

Singular Combinatorics

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Abstract

Combinatorial enumeration leads to counting generating functions presenting a wide variety of analytic types. Properties of generating functions at singularities encode valuable information regarding asymptotic counting and limit probability distributions present in large random structures. “Singularity analysis” reviewed here provides constructive estimates that are applicable in several areas of combinatorics. It constitutes a complex-analytic Tauberian procedure by which combinatorial constructions and asymptotic–probabilistic laws can be systematically related.

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1. Introduction

Large random combinatorial structures tend to exhibit great statistical regularity. For instance, an overwhelming proportion of the graphs of a given large size are connected, and a fixed pattern is almost surely contained in a long random string, with its number of occurrences satisfying central and local limit laws. The objects considered (typically, words, trees, graphs, or permutations) are given by construction rules of the kind classically studied by combinatorial analysts via *generating functions* (abbreviated as GFs). A fundamental problem is then to extract asymptotic information on coefficients of a GF either explicitly given by a formula or implicitly determined by a functional equation. In the univariate case, asymptotic counting estimates are derived; in the multivariate case, moments and limit probability laws of characteristic parameters will be obtained.

In what follows, given a combinatorial class \mathcal{C} , we let C_n denote the number

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of objects in \mathcal{C} of size n and introduce the *ordinary* and *exponential* GF (OGF, EGF),

$$\text{OGF: } C(z) := \sum_{n \geq 0} C_n z^n, \quad \text{EGF: } \hat{C}(z) := \sum_{n \geq 0} C_n \frac{z^n}{n!}.$$

Generally, EGFs and OGFs serve for the enumeration of labelled classes (atoms composing objects are distinguished by labels) and unlabelled classes, respectively. One writes $C_n = [z^n]C(z) = n![z^n]\hat{C}(z)$, with $[z^n](\cdot)$ the coefficient extractor.

General rules for deriving GFs from combinatorial specifications have been widely developed by various schools starting from the 1970's and these lie at the heart of contemporary combinatorial analysis. They are excellently surveyed in books of Foata & Schützenberger (1970), Comtet (1974), Goulden & Jackson (1983), Stanley (1986, 1998), Bergeron, Labelle & Leroux (1998). We shall retain here the following simplified scheme relating combinatorial constructions and operations over GFs:

<i>Construction</i>		<i>Labelled case</i>	<i>Unlabelled case</i>	
Disjoint union	$\mathcal{F} + \mathcal{G}$	$f(z) + g(z)$	$\hat{f}(z) + \hat{g}(z)$	(1.1)
Product	$\mathcal{F} \times \mathcal{G}, \mathcal{F} \star \mathcal{G}$	$f(z) \cdot g(z)$	$\hat{f}(z) \cdot \hat{g}(z)$	
Sequence	$\mathfrak{S}\{\mathcal{F}\}$	$(1 - f(z))^{-1}$	$(1 - \hat{f}(z))^{-1}$	
Set	$\mathfrak{P}\{\mathcal{F}\}$	$\exp(f(z))$	$\exp(\hat{f}(z) + \frac{1}{2}\hat{f}(z)^2 + \dots)$	
Cycle	$\mathfrak{C}\{\mathcal{F}\}$	$\log(1 - f(z))^{-1}$	$\log(1 - \hat{f}(z))^{-1} + \dots$	

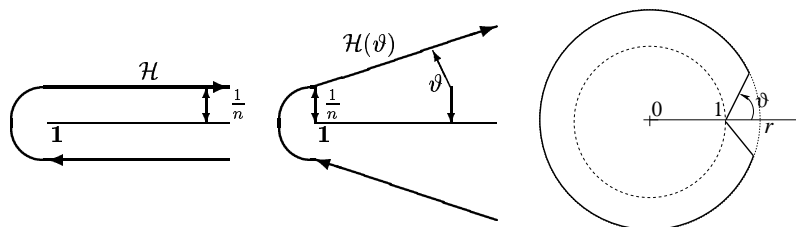
Such operations on GFs yield a wide variety of analytic functions, either given explicitly or as solutions to functional equations in the case of recursively defined classes. It is precisely the goal of singularity analysis to provide means for extracting asymptotic informations. What we termed “singular combinatorics” aims at relating combinatorial form and asymptotic-probabilistic form by exploiting *complex-analytic* properties of generating functions. Classical approaches [2] are Tauberian theory and Darboux’s method, an offspring of elementary Fourier analysis largely developed by Pólya for his programme of combinatorial chemistry [17]. The path followed here, called “singularity analysis” after [7], consists in developing a systematic correspondence between the local behaviour of a function near its singularities and the asymptotic form of its coefficients. (An excellent survey of central aspects of the theory is offered by Odlyzko in [15].)

