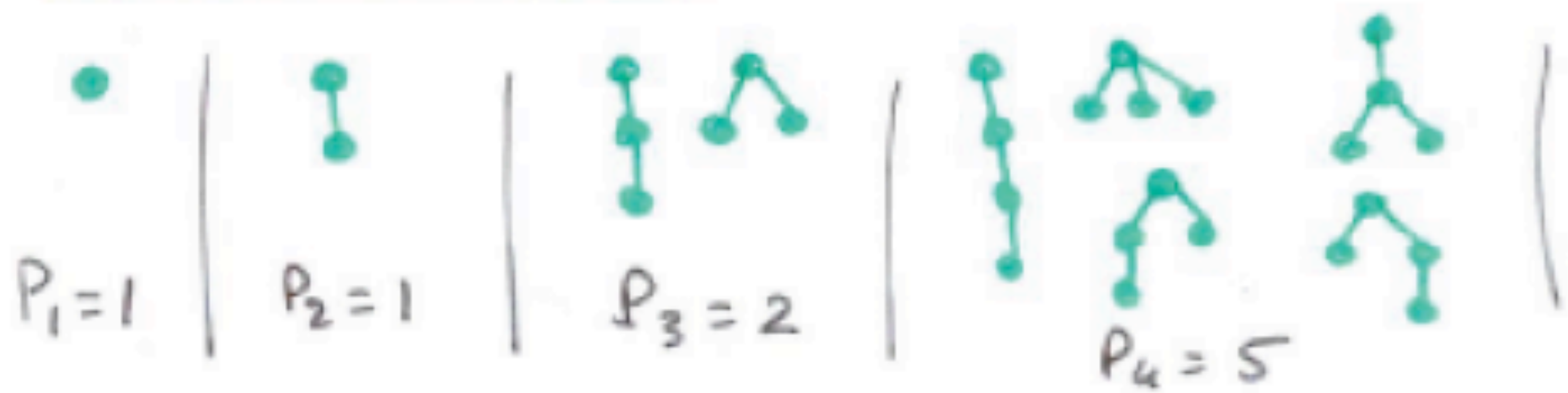
Get $W_n = 2^n$ (!?). Words starting with b and < 4 consecutive a's:

$$\mathcal{W}^\bullet \cong \textsc{Seq}(b\times(1+a+aa+aaa)) \quad \Longrightarrow \quad W^\bullet(z) = \frac{1}{1-(z+z^2+z^3+z^4)}.$$

Longest run statistics lead to rational functions (Feller).

Example 2. Plane trees ("general" = all degrees allowed)

$$\mathcal{P} = \mathcal{Z} \times \mathrm{Seq}(\mathcal{P})$$



$$P(z) = \frac{z}{1 - P(z)} \Rightarrow P(z) = \frac{1 - \sqrt{1-4z}}{2} \qquad P_n = \frac{1}{n}\binom{2n-2}{n-1}$$

Catalan numbers again!

Example 3. Nonplane trees (all degrees allowed)

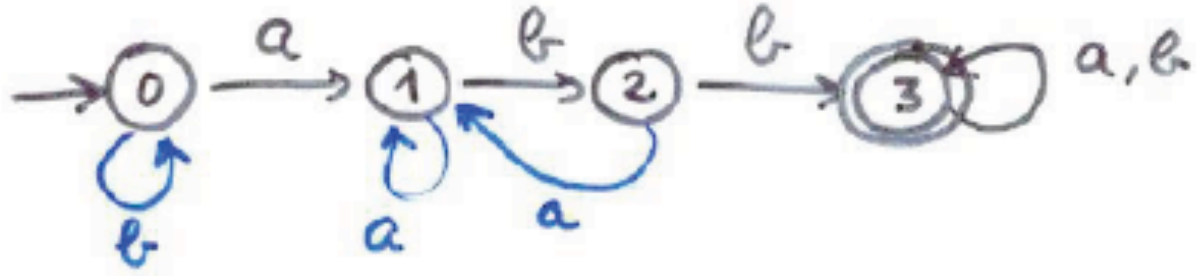
$\mathcal{U} = \mathcal{Z} \times \text{MSET}(\mathcal{U})$. $U_1 = 1,\ U_2 = 1,\ U_3 = 2, U_4 = 5$.

$$U(z) = z \exp\left(\frac{1}{1}U(z) + \frac{1}{2}U(z^2) + \frac{1}{3}U(z^3) + \cdots\right).$$

Cayley: recurrences; Pólya: asymptotics of this infinite functional equation.

Exercise: computable in polynomial time ($O(n^2)$).

Example 4. Words containing a pattern (abb)



$\mathcal{L}_j :=$ language accepted from state j.

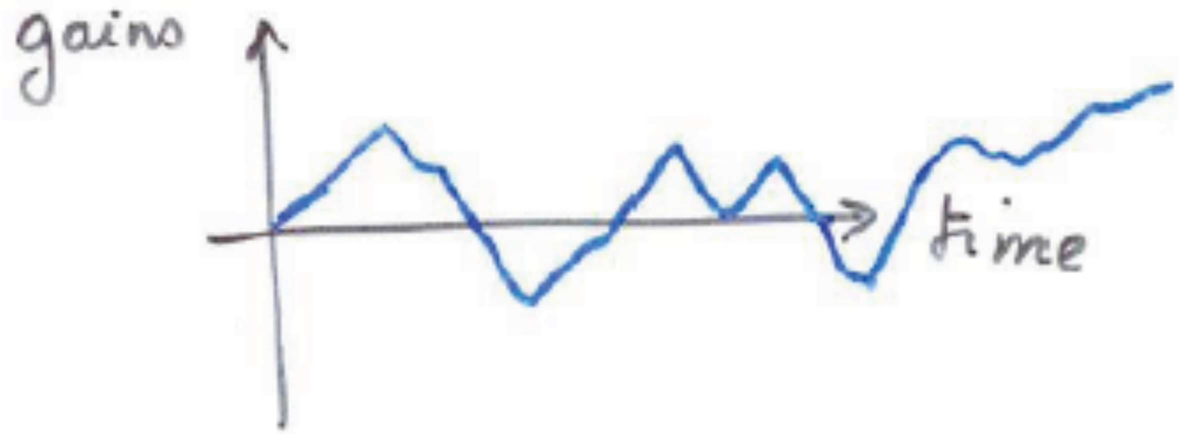
$$\{\mathcal{L}_0 = a\mathcal{L}_1 + b\mathcal{L}_0,\ \mathcal{L}_1 = a\mathcal{L}_1 + b\mathcal{L}_2, \mathcal{L}_2 = a\mathcal{L}_1 + b\mathcal{L}_3, \ldots\}$$

