ANALYTIC URNS

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ABSTRACT. This article describes a purely analytic approach to urn models of the generalized or extended Pólya-Eggenberger type, in the case of *two* types of balls and constant "balance", i.e., constant row sum. (Under such models, an urn may contain balls of either of two colours and a fixed 2×2–matrix determines the replacement policy when a ball is drawn and its colour is observed.) The treatment starts from a quasilinear first-order partial differential equation associated with a combinatorial renormalization of the model and bases itself on elementary conformal mapping arguments coupled with singularity analysis techniques. Probabilistic consequences are new representations for the probability distribution of the urn's composition at any time n, structural information on the shape of moments of all orders, estimates of the speed of convergence to the Gaussian limits, and an explicit determination of the associated large deviation function. In the general case, analytic solutions involve Abelian integrals over the Fermat curve $x^h + y^h = 1$. Several urn models, including a classical one associated with balanced trees (2–3 trees and fringe-balanced search trees) and related to a previous study of Panholzer and Prodinger as well as all urns of balance 1 or 2, are shown to admit of explicit representations in terms of Weierstraß elliptic functions. Other consequences include a unification of earlier studies of these models and the detection of stable laws in certain classes of urns with an off-diagonal entry equal to zero.

In this study, we revisit the most basic urn model, namely the "generalized" (or "extended") Pólya–Eggenberger urn model with *two types of balls*[1], as described in the reference book of Johnson and Kotz [30]. Under this model an urn may contain two types of balls, say "black" (B) and "white" (W), or "type I" and "type II". The composition of the urn at time 0 is fixed. At time n, a ball in the urn is randomly chosen and its colour is *observed* (thus the ball is drawn, looked at and then placed back into the urn): if it is black, then α black and β white balls are subsequently inserted; if it is white, then, γ black balls and δ white balls are inserted. The evolution rule is then summarized by a 2×2-matrix

drawn ↓	added B	added W
B	α	β
W	γ	δ

Negative values of the diagonal entries α, δ are permissible and interpreted as an extraction (rather than an insertion) of balls; a model with both diagonal entries negative will be called here an *urn with subtraction* [of balls of the colour chosen]. The off-diagonal entries β, γ are always taken to be nonnegative.

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[1]We do not address at all in this article models with more than two types of balls; see Smythe's paper [47] for a thorough probabilistic treatment and the works of Aldous and coauthors [1, 2] for an insightful discussion of almost sure convergence issues.

The urn model is said to be *balanced* if $\alpha+\beta=\gamma+\delta$, in which case the common sum of the matrix rows is the *balance*, denoted throughout by s. The model may lead to widely differing behaviours depending on the values of the integer entries $\alpha, \beta, \gamma, \delta$. For instance, Kotz, Mahmoud, and Robert [33] mention the (balanced) urn with matrix $\binom{4\ \ 0}{3\ \ 1}$ for which the number of black balls picked in n steps grows stochastically like $n^{1/4}$. Strikingly, the authors of [33] study the (imbalanced) urn associated to $\binom{1\ \ 0}{1\ \ 1}$ and show the corresponding number to be $\sim n/\log n$ in probability under a Poisson model.

Our interest throughout this article being in urn models that are balanced, the conditions of having a matrix

$$(1) \qquad M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \qquad \text{with} \quad \alpha+\beta=\gamma+\delta=s, \qquad \beta \geq 0, \quad \gamma \geq 0.$$

are invariably assumed. (We also allow ourselves on occasion to describe M linearly as $(\alpha, \beta, \gamma, \delta)$.) In such a case, each elementary action on the urn results in having the total number of balls increase by the fixed quantity s, so that the population at time n has a predictable cardinality, which is exactly t_0+sn if t_0 is the initial size at time 0. For urns involving subtraction, certain simple arithmetic conditions on the parameters, called *tenability* (the Webster dictionary defines "tenable" as meaning "capable of being maintained"), ensure that the process cannot be "blocked"; these conditions are recalled in Section 4, Equation (57), and are assumed to hold.