2. Basic Singularity Analysis

Perhaps the simplest coefficient estimate is $[z^n](1 - z)^{-\alpha} \sim n^{\alpha-1}/\Gamma(\alpha)$, a consequence of the binomial expansion and Stirling’s formula. For the basic scale

$$\sigma_{\alpha,\beta}(z) = (1 - z)^{-\alpha} \left(\frac{1}{z} \log(1 - z)^{-1} \right)^\beta,$$

much more is available and one has a fundamental translation mechanism [7]:

Figure 1: The Hankel contours, \mathcal{H} and $\mathcal{H}(\vartheta)$, and a Δ -domain.

Theorem 1 (Coefficients of the basic scale) For $\alpha \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, $\beta \in \mathbb{C}$, one has

$$[z^n] \sigma_{\alpha, \beta}(z) \underset{n \rightarrow \infty}{\sim} \frac{n^{\alpha-1}}{\Gamma(\alpha)} (\log n)^\beta. \quad (2.1)$$

Proof. The estimate is derived starting from Cauchy's coefficient formula,

$$[z^n] f(z) \sim \frac{1}{2i\pi} \int_{\gamma} f(z) \frac{dz}{z^{n+1}},$$

instantiated with $f = \sigma_{\alpha, \beta}$. The idea is then to select for γ a contour \mathcal{H} that is of Hankel type and follows the half-line $(1, +\infty)$ at distance exactly $1/n$ (Fig. 1). This superficially resembles a saddle-point contour, but with the integral normalizing to Hankel's representation of the Gamma function, hence the factor $\Gamma(\alpha)^{-1}$. \square

The method of proof is very flexible: it applies for instance to iterated logarithmic terms ($\log \log$) while providing full asymptotic expansions; see [7] for details.

A remarkable fact, illustrated by Theorem 1, is that larger functions near the singularity $z = 1$ give rise to larger coefficients as $n \rightarrow \infty$. This is a general phenomenon under some suitable auxiliary conditions, expressed here in terms of *analytic continuation*: a Δ -domain is an indented disc defined by $(r > 1, \vartheta < \pi/2)$

$$\Delta(\vartheta, r) := \{ z \mid |z| < r, \vartheta < \text{Arg}(z-1) < 2\pi - \vartheta, z \neq 1 \}.$$

Theorem 2 (O-transfer) With $f(z)$ continuable to a Δ -domain and $\alpha \notin \mathbb{Z}_{\leq 0}$:

$$f(z) \underset{z \rightarrow 1, z \in \Delta}{=} O(\sigma_{\alpha, \beta}(z)) \quad \implies \quad [z^n] f(z) = O([z^n] \sigma_{\alpha, \beta}(z)).$$

Proof. In Cauchy's coefficient formula, adopt an integration contour $\mathcal{H}(\vartheta)$ passing at distance $1/n$ left of the singularity $z = 1$, then escaping outside of the unit disc within Δ . Upon setting $z = 1 + t/n$, careful approximations yield the result [7]. \square

This theorem allows one to transfer error terms in the asymptotic expansion of a function at its singularity (here $z = 1$) to asymptotics of the coefficients. The Hankel contour technique is quite versatile and a statement similar to Theorem 2 holds with $o(\cdot)$ -conditions. replacing $O(\cdot)$ -conditions. The case of α being a negative integer is covered by minor adjustments due to $1/\Gamma(\alpha) = 0$; see [7]. In concrete terms: *Hankel contours combined with Cauchy coefficient integrals accurately "capture" the singular behaviour of a function.*

By Theorems 1 and 2, whenever a function admits an asymptotic expansion near $z = 1$ in the basic scale, one has the implication, with $\sigma \succ \tau \succ \dots \succ \omega$,

$$f(z) = \lambda\sigma(z) + \mu\tau(z) + \dots + O(\omega(z)) \implies f_n = \lambda\sigma_n + \mu\tau_n + \dots + O(\omega_n),$$

where $f_n = [z^n]f(z)$. In other words, a *dictionary* translates singular expansions of functions into the asymptotic forms of coefficients. Analytic continuation and validity of functions' expansions outside of the unit circle is a logical necessity, but once granted, application of the method becomes quite mechanical.