Theorem. Regular language (finite automaton) has rational GF.

$$Reg \quad \mapsto \quad \mathbb{Q}(z).$$

Patterns of all sorts in words. Applications in pattern matching algorithms and computational biology.

Example 5. Walks and excursions.



walks $\mapsto Q(z, \sqrt{1-4z^2})$

Excursion = 

$= \text{Seq}(\nearrow \text{Excursion} \searrow)$

Positive path = Excursion × Seq(↗ Excursion)

Draw game = $\text{Seq}(\searrow \overline{\text{Excursion}} \nearrow + \nearrow \text{Excursion} \searrow)$ etc

Simple families of plane trees.

Let $\Omega \subseteq \mathbb{Z}_{\geq 0}$ be the set of allowed (out)degrees. Define

$$\phi(y) := \sum_{w \in \Omega} y^{\omega}.$$

Then the simple family $\mathcal{Y}$ has OGF:

$$Y(z) = z\phi(Y(z)).$$

If ϕ is finite, get an algebraic function.

Lagrange Inversion Theorem.

$$[z^n]Y(z) = \frac{1}{n} \operatorname{coeff}[w^n]\phi(w)^n.$$

If ϕ is finite, get multinomial sums.

A variety of classes => a variety of "special functions"

Some constructible families and generating fuctions

- Regular languages, FA, paths in graphs: $\rightsquigarrow$ rational fns
- Unambiguous context-free languages $\rightsquigarrow$ algebraic functions.
- Terms trees $\rightsquigarrow$ [+Pólya operators] implicit functions

Tree = 

$\Longrightarrow$ $T = z\,\Phi(T)$

Algebraic functions (1)

- Arise from specifications (CF grammars), with +, x, Seq
- *Elimination*: system -> single equation P(x,y)=0
- Coefficients are "combinatorial sums"

[e.g., Sokal, *SLC* 2009]

Algebraic functions (2)

- MAPS: Tutte's quadratic method;
  cf Cori, Bousquet-Mélou et al., Bordeaux School...
- EXCURSIONS: the kernel method;
  cf Lalley 1993, Banderier-F 2001, MBM

$$F(z,u) = 1 + z(u^{-2} + u^0 + u^3)F(z,u) - \text{coeff}\,[u^{<0}]u^{-2}F(z,u)$$

$$\Longrightarrow \text{solve} \quad 1 - z(u^{-2} + u^0 + u^3) = 0.$$

Chapter 2
Labelled structures and EGFs

2 LABELLED STRUCTURES AND EGFS

EGF = exponential generating function

$$(f_n) \qquad \longrightarrow f(z) = \sum_{n \geq 0} f_n \frac{z^n}{n!}.$$

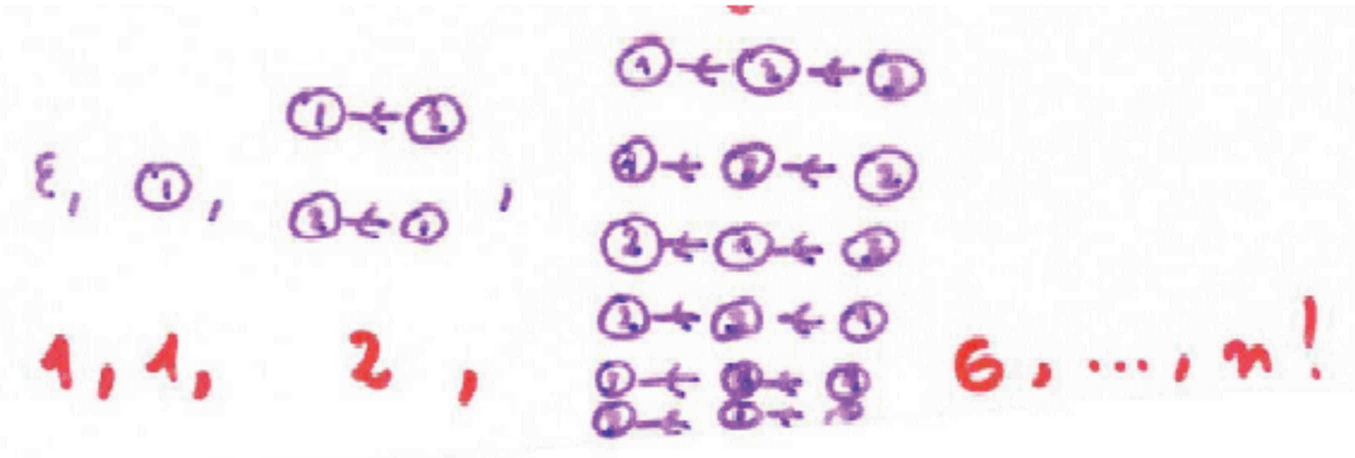
A labelled object has atoms that bear distinct integer labels (canonically numbered on $[1 \,..\, n]$*).*

Example. How many (undirected) graphs on n (distinguish-able) vertices? $G^n = 2^{n(n-1)/2}$.