Balanced 2×2 urn models have been in particular considered by Bagchi and Pal [4] who showed the following: under a supplementary technical condition, namely that the ratio between eigenvalues of the matrix, $(\alpha-\beta)/(\alpha+\beta)$, lies in $(-\infty, \frac{1}{2})$, the distribution of the number of balls of one colour obeys in the limit a normal distribution. Gouet [23] further shows, under the assumptions of Bagchi and Pal, convergence of the discrete urn evolution to a stochastic Gaussian process, and he also investigates other cases using martingale arguments. Aldous, Flannery, and Palacios [2] observe that such results can be supplemented by almost sure convergence properties—their treatment extends the relation between branching processes and urn models to be found in Athreya and Ney's book [3, Sec. V.9]. Thanks to the works of these and many other authors, the normal evolution of the process in the central regime can thus be regarded as well characterized by previous works.

In this article, we revisit urn models under the radical angle of analysis. (Aldous otherwise provides an insightful comparison of the scopes of the traditional probabilistic approach and the modern methods of analysis of algorithms in the introductory section of [1].) Our main results provide a complete analytic solution describing the composition of the urn at each instant, and they apply to virtually all the 2×2 balanced urn schemes, although we predominantly focus attention in this paper on urns involving *subtraction*, that is, having negative diagonal entries.

Our investigations were initially motivated by a desire to understand the specific urn model

$$\mathcal{T}_{2,3} = \begin{pmatrix} -2 & 3 \\ 4 & -3 \end{pmatrix}.$$

In the *first part* of the paper (Sections 1, 2, 3), we take this model as a leading example, since it illustrates in a forthright manner all the basic principles of our approach, whereas the general model involves three parameters (the diagonal entries α, γ of the matrix and the row sum s) plus two initial conditions, a_0, b_0, describing the composition of the urn at the beginning of time. As shown by Aldous *et al.*, Bagchi–Pal, Eisenbarth *et al.*, Panholzer–Prodinger, and Yao, this particular urn process intervenes as a model of several schemes for managing an important data structure of computer science known as the search tree [32, 38] and it surfaces in the analysis of 2–3 trees and fringe-balanced binary search trees [2, 4, 17, 43, 52]. Here, we show, amongst other things, that the probability of large deviations at time n is exponentially small in n with a rate function that is a simple transform of the explicit function

$$K(u) := \frac{1}{(1-u^6)^{1/6}} \int_u^1 \frac{t}{(1-t^6)^{5/6}}\, dt. \tag{2}$$

Our approach starts with a *partial differential equation* (PDE) that is linear of the first order and that describes exactly snapshots of the urn compositions at all times. The solution of this partial differential equation, obtained by standard methods, provides an indirect expression for a bivariate generating functions that encodes the possible configurations of the urn at each time n; see Theorem 1. In the case of the $\mathcal{T}_{2,3}$ model, elementary complex analysis (conformal mapping arguments, essentially) then permits us to analyse *singularities* and eventually relate the expression obtained to classical elliptic functions of the Weierstraß type, $\wp$ and $\zeta = -\int \wp$, as summarized by Theorem 2. This provides a complete analytic representation of the composition of $\mathcal{T}_{2,3}$ urns at any time n including: (*i*) an exact representation of the probability generating function as a lattice sum; (*ii*) an explicit form of all moments of all orders; (*iii*) a new derivation of the existence of a Gaussian limit law accompanied by precise speed-of-convergence estimates; (*iv*) explicit large deviation estimates with a rate that is simply related to (2).

A parallel elliptic connection has been discovered earlier by Panholzer and Prodinger [43] using rather different methods. Their beautiful analysis depends on the specific relationship that their model entertains with a special type of "fringe balanced" search trees—a root decomposition of the tree then leads to a perturbed nonlinear ordinary differential equation (of the form $Y''' = Y'^2 + \cdots$) akin to the one satisfied by the Weierstraß $\wp$–function. In fact, our elliptic connection for the $\mathcal{T}_{2,3}$ urn model could alternatively be deduced by reverse-engineering of the Panholzer–Prodinger treatment, combined with an easy reduction of a special urn model studied by Mahmoud in [37]. We do not proceed along those lines since Panholzer and Prodinger's nonlinear differential approach is problem-specific: it does not appear to generalize to a general class of urn models and, e.g., it would not yield the other elliptic cases listed in (4) below.