For combinatorics, singularities need not be placed at $z = 1$. But since $[z^n]f(z) \equiv \rho^{-n}[z^n]f(z\rho)$, the dictionary can be used for (dominant) singularities that lie anywhere in the complex plane. The case of finitely many dominant singularities can also be dealt with (via composite Hankel contours) to the effect that the translations of local singular expansions get composed additively. In summary one has from function to coefficients:

location+nature of singularity (fn.) \implies exponential+polynomial asymptotics (coeff.)

EXAMPLE 1. *2-Regular graphs* (Comtet, 1974). The class \mathcal{G} of (labelled) 2-regular graphs can be specified as sets of unordered cycles each of length at least 3. Symbolically:

$$\mathcal{G} \cong \mathfrak{P}\left\{\frac{1}{2}\mathfrak{C}_{\geq 3}\{\mathcal{Z}\}\right\} \quad \text{so that} \quad \widehat{G}(z) = \exp\left(\frac{1}{2}\log(1-z)^{-1} - \frac{z}{2} - \frac{z^2}{4}\right) = \frac{e^{-z/2-z^2/4}}{\sqrt{1-z}}.$$

\mathcal{Z} represents a single atomic node. The function $\widehat{G}(z)$ is singular at $z = 1$, and

$$G(z) \underset{z \rightarrow 1}{\sim} e^{-3/4}(1-z)^{-1/2} \implies \frac{G_n}{n!} \underset{n \rightarrow \infty}{\sim} \frac{e^{-3/4}}{\sqrt{\pi n}}.$$

This example can be alternatively treated by Darboux's method. □

EXAMPLE 2. The *diversity index* of a tree (Flajolet, Sipala & Steyaert, 1990) is the number of non-isomorphic terminal subtrees, a quantity also equal to the size of maximally compact representation of the tree as a directed acyclic graph and related to common subexpression sharing in computer science applications. The mean index of a random binary tree of size $2n+1$ is asymptotic to $Cn/\sqrt{\log n}$, where $C = \sqrt{8 \log 2/\pi}$. This results from an exact GF obtained by inclusion-exclusion:

$$K(z) = \frac{1}{2z} \sum_{k \geq 0} \frac{1}{k+1} \binom{2k}{k} \left(\sqrt{1-4z+4z^{k+1}} - \sqrt{1-4z} \right).$$

Singularities accumulate geometrically to the right of $1/4$ while $K(z)$ is Δ -continuable. The unusual singularity type $(1/\sqrt{X \log X})$ precludes the use of Darboux's method. □

Rules like those of Table (1.1) preserve analyticity and analytic continuation. Accordingly, generating functions associated with combinatorial objects described by simple construction rules usually have GFs amenable to singularity analysis. The method is systematic enough, so that an implementation within computer algebra systems is even possible as was first demonstrated by Salvy [18].

3. Closure properties

In what follows, we say that a function is *amenable to singularity analysis*, or “of *S.A. type*” for short, if it is Δ -continuable and admits there a singular expansion in

the scale $\mathcal{S} = \{\sigma_{\alpha,\beta}(z)\}$. First, functions of S.A. type include polylogarithms:

Theorem 3 *The generalized polylogarithms $\text{Li}_{\alpha,k}$ are of S.A. type, where $\text{Li}_{\alpha,k}(z) := \sum_{n \geq 1} (\log n)^k n^{-\alpha} z^n$, $k \in \mathbb{Z}_{\geq 0}$.*

The proof makes use of the Lindelöf representation

$$\sum_{n \geq 1} \phi(n)(-z)^n = -\frac{1}{2i\pi} \int_{1/2-\infty}^{1/2+\infty} \phi(s) z^s \frac{\pi}{\sin \pi s} ds,$$

in a way already explored by Ford [10], with Mellin transform techniques providing validity of the singular expansion in a Δ -domain [5].