Graphs: unlabelled problem is harder (Pólya theory). In general, can get unlabelled by identification of labelled.

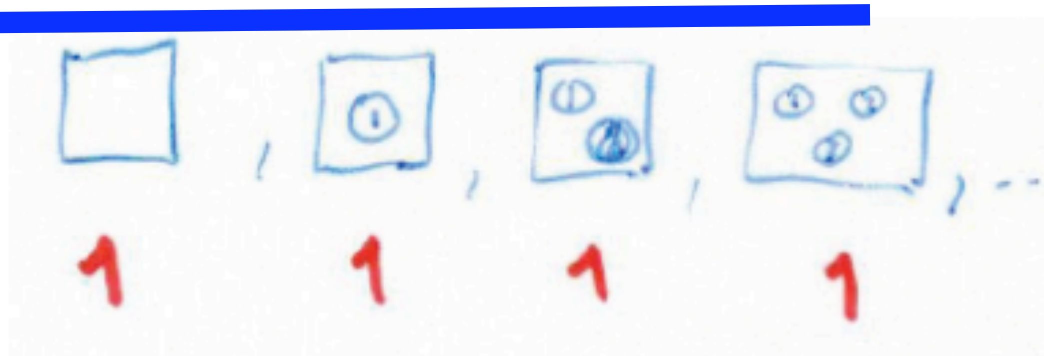
PERMUTATIONS = typical labelled objects: write $\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_n \end{pmatrix}$

as $\sigma_1 \sigma_2 \cdots \sigma_n$ and view as linear digraph that is labelled:



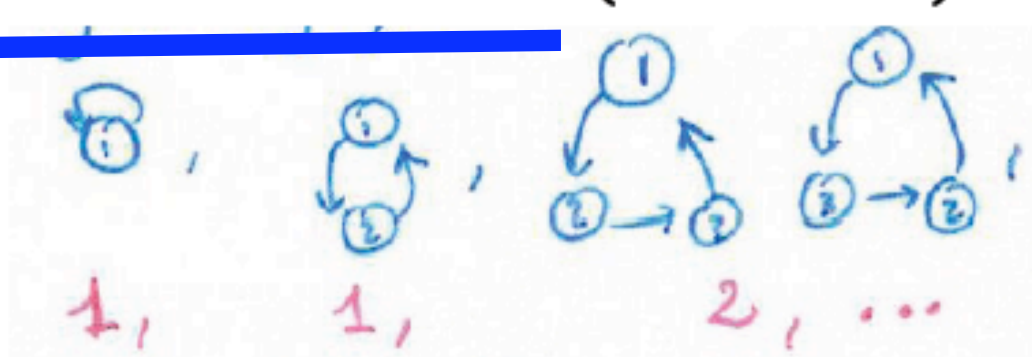
EGF is $\frac{1}{1-z}$ since $P(z) = \sum_n n! \frac{z^n}{n!}$.

DISCONNECTED GRAPHS (labelled) = no edges aka "Urns".



EGF is $U(z) = \exp(z) = e^z$.

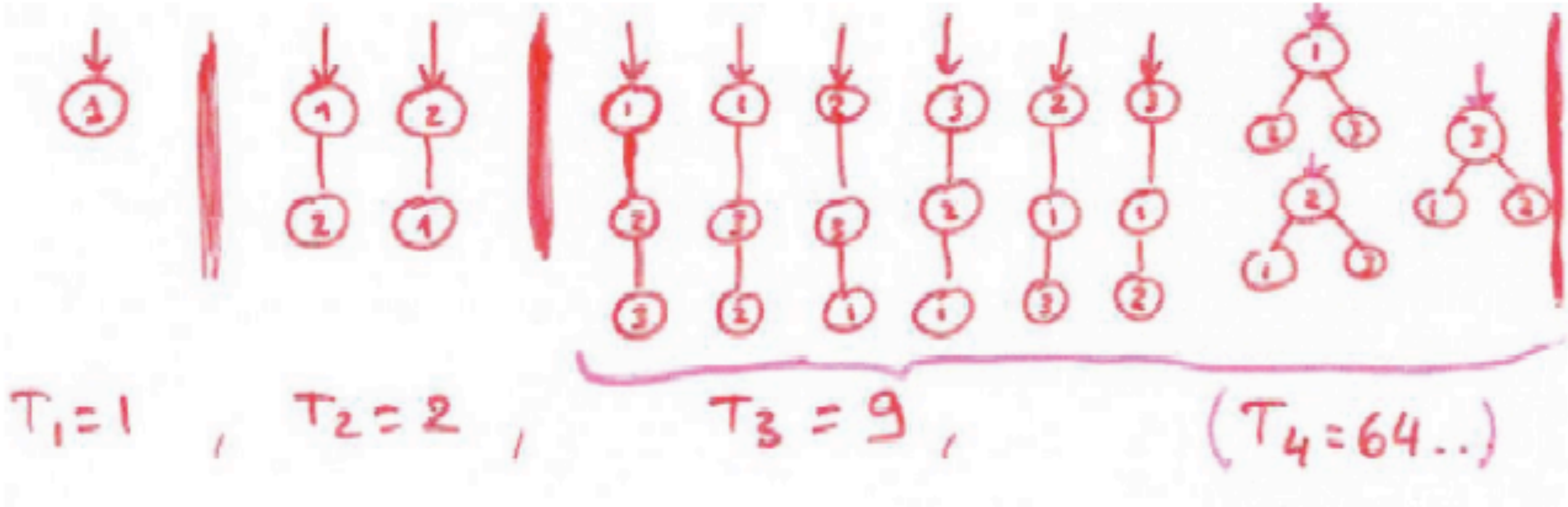
CYCLIC GRAPHS (directed)