In the *second part* of the paper (Sections 4, 5), we adapt the partial differential equation approach to urn models with two types of balls and constant row sum. In fact, to keep the paper within reasonable size, we limit ourselves to a complete discussion of the case where the diagonal entries are negative, the matrix being taken under the form

$$\begin{pmatrix} -a & a+s \\ b+s & -b \end{pmatrix},$$

with balance $s \geq 0$ and diagonal coefficients $-a, -b < 0$. (The urn's initial composition is fixed with t_0 balls in total of which a_0 are of the first type.) Such models with negative diagonal entries are occasionally mentioned by some authors as a harder nut to crack, since the direct embedding of urn schemes into branching processes explained in Athreya and Ney's book [3, Sec. V.9] ceases to be directly applicable. (This position is perhaps to be taken with caution given the discussion in [2] of extensions of the classical probabilistic framework.) Other cases, where the diagonal entries may be 0 or a positive number, are only discussed sporadically, simply to exemplify the wide scope of our methods (Section 5).

It is found that the bivariate generating function describing the composition of the urn at successive times is expressed in terms of a fundamental function ψ. While the $\mathcal{T}_{2,3}$ tree model leads to ψ being an elliptic function, what happens in general is the following. The function ψ is now defined implicitly by an equation of the form

$$\psi(I(u)) = Q(u,v), \tag{3}$$

where (u,v) lies on a Fermat curve $u^h + v^h = 1$ with $h = a + b + s$ a sort of "complexity index", the quantity Q being a rational function on the curve, and $I(u)$ an Abelian integral on that same curve[2]—i.e., the integral of a rational function on the curve. The parameterization suffices in all cases to determine the dominant singularities of ψ together with the associated singular expansions. There results a situation qualitatively similar to the case of the tree model $\mathcal{T}_{2,3}$, albeit with usually weaker error terms. As a consequence, analytic principles provide the Gaussian law together with a precise determination of the speed of convergence and of the large deviation function in the style of the integral representations (2) and (3). In general, ψ is associated with algebraic curves of genus strictly higher than 1, so that elliptic solutions are usually not available. Nonetheless, the general character of our analytic results, Theorems 3 and 4, permits us to single out all the cases where elliptic solutions exist: such a strong elliptic structure is found to prevail for all urns of balance 1 or 2 together with a "sporadic" urn of balance 3, corresponding to the *six* matrices

$$\begin{pmatrix} -2 & 3 \\ 4 & -3 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 3 & -2 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 3 \\ 5 & -3 \end{pmatrix}, \begin{pmatrix} -1 & 4 \\ 5 & -2 \end{pmatrix}, \tag{4}$$

see Subsection 5.3.

We finally examine in Subsections 5.4 and 5.5 specific classes of urns with non-negative diagonal entries, our purpose being simply to point the way to further applications of our general methodology. The original models of Pólya-Eggenberger are easily treated within the framework while we uncover in passing the presence of *stable laws* for models of the form

$$\begin{pmatrix} a+1 & 0 \\ 1 & a \end{pmatrix},$$

previously studied by Kotz, Mahmoud, and Robert. Bagchi and Pal [4, p. 395] describe this case as one which "*presents some curious technical problems and appears to need a separate treatment*" while Gouet [23, Prop. 2.1] singles out a random variable Z from which his functional central limit theorem stems, but stops short of fully characterizing it.

[2]It is to be noted that the Fermat curve is of high topological genus [34], namely $g = (h-1)(h-2)/2$, so that one already has $g = 10$ in the case of the $\mathcal{T}_{2,3}$ model. This makes the existence of elliptic function solutions, which are objects of genus 1, quite remarkable.