EXAMPLE 3. *Entropy computations.* The GF of $\{\log(k!)\}$ is $(1-z)^{-1} \text{Li}_{0,1}(z)$, which is of S.A. type. The entropy of the binomial distribution, $\pi_{n,k} = \binom{n}{k} p^k (1-p)^{n-k}$, results:

$$H_n := -\sum \pi_{n,k} \log \pi_{n,k} \underset{n \rightarrow \infty}{\sim} \frac{1}{2} \log n + \frac{1}{2} + \log \sqrt{2\pi p(1-p)} + \dots$$

Such problems are of interest in information theory, where redundancy estimates precisely depend on higher order asymptotic properties (Jacquet-Szpankowski, 1998). Full expansions for functionals of the Bernoulli distribution are also obtained systematically [5]. \square

As it is well-known, asymptotic expansions can be integrated while differentiation of asymptotic expansions is permissible for functions analytic in sectors:

Theorem 4 *Functions of S.A. type are closed under differentiation and integration.*

Finally, the Hadamard product of two series, $f \odot g$ is defined as the termwise product: $f(z) \odot g(z) = \sum_n f_n g_n z^n$, if $f(z) = \sum_n f_n z^n$, $g(z) = \sum_n g_n z^n$. Hadamard (1898) proved that singularities get composed multiplicatively. Finer composition properties [4] result from an adaptation of Hankel contours to Hadamard's formula

$$f(z) \odot g(z) = \frac{1}{2i\pi} \int_{\gamma} f(t) g\left(\frac{w}{t}\right) \frac{dt}{t}.$$

Theorem 5 *Functions of S.A. type are closed under Hadamard product.*

EXAMPLE 4. *Divide-and-conquer* algorithms solve recursively a problem of size n by splitting it into two subproblems and recombining the partial solutions. Under the assumption of randomness preservation, the expected costs f_n satisfy a “tree recurrence” of the form

$$f_n = t_n + \sum_k \pi_{n,k} (f_k + f_{n-k-a}), \quad a \in \{0, 1\},$$

where the “toll” sequence t_n usually has a simple form (e.g., n^β , $\log n$) and the $\pi_{n,k}$ characterize the stochastic splitting process. The corresponding GFs then satisfy an equation $f(z) = t(z) + \mathcal{L}[f](z)$, where the linear operator \mathcal{L} reflects the splitting probabilities. For instance, binary search trees and the Quicksort algorithm have $\mathcal{L}[f(z)] = 2 \int_0^z f(x) dx / (1-x)$. One then has in operator notation $f(z) = (I - \mathcal{L})^{-1}[t](z)$, where the quasi-inverse acts as a “singularity transformer”. Closure theorems allow for an asymptotic classification of the cost functions induced by various tolls under various probabilistic models mirrored by the splitting probabilities [4]. \square

4. Functional equations

Algebraic functions have expansions at singularities that are expressed by fractional power series (Newton-Puiseux). Consequently, they are of S.A. type with rational exponents; accordingly their coefficient expansions are linear combinations of algebraic elements of the form $\omega^n n^{r/s}$, with ω and algebraic number and $r/s \in \mathbb{Q}$. By Weierstrass preparation, such properties extend to many *implicit* GFs. For instance, GFs of combinatorial families of trees constrained to have degrees in a finite set have a branch point of type \sqrt{X} at their dominant singularity, which in turn corresponds to the “universal” asymptotic form $T_n \sim c\omega^n n^{-3/2}$ for coefficients (Pólya, 1937; Otter 1948; Meir-Moon, 1978).

Order constraints in labelled classes are known to correspond to integral operators and, in the recursive case, there result GFs determined by *ordinary differential equations*. Furthermore, moments (i.e., cumulative values) of additive functionals of combinatorial structures defined by recursion and order constraints have GFs that satisfy *linear differential equations*, for which there is a well-established classification theory going back to the nineteenth century. In particular, in the *Fuchsian case*, singularity analysis applies unconditionally, so that the resulting coefficient estimates are linear combinations of terms $\omega^n n^\alpha (\log n)^k$, with ω, α algebraic and $k \in \mathbb{Z}_{\geq 0}$. This covers a large subset of the class of “holonomic” functions, of which Zeilberger has extensively demonstrated the expressive power in combinatorial analysis [23, 24].