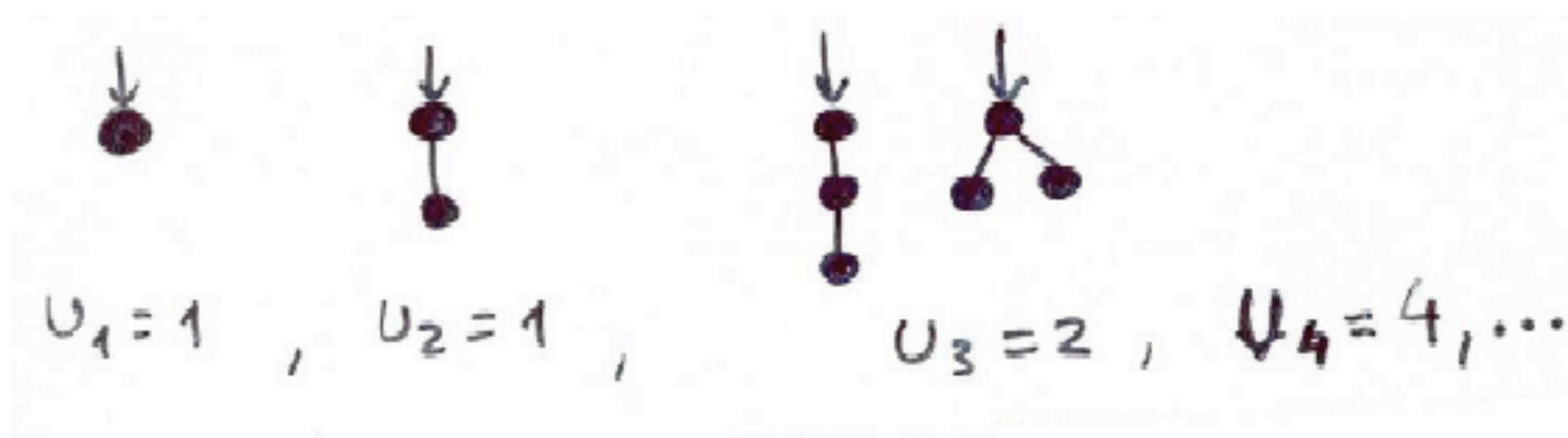
EGF $K(z) = \log \frac{1}{1-z}$.

ROOTED TREES (graphs) nonplane and labelled

$T_n = ??$

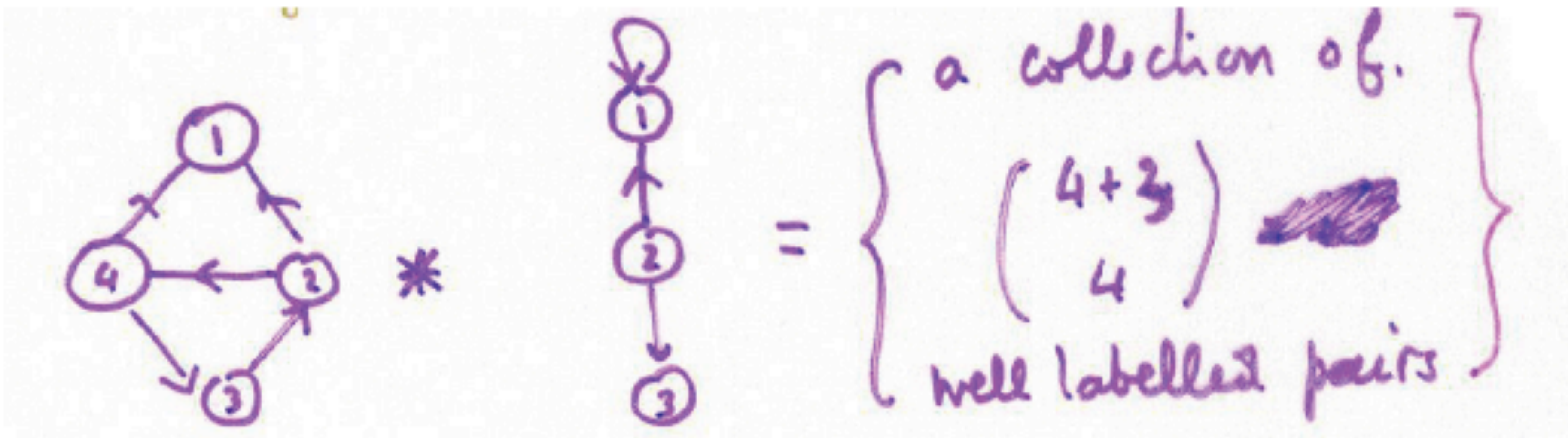


≫ Unlabelled:



Labelled product. Let $\mathcal{A}$ and $\mathcal{B}$ be labelled classes. Then the cartesian product $\mathcal{A} \times \mathcal{B}$ is *not* well-labelled (why?).

Given (β, γ) form all possible *relabellings* that preserve the order structure within β, γ, while giving rise to well-labelled objects.



- **Labelled product of two objects.**

$$(\alpha \star \beta) := \{\gamma \mid \gamma = (\alpha', \beta')\},$$

where γ is well-labelled and $\alpha' \equiv_{\text{order}} \alpha$ and $\beta' \equiv_{\text{order}} \beta$.