A final word is due here about our methodology. The basic PDEs associated with urn models admit solutions expressed in terms of integrals of algebraic functions (the already evoked Abelian integrals) and their inverses. Such solutions in "raw" form are given in the parallel Theorem 1 (for the $\mathcal{T}_{2,3}$ model) and Theorem 3 (in the general case). Analytic combinatorics in its recent developments teaches us the important rôle of singularities as a determinant of asymptotic properties of combinatorial structures. The corresponding singularity analyses are summarized by Theorems 2 ($\mathcal{T}_{2,3}$ model) and 4 (general case), which are focal points of our treatment. (In the case of the $\mathcal{T}_{2,3}$ urn, the stronger elliptic form comes out of an exceptional geometric regularity of certain conformal maps of the complex plane associated with the problem.) Once these two analytic results have been established, probabilistic properties come out by fairly standard developments that take their inspiration from the usual analytic proofs of the central limit theorem, the Berry-Esseen estimates, and basic large deviation theory, all combined with the technique called *singularity analysis* after Flajolet and Odlyzko. It is in this way that probabilistic consequences expressed by Corollaries 1 to 12 are all derived.

1. Basic equations of the $\mathcal{T}_{2,3}$ "tree" model

In order to introduce our methodology as straightforwardly as possible, we quickly proceed to a derivation of the partial differential equation (PDE) of the $\mathcal{T}_{23}$ model. Here, we perform the basic algebra and derive a first representation implicitly in terms of Abelian integrals over the Fermat curve $x^6 + y^6 = 1$. These algebraic results, when supplemented by conformal mapping arguments (presented in the companion Section 2) lead to an exact representation of the probability generating function of urns of one given colour at time n in terms of elliptic functions of the Weierstraß type. Probabilistic consequences, in the form of moment representations and large deviations will be developed next, in Section 3.

1.1. **The basic PDE.**

We first freeze the basic algebra of generating functions that intervene in the fringe analysis of 2–3 trees, that is, the $\mathcal{T}_{2,3}$ model of the introduction. Calculations leading to the main partial differential equation (9) are not detailed since they are routine generating function manipulations and are anyhow subsumed by the more general discussion of Section 4. The model is determined by its matrix and the initial conditions

$$\mathcal{T}_{2,3} = \begin{pmatrix} -2 & 3 \\ 4 & -3 \end{pmatrix}, \qquad a_0 = 2, \quad b_0 = 0.$$

In words, let the two colours be called black and white respectively: if a black ball is drawn (but not withdrawn), two black balls are subsequently pulled out of the urn, and three white balls are inserted; if a white ball is drawn, three black balls are pulled out and four white balls are inserted. Conventionally at time 1, the system is initialized with a configuration of two balls, all of them black, which is translated as $a_0 = 2$ and $b_0 = 0$. (It is easily checked that this model is tenable in the sense of [4]: at any given time, the number of black balls is always of multiple of 2, the number of white balls a multiple of 3.) At any given stage, the size of the urn increases by $s = 1$, being equal to $n + 1$ at time n.

Let X_n be the random variable representing the number of black balls at time n. The dynamics of the process is expressed by the stochastic recurrence

$$(5) \qquad X_1 = 2; \qquad X_n - X_{n-1} = \begin{cases} -2 & \text{with probability } \dfrac{X_{n-1}}{n} \\ +4 & \text{with probability } 1 - \dfrac{X_{n-1}}{n}. \end{cases}$$

Let $p_{n,k} = \mathbb{P}(X_n = k)$; we introduce the *probability generating function* (PGF) of X_n,

$$p_n(u) := \sum_k p_{n,k} u^k = \mathbb{E}(u^{X_n}).$$

The relation (5) translates into a recurrence on the triangular array $\{p_{n,k}\}$,

$$(6) \qquad p_{n,k} = \frac{k+2}{n} p_{n-1,k+2} + \left(1 - \frac{k-4}{n}\right) p_{n-1,k-4}.$$

The collection of all such probability distributions is then encoded by the *bivariate generating function* (BGF),

$$F(z,u) := \sum_{n \geq 1} p_n(u) u^n = \sum_{n,k} p_{n,k} u^k z^n,$$

whose elicitation is our main target.