EXAMPLE 5. *Quadrees* are a way to superimpose a hierarchical partitioning on sequences of points in d -dimensional space: the first point is taken as the root of the tree and it partitions the whole space in 2^d orthants in which successive points are placed and then made to refine the partition [13]. The problem is expressed by a linear differential equation with coefficients in $\mathbb{C}(z)$. The average cost of finding a point knowing only k of its coordinates is of the asymptotic form $c n^\alpha$ with $\alpha = \alpha(k, d)$ an algebraic number of degree d . For instance $k = 1$ and $d = 2$ yield a solution involving a ${}_2F_1$ hypergeometric function as well as $\alpha = (\sqrt{17} - 3)/2 \doteq 0.56155$, in contrast to an exponent $\frac{1}{2}$ that would correspond to a perfect partitioning, i.e., a regular grid (Flajolet, Gonnet, Puech & Robson, 1993). \square

Substitution equations correspond to “balanced structures” of combinatorics. An important rôle in the development of the theory has been played by Odlyzko’s analysis [14] of 2-3 trees (such trees have internal nodes of degree 2 and 3 only and leaves are all at the same depth). The OGF satisfies the equation $T(z) = z + T(\tau(z))$, with $\tau(z) := z^2 + z^3$, and has a singularity at $z = 1/\phi$, a fixed point of τ , ($\phi = (1 + \sqrt{5})/2$). The singular expansion involves periodic oscillations, corresponding to infinitely many singular exponents having a common real part. Singularity analysis extends to this case and the number of balanced 2-3 trees is found to be of the form $T_n \sim \frac{1}{n} \phi^n \Omega(\log n)$, for some nonconstant smooth periodic function Ω .

A similar problem of *singular iteration* arises in the analysis of the height of binary trees [6]. The GF $y_h(z)$ of trees of height at most h satisfies the Mandelbrot recurrence $y_h = z + y_{h-1}^2$, with $y_0 = z$. The fixed point is the GF of binary trees, that is, of Catalan numbers, $y_\infty = (1 - \sqrt{1 - 4z})/2$ which has its dominant singularity at $1/4$. The analysis of moments of the distribution of height turns out to be equivalent to developing *uniform* approximations to $y_h(z)$ as $z \rightarrow 1/4$ and $h \rightarrow \infty$

simultaneously, this for z in a Δ -domain. The end result, by singularity analysis and the moment method, is: *the height of a random binary tree with n external nodes when normalized by a factor of $1/(2\sqrt{n})$ converges in distribution to a theta law defined by the density $4x \sum_{k \geq 1} k^2(2k^2x^2 - 3)e^{-k^2x^2}$.* The result extends to all simple families of trees in the sense of Meir and Moon and it provides pointwise estimates of the proportion of trees of given height, that is, a *local limit law*.

Generalized digital trees (Flajolet & Richmond, 1992) correspond to a *difference differential equation*, $\partial_z^k \varphi(z) = t(z) + 2e^{z/2} \varphi(z/2)$, whose solution involves basic hypergeometric functions. Catalan sums of the form $\sum_k \binom{2n}{n-k} \nu(k)$, with ν an arithmetical function, arise in the statistics of “order” (also known as Horton-Strahler number) of trees (Flajolet & Prodinger, 1986). Both cases are first subjected to a Mellin transform analysis, which provides the relevant singular expansions. Periodic fluctuations similar to the case of balanced trees then result from singularity analysis.

A notable parallel to the paradigm of generating functions and singularity analysis has been developed by Vallée in a series of papers. In her framework, singularities of certain transfer operators (of Ruelle type) replace singularities of generating functions. See, *e.g.*, [21, 22] for applications to Euclidean algorithms and statistics on sequences produced by a general model of dynamical sources.