- **Labelled product of two classes.**

$$\mathcal{C} := \bigcup_{\alpha \in \mathcal{A}, \beta \in \mathcal{B}} (\alpha \star \beta).$$

$$\mathcal{C} = \mathcal{A} * \mathcal{B}$$

$$C_n = \sum_{k=0}^{n} \binom{n}{k} A_k B_{n-k}$$

$$\frac{C_n}{n!} = \sum_k \frac{A_k}{k!} \cdot \frac{B_{n-k}}{(n-k)!}$$

$$\Rightarrow \boxed{\hat{C}(z) = \hat{A}(z) \cdot \hat{B}(z)} \leftarrow \text{EGF}$$

Sequences, Sets, Cycles

- $\mathcal{E}$ (or $\mathbf{1}$): neutral class.
- $\mathcal{Z}$: atomic class $\equiv \boxed{1}$.
- Define $\textsc{Seq}(\mathcal{A})$, $\textsc{Set}(\mathcal{A})$, $\textsc{Cyc}(\mathcal{A})$ by relabellings:

$$\textsc{Seq}(\mathcal{A}) = \mathbf{1} + \mathcal{A} + (\mathcal{A} \star \mathcal{A}) + \cdots .$$

Sets: quotient up to perms. Cyc: up to cyclic perms.

Theorem. *There exists a dictionary:*

Construction	EGF
$\mathcal{C} = \mathcal{A} + \mathcal{B}$	$C(z) = A(z) + B(z)$
$\mathcal{C} = \mathcal{A} \star \mathcal{B}$	$C(z) = A(z) \cdot B(z)$
$\mathcal{C} = \textsc{Seq}(\mathcal{A})$	$C(z) = \dfrac{1}{1 - A(z)}$
$\mathcal{C} = \textsc{Set}(\mathcal{A})$	$C(z) = \exp(A(z))$
$\mathcal{C} = \textsc{Cyc}(\mathcal{A})$	$C(z) = \log \dfrac{1}{1 - A(z)}$

$\mathcal{E}$ or $\mathbf{1}$: "neutral class" formed with element of size $0 \mapsto E(z) = 1$.
$\mathcal{Z}$: "atomic class" formed with element of size $1 \mapsto E(z) = 1$.

$$\textsc{Seq}: 1 + A + A^2 + \cdots = \frac{1}{1-A}.$$

$$\textsc{Set}: 1 + \frac{A}{1!} + \frac{A^2}{2!} + \cdots = \exp(A).$$

$$\textsc{Cyc}: 1 + \frac{A}{1} + \frac{A^2}{2} + \cdots = \log\frac{1}{1-A}.$$

(End of proof of Theorem)

— Perms $\mathcal{P} \cong \text{SEQ}(\mathcal{Z})$

— Urn $\mathcal{U} \cong \text{SET}(\mathcal{Z})$

— Circulars graphs $\mathcal{K} \cong \text{CYC}(\mathcal{Z})$

— m–functions: $\mathcal{F}^{[m]} \cong \overbrace{\mathcal{U} \star \cdots \star \mathcal{U}}^{m \text{ times}} \equiv \text{SEQ}_m(\mathcal{U})$

— m–surjections: $\text{SEQ}(\mathcal{V}),\ \mathcal{V} = \text{SET}_{\geq 1}(\mathcal{Z})$

— Set partitions: $\text{SET}(\text{SET}_{\geq 1}(\mathcal{Z}))$

— Lab. trees: $\mathcal{T} = \mathcal{Z} \star \text{SET}(T)$.

Example 1. Permutations and cycles:

$$\mathcal{P} = \textsc{Set}(\textsc{Cyc}(\mathcal{Z})) \quad \Longrightarrow \quad P(z) = \exp\left(\log \frac{1}{1-z}\right) = \frac{1}{1-z}.$$

Derangements (no fixed point)

$$\mathcal{D} = \textsc{Set}(\textsc{Cyc}(\mathcal{Z}) \setminus \mathcal{Z}) \quad \Longrightarrow \quad D(z) = \exp\left(\log \frac{1}{1-z} - z\right) \equiv \frac{e^{-z}}{1-z}.$$

Thus $\boxed{\frac{D_n}{n!} = 1 - \frac{1}{1!} + \frac{2}{2!} - \cdots + \frac{(-1)^n}{n!}} \sim e^{-1}$.

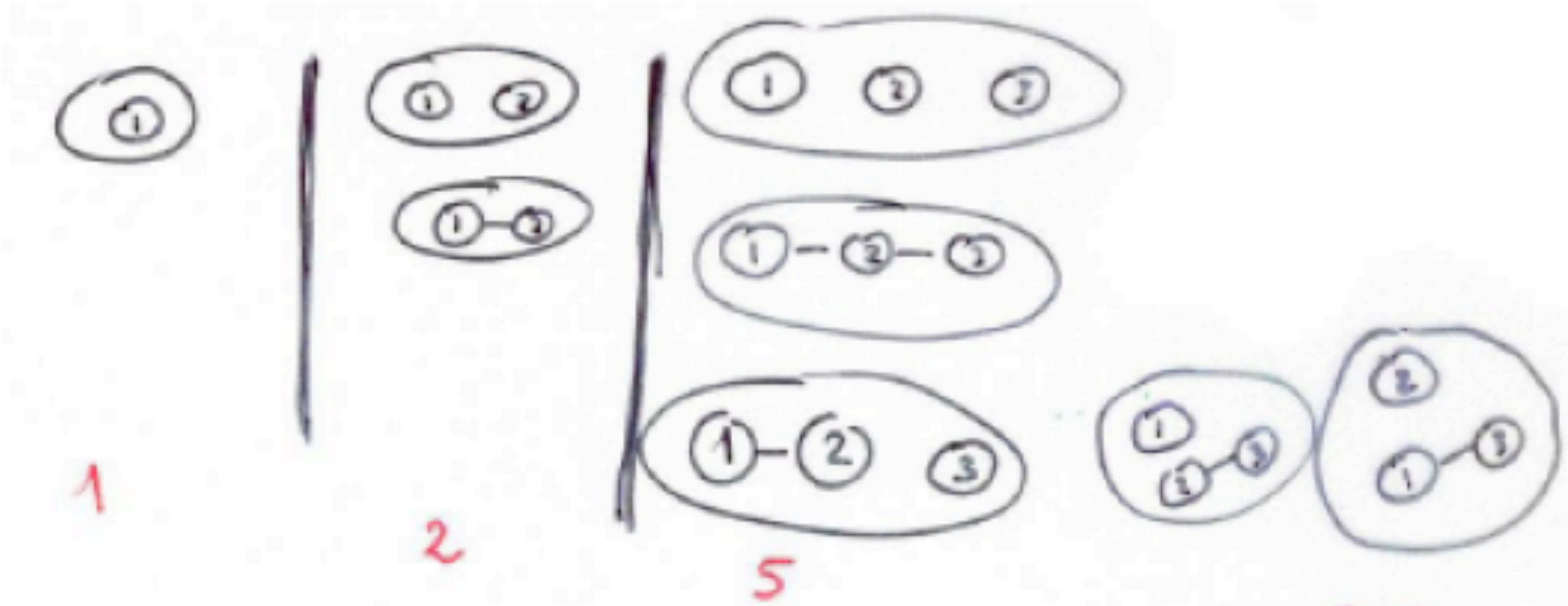
Example 2. Labelled (Cayley) trees:

