Pseudo-factorials, elliptic functions & continued fractions

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joint with Roland Bacher

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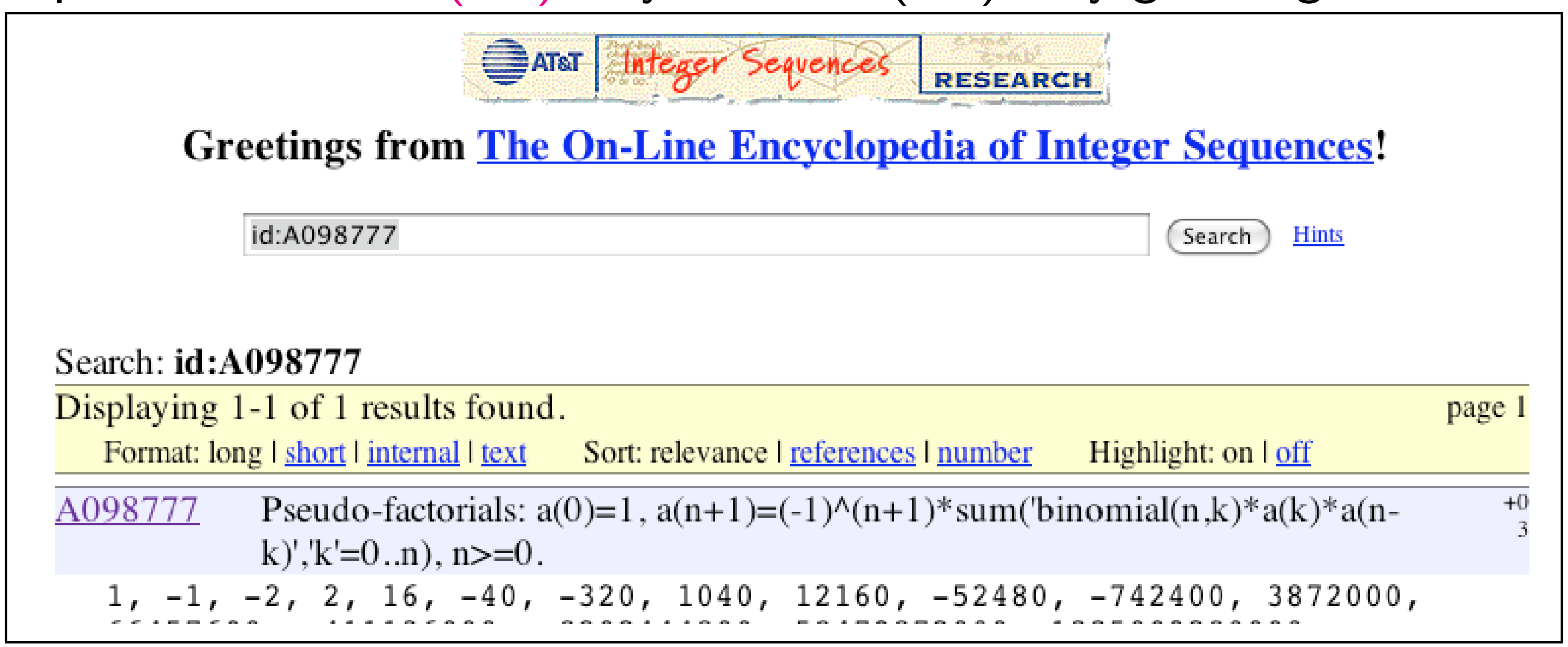
Factorials: the sequence $(\beta_n) = (n!)$ satisfies the recurrence

$$\beta_{n+1} = \sum_{k=0}^{n} \binom{n}{k} \beta_k \beta_{n-k}, \qquad \beta_0 = 1.$$

Pseudo-factorials: they satisfy the twisted recurrence

$$\alpha_{n+1} = (-1)^{n+1} \cdot \sum_{k=0}^{n} \binom{n}{k} \alpha_k \alpha_{n-k}, \qquad \alpha_0 = 1.$$

Equivalent choice: $(-1)^n$; by contrast (-1) only gives signed factorials.


AT&T Integer Sequences RESEARCH

Greetings from The On-Line Encyclopedia of Integer Sequences!

id:A098777 Search Hints

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Displaying 1-1 of 1 results found. page 1

Format: long | short | internal | text Sort: relevance | references | number Highlight: on | off

A098777 Pseudo-factorials: a(0)=1, a(n+1)=(-1)^(n+1)*sum('binomial(n,k)*a(k)*a(n-k)','k'=0..n), n>=0. +0 3

1, -1, -2, 2, 16, -40, -320, 1040, 12160, -52480, -742400, 3872000,

• The **EGF** (exponential generating function) $f(z) := \sum_{n=0}^{\infty} \alpha_n \frac{z^n}{n!}$ must converge in $|z| < 1$. **Growth** is numerically (?):

$$\left|\frac{\alpha_n}{n!}\right|^{1/n} \quad \underset{n\to+\infty}{\longrightarrow} \quad \mathbf{K}, \qquad \mathbf{K} \approx 0.823.$$

We shall see that $\boxed{\mathbf{K} = 2^{7/3}\pi\Gamma(1/3)^{-3}} \doteq 0.8235025.$

• **Sign pattern**: + − − + + − − + + − − + +

• **Congruences**: Period is $1 \pmod{10}$; it is $36 \pmod 7$

$$\begin{aligned}
\alpha_n \text{ mod } 10 &= 1, 9, 8, 2, 6, 0, 0, 0, 0, 0, \ldots, \qquad \text{cf } n!, \\
\alpha_n \text{ mod } 7 &= 1, 6, 5, \underline{2, 2, 2, 2}, 4, 1, \underline{6, 6, 6, 6}, 5, 3, \underline{4, 4, 4, 4}, 1, 2, \underline{5, 5, 5, 5}, \ldots
\end{aligned}$$

Why & How?

• Roland Bacher (Grenoble) was investigating a collection of recurrences loosely suggested by the Dixonian elliptic functions, as appear in [Flajolet+Conrad] *SLC, 2006*.

• He empirically noticed surprising congruences as well as a **continued fraction** that seemed to be of a **new kind**:

$$F(z) \equiv \sum_n \alpha_n z^n = \cfrac{1}{1+1\,z+\cfrac{3\cdot 1^2\cdot z^2}{1-1\,z+\cfrac{2^2\cdot z^2}{1+3\,z+\cfrac{3\cdot 3^2\cdot z^2}{1-3\,z+\cfrac{4^2\cdot z^2}{\ddots}}}}}.$$

1. Elliptic Connexion

~~~ Dixonian functions ~~~
~~~

The (easy) EGF

The recurrence

$$\alpha_{n+1} = (-1)^{n+1} \cdot \sum_{k=0}^{n} \binom{n}{k} \alpha_k \alpha_{n-k}$$

translates into a functional equation for the EGF

$$f(z) := \sum_n \alpha_n \frac{z^n}{n!}!$$

as follows:

$$f'(z) = -f(-z)^2.$$

Solution $\rightsquigarrow$??? . . .

• Start from $\boxed{f'(z) = -f(-z)^2}$

Differentiate: $f''(z) = 2f(-z)f'(-z)$.

Use original equation to get $f(-z) \mapsto \sqrt{-f'(z)}$, hence an **ODE**

$$f''(z) = -2\sqrt{-f'(z)}f(z)^2.$$

• *"Cleverly"* multiply by $\sqrt{-f'(z)}$

$$f''(z)\sqrt{-f'(z)} = 2f(z)^2 f'(z).$$

Integrate (with initial conditions):

$$\boxed{\int_{f(z)}^{1} \frac{dw}{(2-w^3)^{2/3}} = z.}$$

♡♡ *The solution is the inverse of an Abelian integral* $\int 1/(P_3)^{2/3}$.