By standard generating function manipulations, the PGF $p_n(u)$ satisfies the differential recurrence,

$$(7) \qquad p_n(u) = u^4 p_{n-1}(u) + \frac{1}{nu}(1-u^6)\frac{d}{du} p_{n-1}(u),$$

for $n \geq 2$, together with the initial condition $p_1(u) = u^2$. The first few values of the $p_n(u)$ are then:

$$(8) \qquad p_2 = 1,\ p_3 = u^4,\ p_4 = u^2,\ p_5 = \tfrac{3}{5}u^6 + \tfrac{2}{5},\ p_6 = u^4,\ p_7 = \tfrac{3}{7}u^8 + \tfrac{4}{7}u^2.$$

The recurrence (6) is then reflected by a *partial differential equation* (PDE) satisfied by the BGF $F(z,u)$:

$$(9) \qquad (u^5 z - u)\frac{\partial F}{\partial z} + (1-u^6)\frac{\partial F}{\partial u} + u^5 F + u^3 = 0.$$

(Proof: multiply both members of (6) by z^n and sum over n.)

It will prove technically convenient to assign a meaning to $p_0(u)$ that is consistent with the recurrence (7). For this purpose, it suffices to solve for $p_0(u)$ the ordinary differential equation obtained by setting $n = 1$ in Equation (7) while imposing $p_1(u) = u^2$ on the left hand side. This gives a particular solution

$$(10) \qquad p_0(u) = (1-u^6)^{1/6} \int_0^u t^3 (1-t^6)^{-7/6}\, dt,$$

where the integration constant has been chosen to be the "simplest" possible. We have $p_0(1) = 1$, and also

$$(11) \qquad p_0(u) = \frac{1}{4}u^4 + \frac{3}{40}u^{10} + \frac{27}{640}u^{16} + \frac{81}{2816}u^{22} + \frac{243}{11264}u^{28} + \cdots.$$

This suggests considering the modified BGF

$$(12) \qquad G(z,u) := p_0(u) + F(z,u),$$

Start with a *quasi-linear* (bivariate) partial differential equation of the form

$$A(z,u,G)\frac{\partial G(z,u)}{\partial z} + B(z,u,G)\frac{\partial G(z,u)}{\partial u} + C(z,u,G) = 0, \tag{14}$$

where A, B, C are given functions.

1. First look for a solution in implicit form $X(z,u,G) = 0$. A calculation shows that the trivariate X must satisfy the *linear* (trivariate) partial differential equation:

$$A(z,u,w)\frac{\partial X(z,u,w)}{\partial z} + B(z,u,w)\frac{\partial X(z,u,w)}{\partial u} - C(z,u,w)\frac{\partial X(z,u,w)}{\partial w} = 0. \tag{15}$$

2. Next consider the ordinary differential system

$$\frac{dz}{A} = \frac{du}{B} = -\frac{dw}{C}. \tag{16}$$

The solution of two "independent" ordinary differential equations induced by (16), e.g.,

$$\frac{du}{B} = -\frac{dw}{C} \quad \text{and} \quad \frac{dz}{A} = \frac{du}{B}, \tag{17}$$

leads to two families of integral curves known as "first integrals",

$$U(z,u,w) = C_1 \quad \text{and} \quad V(z,u,w) = C_2, \tag{18}$$

with z and w respectively treated as parameters. Assuming nondegeneracy, the generic solution of the PDE (15) is provided by

$$X(z,u,w) = \Phi(U(z,u,w), V(z,u,w)), \tag{19}$$

for an arbitrary bivariate Φ.

3. The trivariate X determines G implicitly by $X(z,u,G) = 0$, that is, one must have $\Phi(U(z,u,G), V(z,u,G)) = 0$ by (19). Solving for G provides a relation $G = R_\Phi(z,u)$, where R_Φ depends upon the arbitrary function Φ. The general solution of (14) is then

$$G(z,u) := R_\Phi(z,u). \tag{20}$$

FIGURE 1. The solution algorithm for quasilinear PDEs of first order.

with $p_0(u)$ as given by (10). The function G is now a solution of the homogeneous equation

$$(u^5 z - u)\frac{\partial G}{\partial z} + (1-u^6)\frac{\partial G}{\partial u} + u^5 G = 0. \tag{13}$$

1.2. **The solution by quadratures.** There exists a well known algorithm [55, Sec. 94], "the method of characteristics", for solving a quasi-linear partial differential equation of the first order, of which (13) is an instance. We summarize the algorithm in Figure 1 since it will also be required in Sections 4 and 5. The general solution involves an arbitrary function; boundary conditions may then be used to identify the particular solution to a given problem.