5. Limit laws

One of the important features of singularity analysis, in contrast with Darboux’s method or (real) Tauberian theory, is to allow for *uniform* estimates. This makes it possible to analyse asymptotically coefficients of multivariate generating functions, $f(z, u)$, where the auxiliary variable u marks some combinatorial parameter χ . One *first* proceeds to extract $f_n(u) := [z^n]f(z, u)$ by considering $f(z, u)$ as a parameterized family of univariate GFs to which singularity analysis is applied. (The coefficients $f_n(u)$ are, up to normalization, probability generating functions of χ .) A *second* level of inversion is then achieved by the standard continuity theorems for probability characteristic functions (equivalently Fourier transforms). Technically, consideration of a (small) neighbourhood of $u = 1$ is normally sufficient for extracting central limit laws.

Two important cases are those of a smoothly varying singularity and of a smoothly varying exponent. In the first case, $f(z, u)$ has a constant singular exponent α_0 and one has $f(z, u) \sim c(u)(1 - z/\rho(u))^{-\alpha_0}$. Then, uniformity of singularity analysis implies the estimate $f_n(u)/f_n(1) \sim (\rho(1)/\rho(u))^n$. In other words, the probability generating function of χ over objects of size n is analytically similar to the GF of a sum of independent random variables—this situation is described as a “*quasi-powers*” approximation. A Gaussian limit law for χ results from the continuity theorem, with mean and variance that grow in proportion to n . The other case of a smoothly varying exponent is dealt with similarly: one has $f(z, u) \sim c(u)(1 - z/\rho)^{-\alpha(u)}$ implying $f_n(u)/f_n(1) \sim n^{\alpha(u) - \alpha(1)}$; this is once more a quasi-power approximation, but with the parameter now in the scale of $\log n$. (See Gao & Richmond, 1992, for hybrid cases.)

The technology above builds on early works of Bender [1], continued by Flajolet & Soria [9, 19], and H. K. Hwang [11]. In particular, under general conditions, the following hold: a local limit law expresses convergence to the Gaussian density; speed of convergence estimates result from the Berry–Esseen inequalities; large deviation estimates derive from singularity analysis applied at fixed real values $u \neq 1$.

EXAMPLE 6. Polynomials over finite fields. Consider the family \mathcal{P} of all polynomials with coefficients in the Galois field \mathbb{F}_q . A polynomial being determined by its sequence of coefficients, the GF $P(z)$ of all polynomials has a polar singularity. Furthermore, the unique factorization property implies that \mathcal{P} is isomorphic to the class of all multisets (\mathfrak{M}) of the irreducible polynomials \mathcal{I} : $\mathcal{P} \simeq \mathfrak{M}\{\mathcal{I}\}$. Since taking multisets corresponds to exponentiating singularities of GFs, the singularity of the GF $I(z)$ is logarithmic. By singularity analysis, the number of irreducible polynomials is asymptotic to q^n/n —this is an analogue of the prime number theorem, which was already known to Gauß. The bivariate GF of the number of irreducible factors in polynomials turns out to be of the singular type $(1-qz)^{-u}$, with a smooth variable exponent, so that: *the number of irreducible factors of a random polynomial over \mathbb{F}_q is asymptotically Gaussian*. This parallels the Erdős–Kac theorem for integers. Similar developments lead to a complete analysis of a major polynomial factorization algorithm (Flajolet, Gourdon & Panario, 2001). \square

Movable singularities and exponents occur frequently in the analysis of parameters defined by recursion, leading to algebraic or differential equations, which “normally” admit a smooth perturbative analysis.

EXAMPLE 7. Patterns in random strings. Let Ω be the total number of occurrences of a fixed pattern (as a contiguous block) in a random string over a finite alphabet. For either the Bernoulli model, where letters are independently identically distributed, or the Markov model, the bivariate GF, with z marking the length of the random string and u the number Ω of occurrences, is a *rational function*, as it corresponds to a finite-state device. Perron–Frobenius properties apply for positive u . Therefore the bivariate GF viewed as a function of z has a simple dominant pole at some $\rho(u)$ that is an algebraic (and holomorphic) function of u , for $u > 0$. Quasi-powers approximations therefore hold and the limit law of Ω in random strings of length n is Gaussian. Such facts hold for very general notions of patterns and are developed systematically in Szpankowski’s book [20]. \square