Standardization

Theorem

The EGF of pseudo-factorials satisfies

$$f(z) = 2^{1/3}\,\mathrm{sm}\left(\frac{\pi_3}{3} - 2^{1/3} z\right),$$

where the Dixon function sm *is defined by*

$$\int_0^{\mathrm{sm}\, z} \frac{dy}{(1-y^3)^{2/3}} = z,$$

and $\pi_3 = 3\int_0^1 \frac{dy}{(1-y^3)^{2/3}} = \frac{\sqrt{3}}{2\pi}\Gamma\left(\frac{1}{3}\right)^3.$

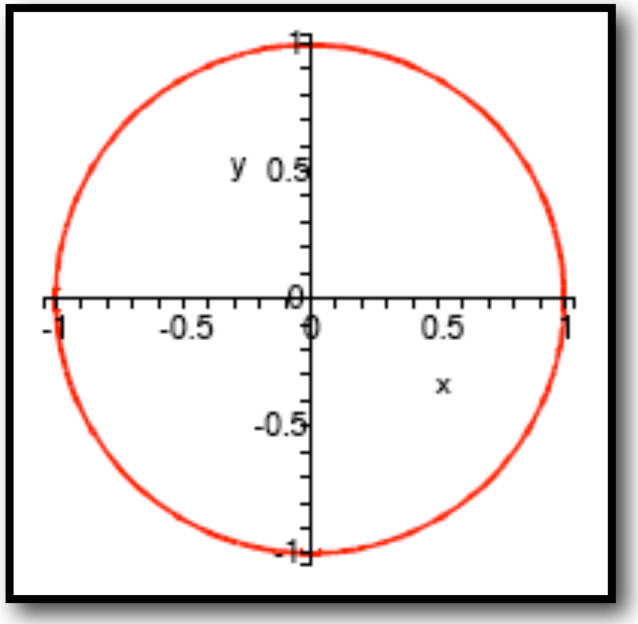
Also: $\mathrm{sm}(z) = z\ \mathbf{Inv}\ {}_2F_1\left[\frac{1}{3}, \frac{2}{3}; \frac{4}{3}; z^3\right]$, with

$F[a, b; c; z] := 1 + \frac{a \cdot b}{c}\frac{z^1}{1!} + + \frac{a(a+1)\cdot b(b+1)}{c(c+1)}\frac{z^2}{2!} \cdots.$

Lundberg

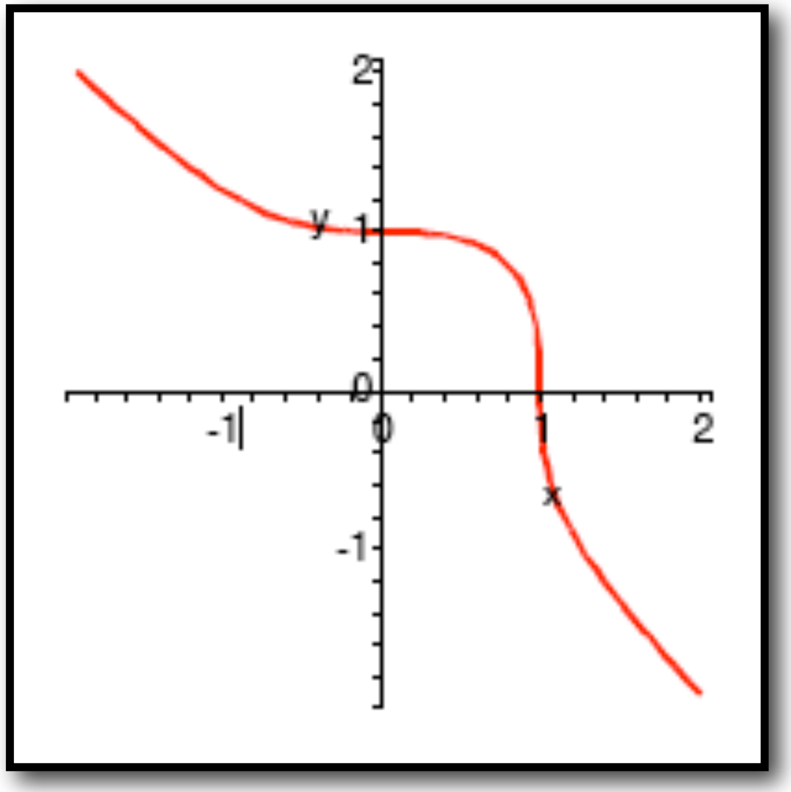
• The differential system $\boxed{s' = c, \quad c' = -s}$ gives rise to *entire functions* that parameterize the circle $\mathbf{F}_2 : X^2 + Y^2 = 1$.

$$s = \sin(\cdot); \qquad c = \cos(\cdot).$$



• The differential system $\boxed{s' = c^2, \quad c' = -s^2}$ gives rise to *meromorphic functions* that parameterize the Fermat cubic $\mathbf{F}_3 : X^3 + Y^3 = 1$.

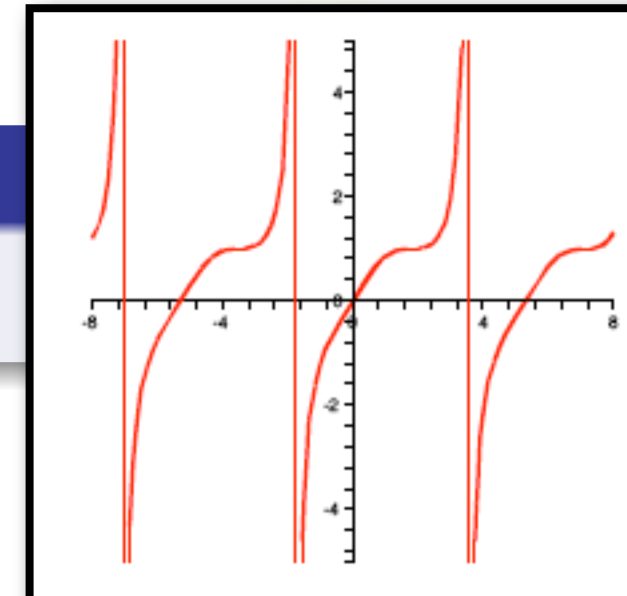
$$s = \mathrm{sm}(\cdot); \qquad c = \mathrm{cm}(\cdot).$$



— Lundberg's *hypergoniometric functions*.
— A. C. Dixon as a simple basis of elliptic functions.

Elliptic functions I

Def. *An <u>elliptic function</u> is a doubly periodic meromorphic function.*



Proposition

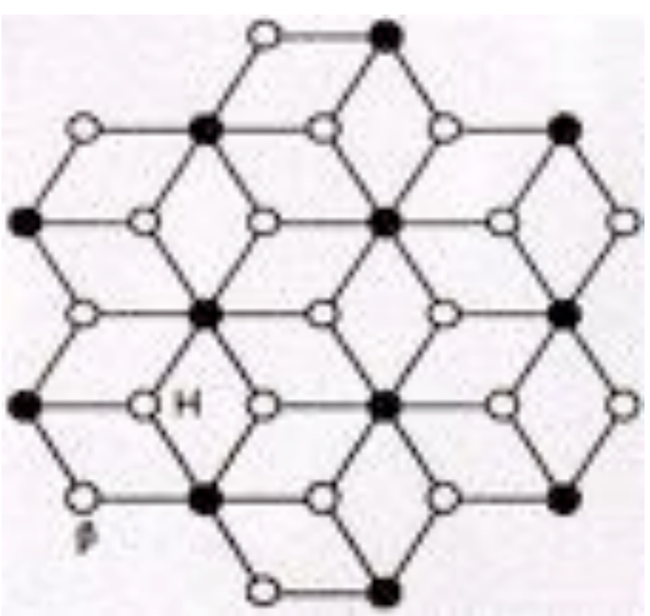
The function sm(·) *is* **elliptic**.