We are now going to carry out this programme. One can proceed formally without concern of analytic details since, once found by whatever means, a candidate solution can always be checked by back-substitution into the defining equations.

Following the general principles of Figure 1 applied to the PDE (13), we aim at solving the ordinary differential system:

$$\frac{du}{1-u^6} = \frac{dz}{u^5 z - u} = -\frac{dw}{u^5 w}. \tag{21}$$

The ordinary differential equation

$$\frac{du}{1-u^6} = -\frac{dw}{u^5 w}$$

admits, by separation of variables, the solution $w = w(u)$, where

$$w(1-u^6)^{-1/6} = C_1, \tag{22}$$

for some arbitrary integration constant C_1. The ordinary differential equation

$$\frac{du}{1-u^6} = \frac{dz}{u^5 z - u},$$

is readily solved by the variation-of-constant method: for $z = z(u)$, one finds

$$z(1-u^6)^{1/6} + \int_0^u \frac{t}{(1-t^6)^{5/6}}\,dt = C_2. \tag{23}$$

The general theory of first order PDE's tells us that the generic solution of the basic PDE is of the form $\Phi(U, V) = 0$, where U, V are the left hand sides of (22), (23). Thus, one has (G is represented by w):

$$\Phi\left(\frac{G}{(1-u^6)^{1/6}},\ z(1-u^6)^{1/6} + \int_0^u \frac{t}{(1-t^6)^{5/6}}\,dt\right) = 0,$$

for some arbitrary dependency relation Φ. We set

$$I(u) := \int_0^u \frac{t}{(1-t^6)^{5/6}}\,dt, \qquad \delta(u) := (1-u^6)^{1/6}.$$

Then, provided it is legitimate to solve for the first argument, G must be of the form

$$G(z,u) = \delta(u)\Psi(\delta(u)z + I(u)), \tag{24}$$

with Ψ is an *arbitrary* function. This is the fundamental functional form that we shall operate with.

There finally remains to identify the unknown function Ψ from the template (24). The boundary condition,

$$G(0,u) = p_0(u), \tag{25}$$

implies, assuming that (24) remains valid in this boundary case:

$$\frac{p_0(u)}{\delta(u)} = \Psi(I(u)) \quad \text{or} \quad \int_0^u t^3(1-t^6)^{-7/6}\,dt = \Psi(I(u)). \tag{26}$$

The relation (26) then provides a *parameterization* of Ψ, hence it eventually determines a plausible value for G. We state:

Theorem 1. *Define the quantities*

$$\begin{aligned} \delta(u) &= (1-u^6)^{1/6}, \\ I(u) &= \int_0^u \frac{t}{(1-t^6)^{5/6}}, \qquad J(u) = \int_0^u \frac{t^3}{(1-t^6)^{7/6}}\,dt. \end{aligned} \tag{27}$$

Then, the bivariate generating function of the probabilities is

$$G(z,u) = \delta(u)\Psi\left(z\delta(u) + I(u)\right), \tag{28}$$

where Ψ is the function defined parametrically for $|u| < 1$ by

$$\Psi(I(u)) = J(u). \tag{29}$$

Proof. This is merely a verification. As a function analytic in $|z| < 1$ and $|u| < 1$, the probability BGF $G(z, u)$ is uniquely determined by the PDE (13) and the initial condition $G(0, u) = p_0(u)$: this is because the PDE (13) is algebraically[3] equivalent to the recurrence (7) on $p_n(u) := [z^n]G(z, u)$, and the recurrence itself uniquely determines the coefficients in the z expansion of G. In other words, any bivariate analytic solution of the PDE (13) satisfying the boundary condition (25) *must* coincide with the sought BGF of probabilities.