EXAMPLE 8. Non-crossing graphs. Consider graphs with vertex set the n th roots of unity, constrained to have only non-crossing edges; let the parameter χ be the number of connected components. The bivariate GF $G(z, u)$ is an *algebraic function* satisfying

$$G^3 + (2w^3z^2 - 3w^2z + w - 3)G^2 + (3w^2z - 2w + 3)G + w - 1 = 0.$$

$G(z, 1)$ has a dominant singularity at $\rho(1) = 3/2 - \sqrt{2}$ which gets smoothly perturbed to $\rho(u)$ for u near 1. The singularity type is consistently of the form $(1 - z/\rho(u))^{1/2}$. A central limit law results for the number of components in such graphs (Flajolet–Noy, 1999). Drmota has given general conditions ensuring Gaussian laws for problems similarly modelled by multivariate algebraic functions [3]. \square

EXAMPLE 9. Profile of quadrees. Refer to Example 5. The bivariate GF $f(z, u)$ of node levels in quadrees satisfies an equation, which, for dimension $d = 3$ reads

$$f(z, u) = 1 + 2^3u \int_0^z \frac{dx_1}{x_1(1-x_1)} \int_0^{x_1} \frac{dx_2}{x_2(1-x_2)} \int_0^{x_2} f(x_3, u) \frac{dx_3}{1-x_3}.$$

This corresponds to a *linear differential equation* with coefficients in $\mathbb{C}(z, u)$ and a fixed singularity at $z = 1$. The indicial equation is an algebraic one parameterized by u and, when $u \approx 1$, there is a unique largest branch $\alpha(u)$ that determines the dominant regime of

the form $(1-z)^{-\alpha(u)}$. This is the case of a movable exponent inducing a central limit law: *The level profile of a d -dimensional quadtree is asymptotically Gaussian*. Such properties are expected in general for models that are perturbations of linear differential equations with a fixed Fuchsian singularity (Flajolet & Lafforgue, 1994). \square

Finally, singularity analysis also intervenes by making it possible to “pump” moments of combinatorial distributions. Examples include the height of trees discussed earlier, as well as tree path length (Louchard 1983, Takács 1991) and the construction cost of hashing tables (Flajolet, Poblete & Viola, 1998). The latter problems were first shown in this way to converge to Brownian Excursion Area.

6. Conclusions

Elementary combinatorial structures are enumerated by generating functions that satisfy a rich variety of functional relations. However, the singular types that are observed are usually somewhat restricted, and *driven by combinatorics*. In simpler cases, the generating functions are explicit combinations of a standard set of special functions. Next, implicitly defined functions (associated with recursion) have singularities that arise from failures of the implicit function theorem and are consequently of the algebraic type, often with exponent $\frac{1}{2}$. Linear differential equations have a well-established classification theory that, in the Fuchsian case, leads to algebraic-logarithmic singularities. In all such cases, the singular expansion is known to be valid outside of the original disc of convergence of the generating function. This means that singularity analysis is *automatically* applicable, and precise asymptotic expansions of coefficients result.

Parameters of combinatorial structures, provided they remain “simple” enough, lead to local deformations (via an auxiliary variable u considered near 1) of the functional relations defining univariate counting generating functions. Under fairly general conditions, such deformations are amenable to perturbation theory and admit of uniform expansions near singularities. Then, since the singularity analysis process preserves uniformity, limit laws result via the continuity theorem for characteristic functions. In this way, the behaviour of a large number of parameters of elementary combinatorial structures becomes predictable. (The theory of functions of several complex variables is thus bypassed. See Pemantle’s recent work [16] based on this theory for a global characterization of all the asymptotic regimes involved.)

The generality of the singular approach makes it even possible to discuss *combinatorial schemas* at a fair level of generality [8, 9, 11, 19], the case of polynomial factorization (Ex. 6) being typical. Roughly, combinatorial constructions viewed as “singularity transformers” dictate asymptotic regimes and probabilistic laws. Analytic combinatorics then represents an attractive alternative to probabilistic methods, whenever a strong analytic structure is present—this is the case for most combinatorial problems that are “decomposable” and amenable to the generating function methodology. Very precise asymptotic information on the randomness properties of large random combinatorial objects results from there. This in turn has useful implications in the analysis of many fundamental algorithms and data structures of computer science, following the steps of Knuth’s pioneering works [12].

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