Proof (Exercise: bare-handed).

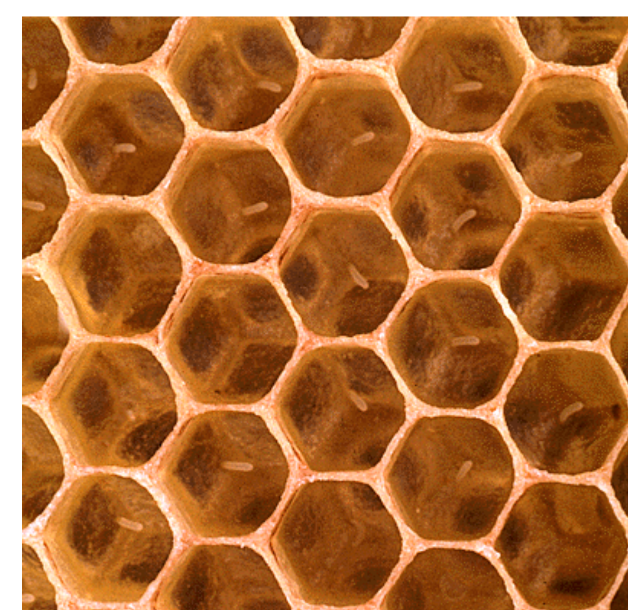
— analytic near 0; a *polar-like singularity* along real axis;

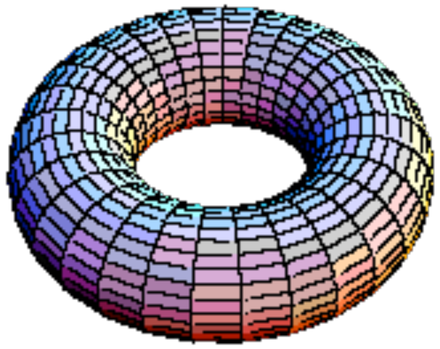
— is *simply periodic* with a real period $2^{-4/3}3^{1/2}\pi^{-1}\Gamma(1/3)^3$;

— satisfies *invariance by rotation* $\pm\frac{2\pi}{3}$, hence second period:



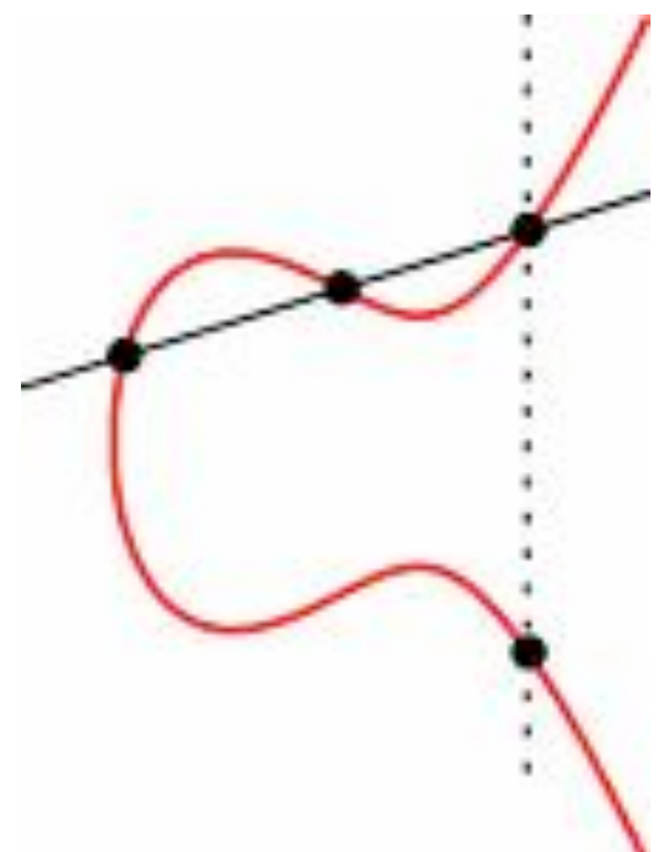
$$\mathrm{sm}(z) = z^1 - 4\frac{z^4}{3!} + 160\frac{z^7}{7!} - \cdots .$$





- **Algebraic curves of genus 1** are doughnuts. The integrals have two "periods". The inverse functions are **elliptic functions**; i.e., doubly periodic meromorphic.
- Weierstraß $\wp$ arises from $y^2 = P_3(z)$;
- Jacobian sn, cn arise from $y^2 = (1 - z^2)(1 - k^2 z^2)$;
- Dixonian sm, cm arise from $y^3 + z^3 = 1$.

They satisfy addition formulae!



2. Elliptic Connexions

~~~ Weierstraß forms & lattices ~~~
~~~

The Weierstraß $\wp$ function

Λ being a lattice of $\mathbb{C}$, consider $\boxed{\wp(z\,|\Lambda) := \sum{}' \frac{1}{(z-\Omega)^2} - \frac{1}{\Omega^2}}$.

- Clearly *meromorphic* and *doubly periodic*.
- Coefficients in $\wp(z) = z^{-2} + ?\, z^2 + ?\, z^4 + \ldots$ are *lattice invariants*:

$$g_2 = 60 \sum{}' \Omega^{-4}; \qquad g_4 = 140 \sum{}' \Omega^{-4}, \ \ldots$$

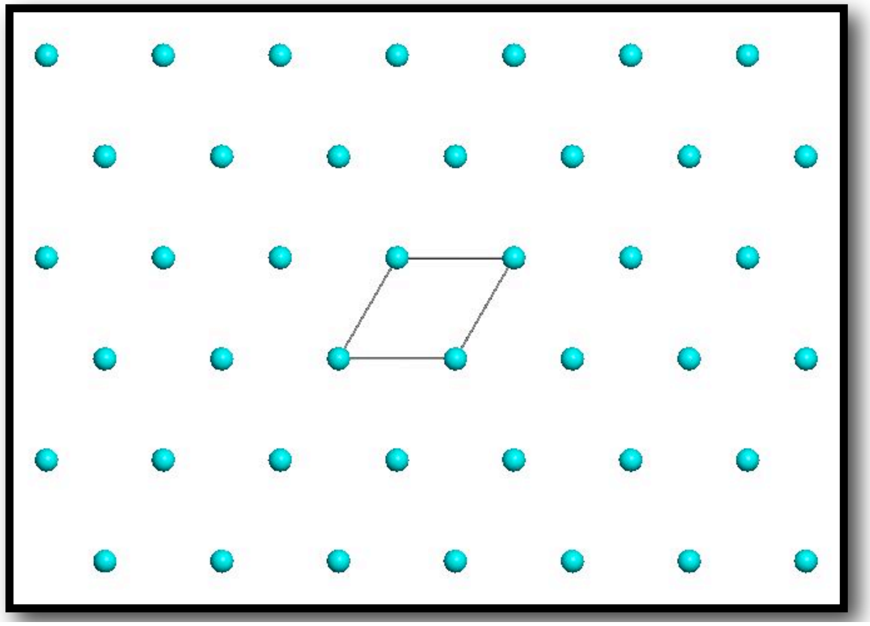
- It satisfies the *differential equation*

$$\wp'(z) = 4\wp(z)^3 - g_2\wp(z) = g_3$$

and it *parametrizes the curve $Y^2 = 4X^3 - g_2X - g_3$.*

- It is the inverse of an *elliptic integral*:

$$\mathrm{Inv} \int_y^\infty \frac{ds}{\sqrt{4s^3 - g_2 s - g3}},$$



Proof: match zeros and poles; use (as usual) Liouville's Theorem.