$$\mathcal{T} = \mathcal{Z} \star \textsc{Set}(\mathcal{T}) \quad \Longrightarrow \quad T(z) = ze^{T(z)}.$$

Thus $\boxed{T_n = n^{n-1}}$ by Lagrange Inversion Th.

Example 3. Set partitions:

$$\mathcal{B} = \text{SET}(\text{SET}_{\geq 1}(\mathcal{Z})) \implies B(z) = e^{e^z - 1}.$$



Bell numbers: $$B_n = e^{-1} \sum_{k \geq 0} \frac{k^n}{k!}.$$

Example 4. Allocations to $[1\,..\,m]$:

— all: $e^{mz} \leadsto F_n = m^n$.

— surjective: $(e^z - 1)^m \leadsto$ *Stirling numbers*, $m!\left\{{m \atop n}\right\} = \sum \binom{m}{k}(-1)^{m-k}k^n$.

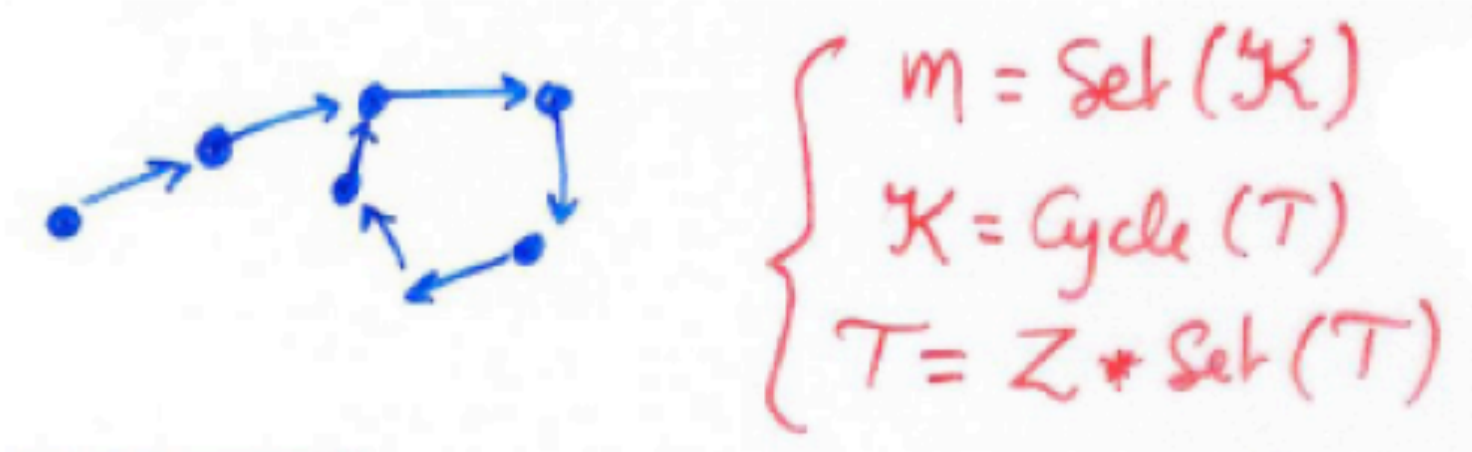
— injective: $(1+z)^m \leadsto \binom{m}{n} n!$ (arrangement #).

Exercise: Birthday Problem and Coupon Collector.

$$\mathbb{E}(B) = \int_0^\infty \left(1 + \frac{t}{m}\right)^m e^{-t}\,dt, \qquad \mathbb{E}(C) = \int_0^\infty \left(e^t - (e^{t/m} - 1)^m\right) e^{-t}\,dt.$$

Multiple birthdays, multiple collections. (Cf Poissonization.)

Example 5. Mappings aka functional graphs = endofunctions of finite set.



$T = ze^T, K = \log(1-T)^{-1}, M = e^K$: $\boxed{M_n = n^n}$. $\mathbb{P}$(connected)=$O\left(\frac{1}{\sqrt{n}}\right)$.

Exercise: A binary functional graph is such that each x has either 0 or 2 preimages (cf $x^2 + a \mod p$). **Q1.** Construct; **Q2.** enumerate.

Exercise: All graphs $G(z) = 1 + \sum_{n=1}^{\infty} 2^{n(n-1)/2} z^n/n!$. **Q1.** EGF $K(z)$ of connected graphs? **Q2.** Probability of connectedness. **Q3*** Prove not constructible.