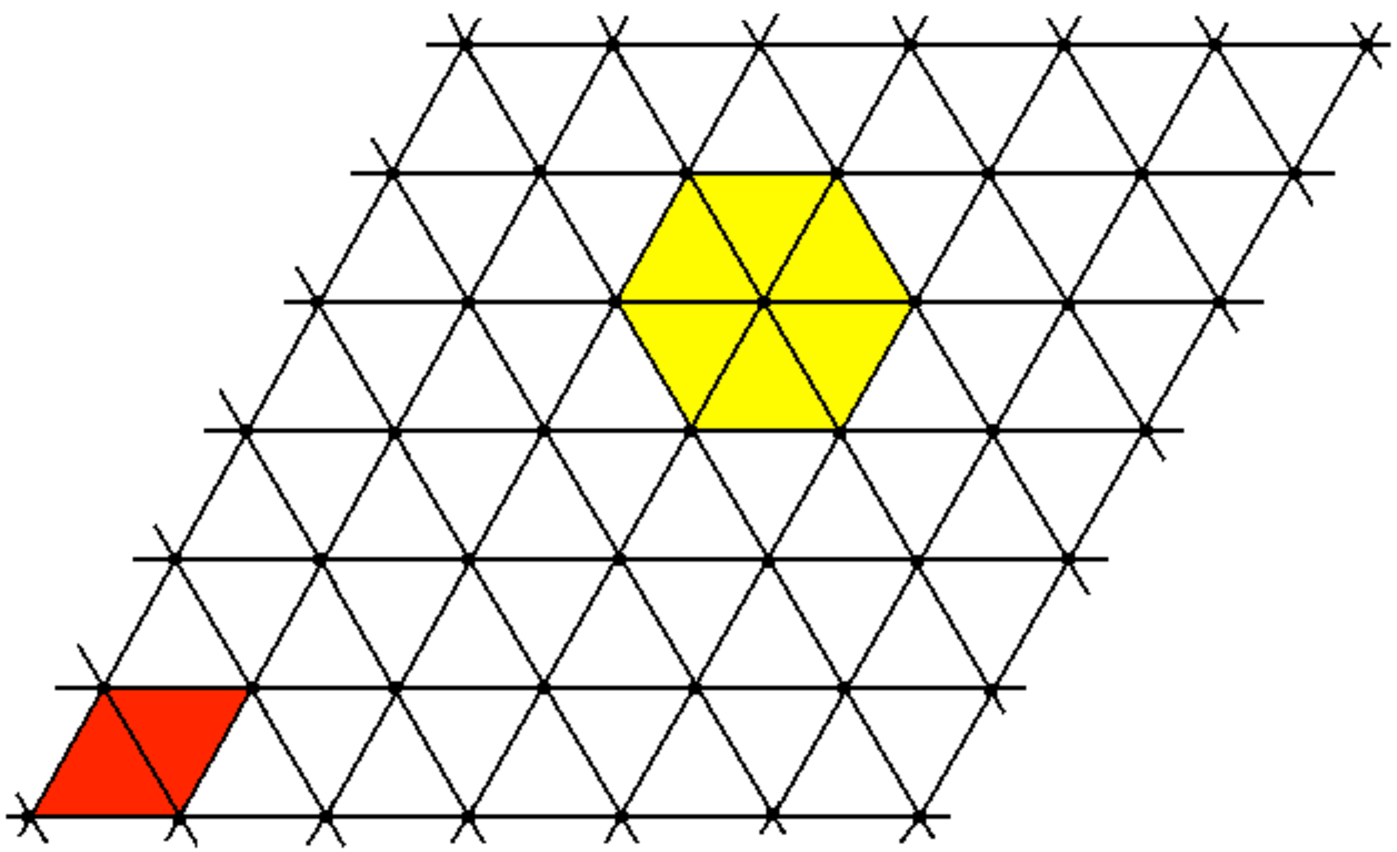
Theorem

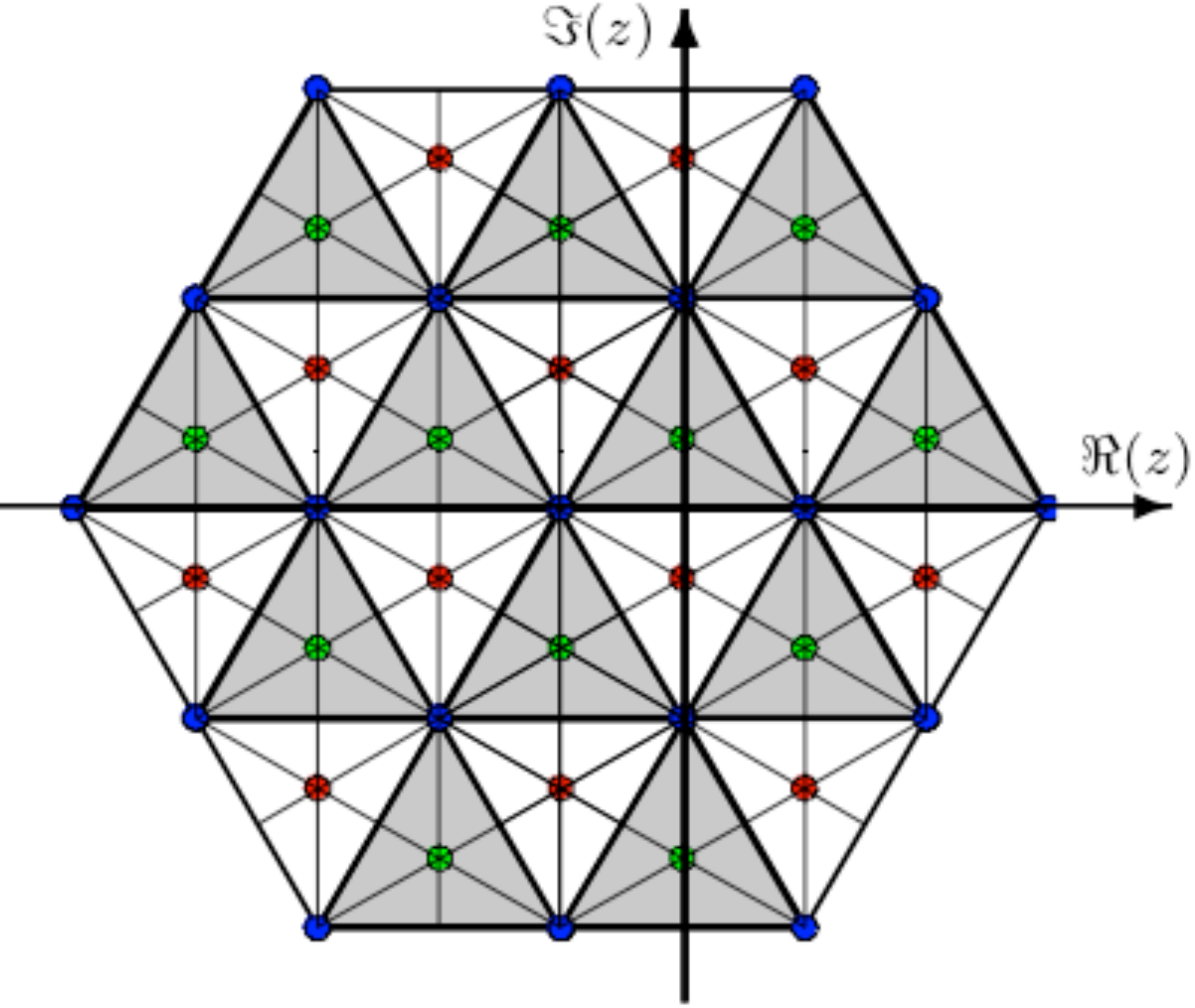
With $f(z)$ the EGF of pseudo-factorials:

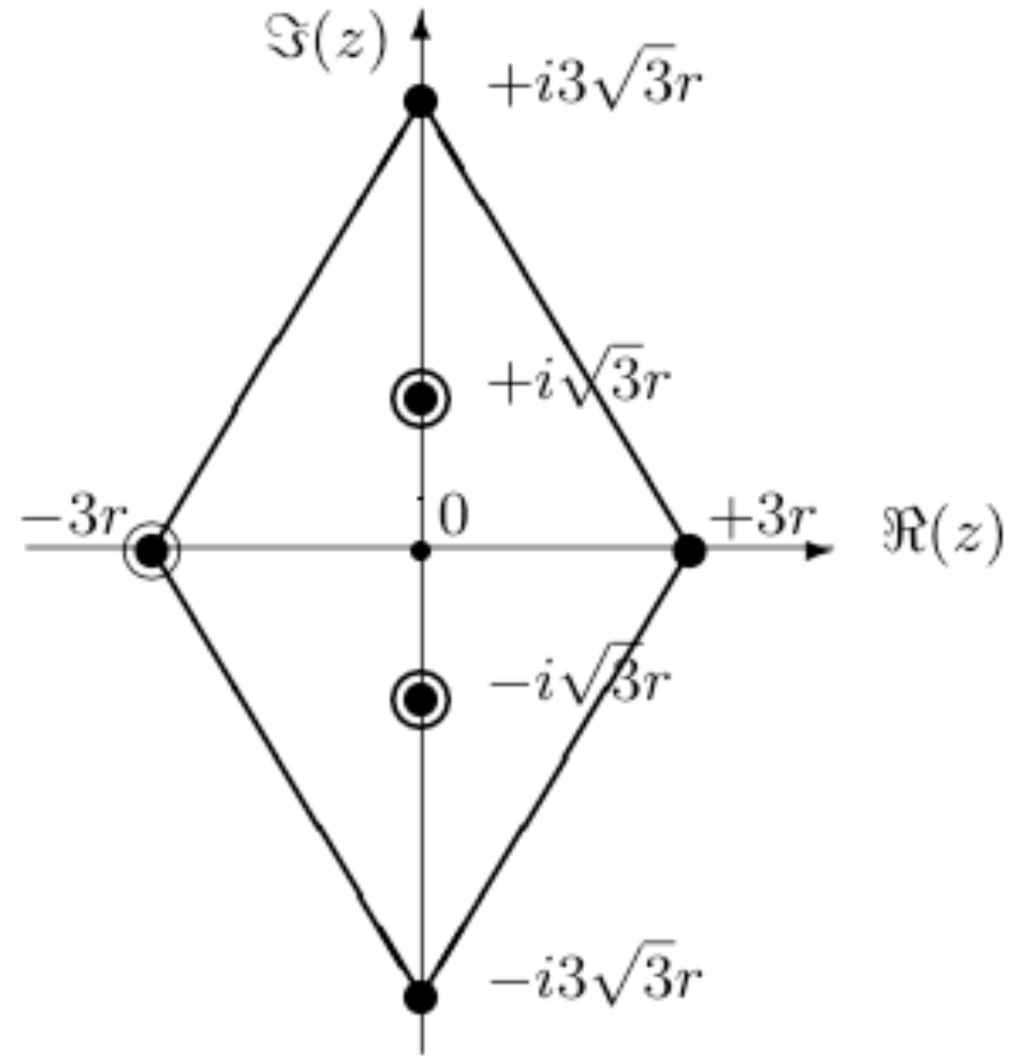
$$f(i\sqrt{3}z) = \frac{-\wp'(z+3r) - 2i\sqrt{3}}{2i\sqrt{3}\wp(z+3r)}, \qquad \wp(z) \equiv \wp(z;0,-4),$$

where $6r = \pi_3 2^{-1/3} = 2^{-4/3}3^{1/2}\pi^{-1}\Gamma(1/3)^3$.

Notation: $\wp(z; g_2, g_3)$. Here: $\wp(z; \mathbf{0}, -4)$ relative to hex lattice.







Theorem 3. *The pseudo-factorials are expressible as lattice sums involving a twelfth root of unity: with* $\rho = 2r\sqrt{3}$ *and* r *as in Eq.* (22)*, one has, for any* $n \geq 2$*:*

$$(26) \qquad \alpha_n = -\frac{n!}{\rho^{n+1}} \sum_{\lambda,\mu \in \mathbb{Z}} \frac{\zeta^{8\lambda+4\mu}}{\left[\left(\lambda - \frac{1}{2}\right)\zeta + \left(\mu - \frac{1}{2}\right)\zeta^{-1}\right]^{n+1}}, \qquad \zeta := e^{i\pi/6}.$$

Asymptotics...

3. Continued Fractions

~~` The Stieltjes-Rogers addition theorem ~~~

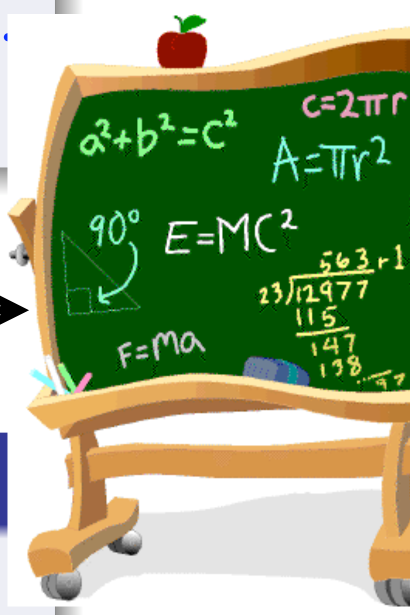
The Stieltjes–Rogers Theorem (I)

Definition

The function $\varphi(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{n!}$ satisfies an addition formula iff

$$\varphi(x+y) = \sum_k \omega_k \varphi_k(x)\varphi_k(y), \quad \text{where} \quad \varphi_k(x) = \frac{x^k}{k!} + O(x^{k+1}).$$

Examples: $\dfrac{1}{1-z}, \quad \dfrac{1}{\cos(z)}.$



Theorem

An addition formula automatically gives a continued fraction for

$$F(z) = \sum_{n=0}^{\infty} a_n z^n = \left\langle\!\!\left\langle \int_0^{\infty} e^t \varphi(zt)\, dt \right\rangle\!\!\right\rangle.$$

The Stieltjes–Rogers Theorem (II)

Data: $\varphi(z) = 1 + \sum_n a_n \frac{z^n}{n!}$; $\quad F(z) = 1 + \sum_n a_n z^n$

SR : $\varphi(x+y) = \sum_k \omega_k \varphi_k(x)\varphi_k(y)$; $\quad \varphi_k(x) = \frac{x^k}{k!} + \varphi_{k,k+1}\frac{x^{k+1}}{(k+1)!} + \cdots$

Continued fraction :

$$F(z) = \cfrac{1}{1 - c_0 z - \cfrac{a_1 z^2}{1 - c_1 z - \cfrac{a_2 z^2}{\ddots}}}$$

Dictionary :

$$\omega_k = a_1 a_2 \cdots a_k; \qquad c_k = \varphi_{k,k+1} - \varphi_{k-1,k}$$