- Additional constructions: *substitution, pointing, order constraints*:

$$f \circ g, \qquad \partial f, \qquad \int f.$$

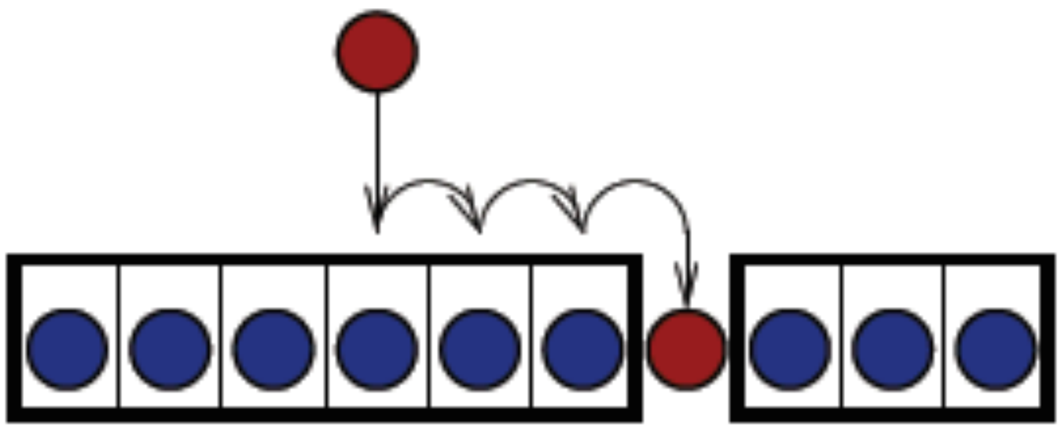
Linear probing hashing: From Knuth's original derivation (rec.):

We have

$$M(N,k) = \sum_{m=1}^{N} m\, P(N,k,m) = \frac{N(N+1)}{2}\left(1-\frac{k-1}{N}\right) - \frac{N-k}{N^2}\sum_{r=1}^{N}\left(\frac{N(N+1)}{2} - \frac{r(r+1)}{2}\right)R(N,k,r)$$

$$= \frac{1}{2}\left\{ N(N+1)\left(1-\frac{k-1}{N}\right) - \left(1-\frac{k}{N}\right)\sum_{r=1}^{N}\left[(2N+1)\left(1-\frac{r}{N}\right) - N\left(1-\frac{r}{N}\right)^2\right]R(N,k,r).\right.$$

to *symbolic GFs*:

Island = □ + 

$$I(z) = 1 + \int \frac{\partial}{\partial z}(zI(z)) \times I(z)$$

Get nonempty island by joining two islands by means of a gluing element.

⇝ wide encompassing extensions of original analyses [F-Poblete-Viola, Pittel, Knuth 1998, Janson, Chassaing-Marckert, ...].

Some constructible families and generating fuctions

labelled

Tree = 

$\Rightarrow$ $T = z\,\Phi(T)$

implicit functions

- Increasing trees ⇝ $Y = \int \Phi(Y)$

differential equation

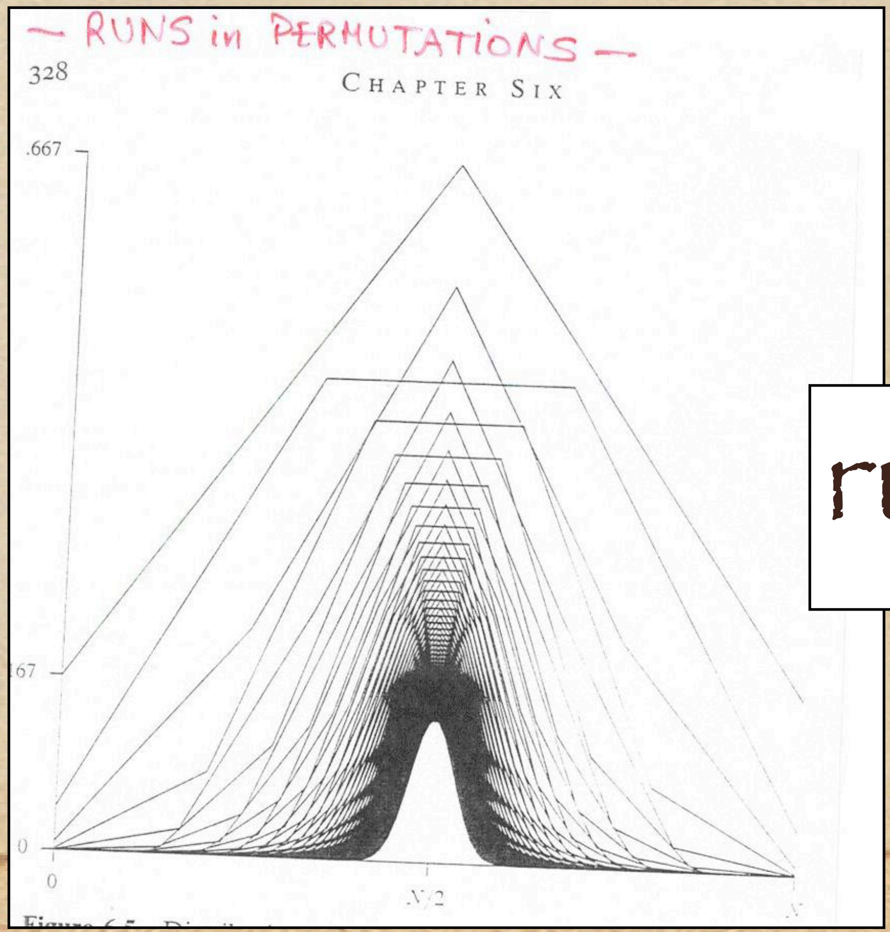
- Mappings ⇝

exp ∘ log ∘ implicit

$$\begin{cases} M = \mathrm{Set}(K) \\ K = \mathrm{Cycle}(T) \\ T = Z \star \mathrm{Set}(T) \end{cases}$$