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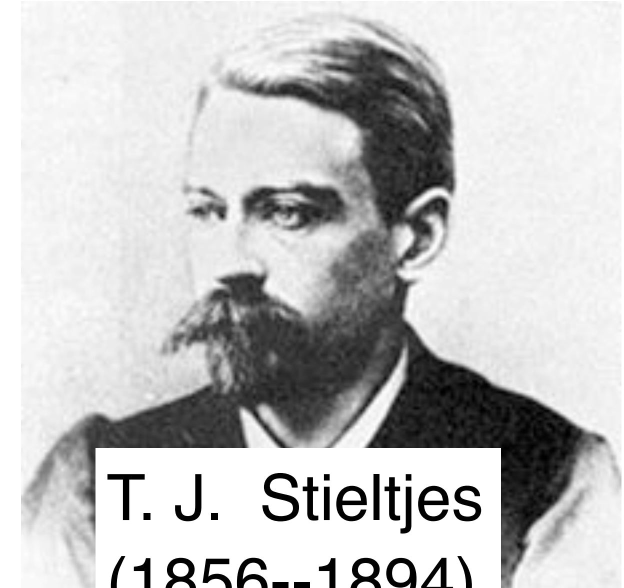
$$1 - 1x + 2x^2 - 6x^3 + 24x^4 - 120x^5 + 720x^6 - 5040x^7 + \text{etc.} =$$

$$A = \cfrac{1}{1 + \cfrac{x}{1 + \cfrac{x}{1 + \cfrac{2x}{1 + \cfrac{2x}{1 + \cfrac{3x}{1 + \cfrac{3x}{1 + 4x}}}}}}}$$

$$\int_0^\infty \operatorname{sech}^k u \, e^{-zu} du = \cfrac{1}{z + \cfrac{1 \cdot k}{z + \cfrac{2(k+1)}{z + \cfrac{3(k+2)}{z + \cdot_{\cdot_{\cdot}}}}}}$$



T. J. Stieltjes
(1856--1894)

Ainsi nous avons

$$) \quad . \int_0^\infty c \, \frac{\sinh(a u) \sinh(b u)}{\sinh(c u)} e^{-x u} \, d u = \frac{a b}{x^2 + \lambda_0 -} \; \frac{\mu_1}{x^2 + \lambda_1 -} \; \frac{\mu_2}{x^2 + \lambda_2 -} \cdots$$

$$\lambda_n = (2 n^2 + 2 n + 1) c^2 - a^2 - b^2,$$

$$\mu_n = \frac{4 n^2}{4 n^2 - 1} (n^2 c^2 - a^2)(n^2 c^2 - b^2).$$

R. Apéry

On peut déduire de la formule (21) d'une façon analogue

$$) \quad . \quad . \; 4 x^3 \Sigma_0^\infty \frac{1}{(x+n)^3} - 2x - 2 = \frac{1}{x+} \; \frac{p_1}{x+} \; \frac{q_1}{x+} \; \frac{p_2}{x+} \; \frac{q_2}{x+}$$

$$p_n = \frac{n^2 (n+1)}{4n+2},$$

$$q_n = \frac{n (n+1)^2}{4n+2},$$

4. The addition formula

~~~ relative to the OGF $F(z)$ ~~
~~~

Addition formula: Approaches

Lemma

In an addition formula,

$$\mathbf{SR}: \varphi(x+y) = \sum_k \omega_k \varphi_k(x)\varphi_k(y); \quad \varphi_k(x) = \frac{x^k}{k!} + \cdots$$

the $\varphi_k(x)$ *lie in* $\mathbb{C}[\varphi(x), \varphi'(x), \varphi''(x), \ldots]$.

E.g.: $\sec(z) \quad \rightsquigarrow \quad \frac{1}{k!}\sec(z)\cdot\tan(z)^k$

Thus: for pseudo-factorials, the φ_k *must be* elliptic functions.

Caveat: The usual elliptic function formulae *cannot* be imported verbatim $\not\equiv$ **SR**:

$$\wp(x+y) = \frac{1}{4}\left(\frac{\wp'(x)-\wp'(y)}{\wp(x)-\wp(y)}\right)^2 - \wp(x) - \wp(y).$$

The Meixner class

Definition (The Meixner Class)

$$\varphi(x+y) \;=\; \varphi(x)\varphi(y)\,\Psi\big(\sigma(x)\cdot\sigma(y)\big), \qquad \sigma(0)=0,\ \sigma'(0)\neq 0.$$

Property: GF of orthopolys $\sum Q_k(z)t^k/k!$ is $A(t)e^{zH(t)}$.

Theorem [Meixner]: *There are only five cases reducible to*

$$\sec(z),\quad \frac{1}{1-z},\quad e^{e^z-1},\quad e^{z^2/2},\quad \frac{1}{2-ez}.$$

No elliptic function!! ... **but** ...

The Jacobian elliptic functions

~ ODD/EVEN Meixner

$$\mathrm{sn}(z) := \mathbf{Inv} \int_0^y \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}.$$

with $\mathrm{cn}(z) := \sqrt{1-\mathrm{sn}(z)^2}$; $\mathrm{dn}(z) := \sqrt{1-k^2\,\mathrm{sn}(z)^2}$.

They satisfy an addition formula with *odd/even alternation*

$$\mathrm{cn}(x+y) = \frac{\mathrm{cn}\,x\,\mathrm{cn}\,y - \mathrm{sn}\,x\,\mathrm{dn}\,x\;\mathrm{sn}\,y\,\mathrm{dn}\,y}{1-k^2\,\mathrm{sn}^2\,x\,\mathrm{sn}^2\,y}$$

I.e.: cn; $-\mathrm{cn}\,\mathrm{sn}\,\mathrm{dn}$; $k^2\,\mathrm{cn}\,\mathrm{sn}^2$; $-k^2\,\mathrm{cn}\,\mathrm{sn}^3\,\mathrm{dn}$; $k^2\,\mathrm{cn}\,\mathrm{sn}^4$; ...

$$\sum \mathrm{cn}_n\,z^n = \cfrac{1}{1-\cfrac{1\cdot z^2}{1-\cfrac{2k^2\cdot z^2}{1-\cfrac{3^2\cdot z^2}{\cdots}}}}.$$

Try a formula à la "cn" ...?

$$f(x+y) = f(x)f(y)\Psi(\sigma(x)\sigma(y)) \; + \; h(x)h(y)\Xi(\tau(x)\tau(y))$$

with $\sigma(x) = O(x^2)$; $\tau(x) = O(x^2)$; $h(x) = O(x)$.

Hope for $\qquad \varphi_{2j} \propto f(x)\sigma(x)^j; \qquad \varphi_{2j+1}(x) \propto h(x)\tau(x)^{j+1}$

- We indeed verify to order 100 and indices till 25 that

$$\frac{\varphi_2(x)}{\varphi_0(x)} \propto \frac{\varphi_4(x)}{\varphi_2(x)} \propto \cdots; \qquad \frac{\varphi_3(x)}{\varphi_1(x)} \propto \frac{\varphi_5(x)}{\varphi_3(x)} \propto \cdots.$$

- We **can't miss** $\sigma(z) = \tau(z) = 3(z^2 - z^4 + z^6 - \frac{6}{7}z^8 + \cdots)$.
- We **know** that $\sigma(z)$ must be linearly related to $f, f', f'', \ldots$
- We can infer that Ψ, Ξ are simple rational functions.

Proposition (Conjecture?)

The EGF of pseudo factorials satisfies

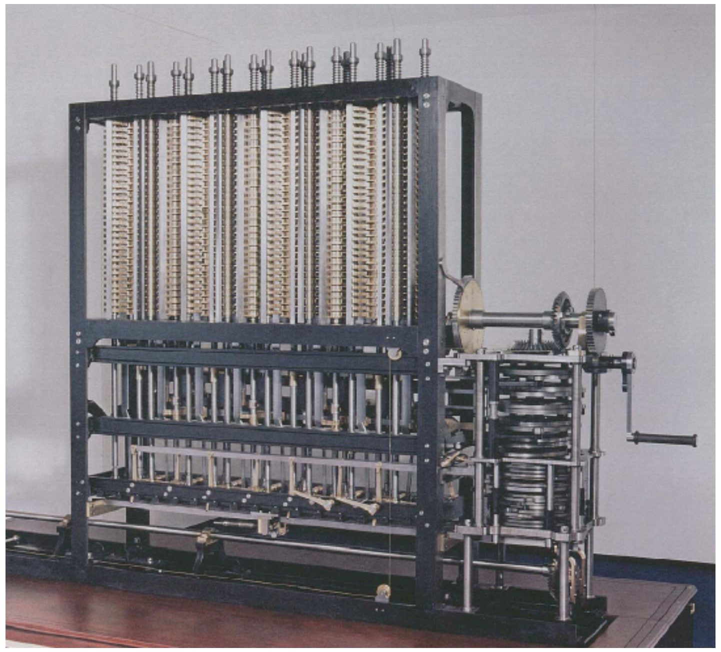
$$f(x+y) = \frac{f(x)f(y) - \frac{1}{3}(f(x)+f'(x))(f(y)+f'(y))}{1 - \frac{1}{3}(1 - f(x)f(-x))(1 - f(y)f(-y))}.$$

Proof. (i) **Reduce** to Weierstraß $\wp$ with known addition formula;

(ii) **Get huge** fraction $\mathcal{D} = A/B$ in X_1, Y_1, X_2, Y_2, where
$X_1 = \wp(x/i\sqrt{3})$, $Y_1 = \wp'(x/i\sqrt{3})$,
$(X_2 = \wp(y/i\sqrt{3})$, $Y_1 = \wp'(x/i\sqrt{3}))$.

(iii) **Check reduction** to 0 modulo

$$\{ Y_1^2 \mapsto 4X_1^3 + 4, \quad Y_2^2 \mapsto 4X_2^3 + 4 \}$$



$\mathcal{D} = \mathcal{A}/\mathcal{B}$, where A has 2388 monomials. Effect multivariate GCD:
$A \mapsto 0$; $B \not\mapsto 0$.

Alin, Frédéric, Bruno...

Theorem

The OGF of pseudo-factorials satisfies

$$F(z) = \cfrac{1}{1 + 1\,z + \cfrac{3 \cdot 1^2 \cdot z^2}{1 - 1\,z + \cfrac{2^2 \cdot z^2}{1 + 3\,z + \cfrac{3 \cdot 3^2 \cdot z^2}{\ddots}}}}.$$

Coefficients:

$$c_j = (-1)^{j-1}\left(j + \frac{1 + (-1)^j}{2}\right); \qquad a_j = j^2(2 - (-1)^j).$$

5. Orthogonal Polynomials

~~~ A new brand of “elliptic” polynomials ~~~
~~~

A family of orthogonal polynomials

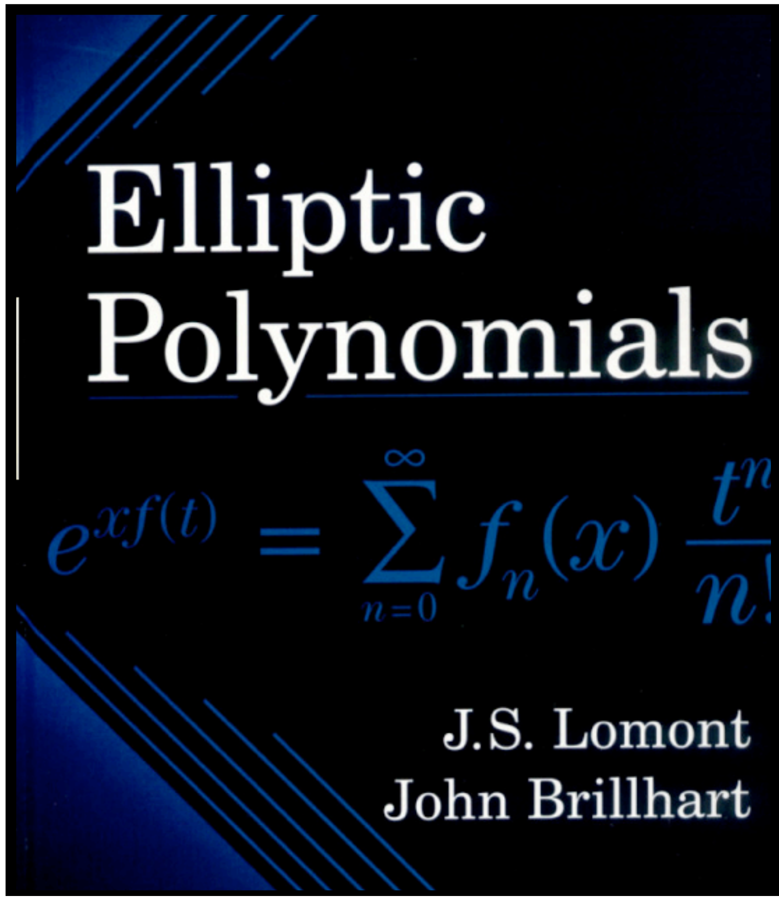
Convergents of the form $\frac{P_k(z)}{Q_k(z)}$ are obtained by truncating the continued fraction at level k:

$$\frac{0}{1}, \quad \frac{1}{1+z}, \quad \frac{1-2z+z^2}{1+3z+6z^2+10z^3}, \quad \ldots$$

Theorem: *The reciprocal polys of the denominators are formally orthogonal w.r.t. a measure whose moments are the a_n.*

What are these polynomials?

Cf **elliptic polynomials** by Carlitz (for "cn") *et al.*



Consider the reciprocal denominators polynomials $q_k(z) := z^k Q_k(1/z)$ and their EGF:

$$\Upsilon(z,t) := \sum_{k=0}^{\infty} q_k(z) \frac{t^k}{k!}$$

Theorem

We have $\Upsilon(z,t) = \eta(t)\cosh(zJ(t)) + \chi(t)\sinh(zJ(t))$, *where*

$$J(t) := \int_0^t \frac{du}{\sqrt{1-3u^2+3u^4}}$$

and $\eta(t) = 1 + t + \cdots$ *is a branch of the genus 0 cubic*

$$2 + 3t + 3t(1+t)\eta - 2(1-3t^2+t^4)\eta^3 = 0.$$

Also: $\chi(t) = \sqrt{\eta(t)^2 - \dfrac{2t(1+t)}{1-3t^2+3t^4}}$.

$$\Upsilon(z,t) = \eta(t)\cosh(zJ(t)) + \chi(t)\sinh(zJ(t)), \quad J(t) := \int_0^t \frac{du}{\sqrt{1-3u^2+3u^4}}$$

Interest: a new brand of elliptic polynomials $\notin$ {Carlitz, Al Salam, Lomont–Brilhart, Ismail–Masson}.

Suggests: A perhaps interesting "bimodal" Meixner class, yet to be studied: $\exp \mapsto \sinh, \cosh$?

Remarkably, for the original Dixon "sm,cm", Gilewicz, Valent *et al.* have a "trimodal" $\approx E_3(zK(t))$ where $E_3(\,\cdot\,)$ is a 3–section of the exponential and $K(t)$ is the elliptic integral $\int (1-t^3)^{-2/3}$.

Elliptic functions/integrals, continued fractions,
& orthogonal polynomials
What goes on here???
[$\rightsquigarrow$ more to come at the end!]

Proof (orthopolys)

- GF of polys is *a priori* within the **holonomic class**.
- But by brute force: degree/order is high and no chance of solving directly. **Need to simplify!**
- Mixture of induction and guessing with Gfun's guess
- Verifications (proof) based on closure prop'ties, also w/ Gfun.

THANKS, BRUNO!!

6. Hankel determinants & Congruences

~~~ plus some (conjectured) goodies ~~~
~~~

Hankel determinants

Corollary 1 *Let m be a positive integer. The Hankel determinant of pseudo-factorials*

$$H_m^{(0)} := \begin{vmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_{m-1} \\ \alpha_1 & \alpha_2 & \cdots & \alpha_m \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m-1} & \alpha_m & \cdots & \alpha_{2m-2} \end{vmatrix}$$

admits the closed form

$$H_m^{(0)} = \prod_{j=1}^{m-1} a_j^{m-j} = \begin{cases} (-1)^{m/2} 3^{m^2/4} \left(\prod_{k=1}^{m-1} k!\right)^2 & (m \text{ even}) \\ (-1)^{(m-1)/2} 3^{(m^2-1)/4} \left(\prod_{k=1}^{m-1} k!\right)^2 & (m \text{ odd}), \end{cases}$$

where the $a_j = -j^2(2-(-1)^j)$ *are the continued fraction numerators of* (43).