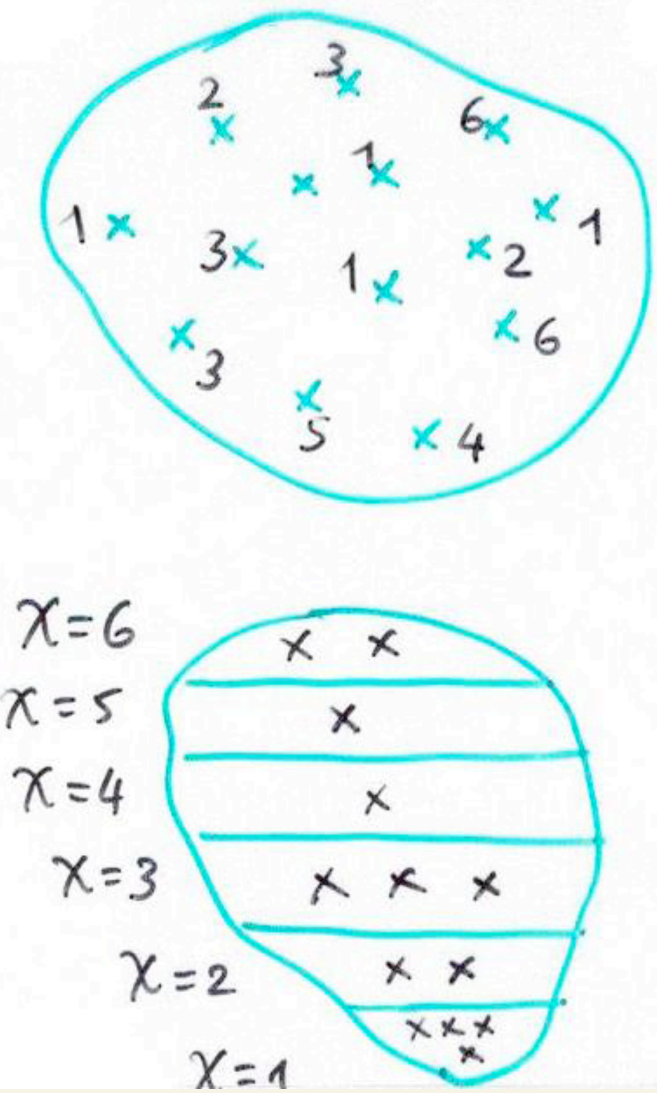
$$\begin{cases} M = \exp(K) \\ K = \log(1-T)^{-1} \\ T = z\exp(T) \end{cases}.$$

Chapter 3. Parameters and Multivariate GFs



runs in perms

Discrete Probabilistic Model



E_N with X

(eg. Trees_N with path length)

$\{E_{N,k}\}$

→ Enumeration / Counting

$$\Pr\{X=k\} = \frac{E_{N,k}}{E_N},$$

$$E[X] = \sum k \cdot \frac{E_{N,k}}{E_N},$$

+ variance, etc.

Bivariate GF (ordinary) $(E_{n,k})$ $\rightsquigarrow$ $E(z,u) = \sum_{n,k} E_{n,k} u^k z^n$.

Bivariate GF (exponential) $(E_{n,k})$ $\rightsquigarrow$ $E(z,u) = \sum_{n,k} E_{n,k} u^k \frac{z^n}{n!}$.

- BGF encodes exact distributions. hence, moments.

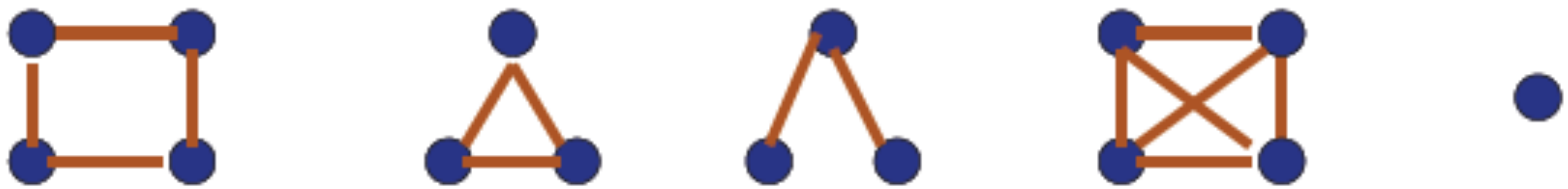
$$\mathbb{E}_{\mathcal{E}_n}[\chi] = \sum_k k \cdot \frac{E_{n,k}}{E_n} = \frac{1}{E_n} \operatorname{coeff}[z^n] \left. \frac{\partial}{\partial u} E(z,u) \right|_{u=1}.$$

Variance & moment of order 2: second derivative, etc.

Chebyshev inequalities: $\sigma_n/\mu_n \to 0$ implies *convergence in probability*.

- **Parameters**: via *multivariate* GFs.

$\mathcal{C}$ =



$$C(z,u) = \frac{z\,z\,z\,z}{u\,u\,u\,u} + \frac{z\,z\,z}{u\,u\,u} + \frac{z\,z\,z}{u\,u} + \frac{z\,z\,z\,z}{u\,u\,u\,u\,u\,u\,u} + \frac{z}{u^0}$$

Also: combinatorial forms. e.g. for ordinary GFs

$$C(z,u) = \sum_{\gamma \in \mathcal{C}} u^{\chi[\gamma]} z^{|\gamma|}.$$

• BGF is reduction of combinatorial structure. Thus expect **multivariate dictionaries.**

PRINCIPLE: *Add variables marking parameters at appropriate places and recycle:*

Theorem (Symbolic method)

A dictionary translates constructions into generating functions:

Union	$+$
Product	$\times$
Sequence	$\frac{1}{1-\cdots}$
Set	Exp
Cycle	Log

Conclusions (Part I)

- [Chapter 3]: Multivariate GFs give access to parameters; those that can be obtained by "marking" in combinatorial constructions.
- [Chapters 1-2-3]: Exploit all this asymptotically?
  counting; mean, variance, distribution?