Congruences

$\overline{6,5,2,2,2,2,4,1,6,6,6,6,5,3,4,4,4,4,1,2,5,5,5,5,3,6,1,1,1,1,2,4,3,3,3,3}$

mod 7

An identity …everyhone should know!

If $F(z)$ has CF convergents $\left(\frac{P_k(z)}{Q_k(z)}\right)$, then

$$F(z) = \frac{P_m(z)}{Q_m(z)} + \sum_{k \geq m} \frac{a_1 a_2 \cdots a_k}{Q_k(z)\, Q_{k+1}(z)}.$$

If the CF's numerators involve integers, then we get **congruences modulo** $A_m := a_1 a_2 \cdots a_m$:
The original sequence (α_n) *is eventually periodic mod* $a_1 \cdots a_m$.

mod 5	mod 11	mod 17	mod 23
$\frac{P_5}{Q_5} \equiv \Pi_4$	$\frac{P_{11}}{Q_{11}} \equiv \Pi_{10}$	$\frac{P_{17}}{Q_{17}} \equiv \Pi_{16}$	$\frac{P_{23}}{Q_{23}} \equiv \Pi_{22}$

mod 7	mod 13	mod 19	mod 31
$\frac{P_7}{Q_7} \equiv \frac{\Pi_6}{1+4z^6}$	$\frac{P_{13}}{Q_{13}} \equiv \frac{\Pi_{12}}{1+11z^{12}}$	$\frac{P_{19}}{Q_{19}} \equiv \frac{\Pi_{18}}{1+11z^{18}}$	$\frac{P_{31}}{Q_{31}} \equiv \frac{\Pi_{30}}{1+4z^{30}}$

So… Where are we?

- **Theorem [F; Dumont 1980].** *Jacobian elliptic functions count alternating perms w/parity of peaks.*

- **Theorem [Conrad+F, 2006].** *Dixonian functions have continued fractions*

$$\int_0^\infty \mathrm{sm}(u)e^{-u/x}\,du = \cfrac{x^2}{1+b_0x^3-\cfrac{1\cdot 2^2\cdot 3^3\cdot 4\,x^6}{1+b_1x^3-\cfrac{4\cdot 5^2\cdot 6^2\cdot 7\,x^6}{\cdots}}},$$

$\equiv$ *levels in trees and an urn model ($\approx$Yule process), &c*

- **Theorem [Bacher+F, 2006].** *Pseudofactorials* $a_{n+1} = (-1)^{n+1}\sum\binom{n}{k}a_k a_{n-k}$ *have a CF*

$$\sum a_n z^n = \cfrac{1}{1+z+\cfrac{3\cdot 1^2\,z^2}{1-z+\cfrac{2^2\,z^2}{1+3z+\ddots}}}.$$

Combinatorics
perms, partitions,
trees,
urn processes...

$$\zeta(3) = \cfrac{6}{\varpi(0)-\cfrac{1^6}{\varpi(1)-\cfrac{2^6}{\varpi(2)-\cfrac{3^6}{\ddots}}}},$$

Merry Christmas

GIFT. *Define the Hurwitz numbers* [Math. Ann. **51** *(1899), 196–226*] *by*

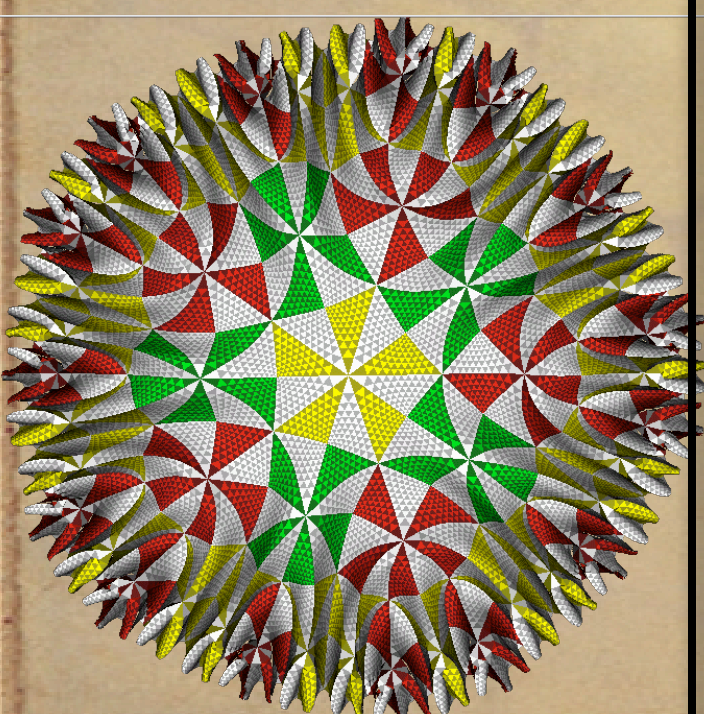
$$E_\nu = \frac{(4\nu)!}{\Omega^\nu} \sum_{r,s\in\mathbb{Z}}^{\star} \frac{1}{(r+s\sqrt{-1})^{4\nu}}, \qquad \Omega := \frac{1}{\sqrt{2\pi}}\Gamma\left(\frac{1}{4}\right)^2.$$

The ordinary generating function of (E_ν) *satisfies*

$$10\sum_{\nu\geq 1} E_\nu z^{\nu-1} = \cfrac{1}{1-\cfrac{c_1\cdot z}{1-\cfrac{c_2\cdot z}{\ddots}}},$$

where $c_1 = 3$, $c_2 = \frac{150}{13}$, *and* $c_n = \begin{cases} \dfrac{1}{16}\dfrac{(2n)(2n+1)^2(2n+2)^3}{(4n+1)(4n+5)} & (n \text{ even}) \\ \dfrac{1}{16}\dfrac{(2n+1)^3(2n+2)^2(2n+3)}{(4n+1)(4n+5)} & (n \text{ odd}). \end{cases}$

Happy New Year 2009



GIFT. *Define the "equiharmonic numbers" by*

$$K_\nu := \frac{(6\nu)!}{\Omega^{6\nu}} \sum_{(n_1,n_2)\in(\mathbb{Z}\times\mathbb{Z})\setminus\{(0,0)\}} \frac{1}{(n_1 e^{-2i\pi/3} + n_2 e^{2i\pi/3})^{6\nu}}, \qquad \Omega := \frac{1}{2\pi}\Gamma\left(\frac{1}{3}\right)^3.$$

The generating function of (K_ν) *admits the continued fraction representation*

$$\frac{7}{36}\sum_{\nu\geq 1} K_\nu z^{\nu-1} = \cfrac{1}{1-\cfrac{d_1\cdot z}{1-\cfrac{d_2\cdot z}{\ddots}}},$$

where $d_1 = \frac{10880}{13}$, $d_2 = \frac{13810240}{247}$, $d_n = \frac{1}{4}\frac{(3n)(3n+1)^2(3n+2)^2(3n+3)^2(3n+4)}{(6n+1)(6n+7)}$.

27 Dec 2008

Facts calling for a theory...

- Classification of orthogonal polynomials, cf Meixner
- Multimodal addition formulae in relation to continued fractions and orthogonal polynomials???
- Understanding (some of) Pollaczek continued fractions???
- Relations between CF & holonomy???
- Elliptic functions, continued fractions, and higher genus???