As regards the initial condition, it is granted by (28) upon setting $z = 0$. A simple calculation then shows that a function G determined by Equations (28) and (29) satisfies ($\delta \equiv \delta(u)$, $I \equiv I(u)$)

$$\begin{array}{lllllll} G & = & \delta\Psi(z\delta + I), & & & \\ G'_z & = & \delta^2\Psi'(z\delta + I), & \qquad G'_u & = & \delta'\Psi(z\delta + I) + \delta(z\delta' + I')\Psi'(z\delta + I). \end{array}$$

The verification of the PDE (13) by substitution is then immediate. □

As a check, the parameterization near $u = 0$ reads

$$\begin{array}{lll} I(u) & = & \frac{1}{2}u^2 + \frac{5}{48}u^8 + \frac{55}{1008}u^{14} + \frac{187}{5184}u^{20} + O\left(u^{26}\right), \\ \dfrac{p_0(u)}{\delta(u)} & & \frac{1}{4}u^4 + \frac{1}{6}u^6 + \frac{3}{40}u^{10} + \frac{1}{12}u^{12} + \frac{27}{640}u^{16} + O(u^{18}). \end{array}$$

The relation $I(u) = z$ is first inverted near 0, yielding $u(z) = \sqrt{2y}(1-5/6y^3+O(y^6))$. Then, the parameterization (29) induces the expansion of Ψ near 0:

$$\Psi(y) = y^2 + \frac{2}{5}\,y^5 + \frac{1}{7}\,y^8 + \frac{4}{77}\,y^{11} + \frac{12}{637}\,y^{14} + \frac{74}{10829}\,y^{17} + O(y^{20}). \tag{30}$$

Finally, substitution into the form (28) yields

$$G - p_0(u) = u^2z + z^2 + u^4z^3 + u^2z^4 + (\frac{2}{5} + \frac{3}{5}u^6)z^5 + u^4z^6 + (\frac{4}{7}u^2 + \frac{3}{7}u^8)z^7 + \cdots,$$

which agrees with the already determined values (8).

1.3. **Dominant singularities.**

The diagram that summarizes the definition of Ψ is

$$\begin{array}{ccc} & u & \\ \swarrow & & \searrow \\ z = I(u) & \xrightarrow{\ \Psi\ } & \Psi(z) = J(u). \end{array} \tag{31}$$

Clearly, the map $u \mapsto I(u)$ is analytic in the unit disc $|u| < 1$. Since $I(u)$ has nonnegative Taylor coefficients at 0, the image of the unit disc in the u-plane is a subset of the disc centred at the origin and having radius ρ defined by $\rho = I(1)$. In particular, as $u \to 1^-$, one has $z \to \rho^-$. On the other hand the integral $J(u)$ diverges to $+\infty$ as $u \to 1^-$. There results that $z = \rho$ is a singularity of $\Psi(z)$. The quantity ρ happens to admit of closed form in terms of Gamma function factors as it is representable as a complete Eulerian beta integral [51, p. 253]:

$$B(\alpha, \beta) := \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

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[3] We make use of the notation $[z^n]f(z)$ to represent the coefficient of z^n in the Taylor expansion of f at 0. This notation due to Goulden and Jackson [24] and popularized by [26] is now a *de facto* standard in combinatorics.

for $\Re(\alpha) > 0$ and $\Re(\beta) = 0$.

Next, $\Psi(z)$ is equal to $G(z, 0)$ which, by the probabilistic origin of the problem, has nonnegative coefficients. (It is the generating function of the quantities $p_n(0)$.) By Pringsheim's theorem [50], the radius of convergence of the series representing Ψ at the origin must be equal to ρ; in other words, there can be no singularities of modulus smaller than ρ. Hence:

Lemma 1. *The function $\Psi(z)$ is analytic in the disc $|z| < \rho$, where*

$$\rho \equiv I(1) = \frac{1}{6}B(\frac{1}{6}, \frac{1}{3}) = \frac{1}{6}\frac{\Gamma(1/3)\Gamma(1/6)}{\Gamma(1/2)} \doteq 1.40218\,21053\,25454. \tag{32}$$

Here is a surprisingly accurate numerical check. Let $\Psi_n = [z^n]\Psi(z)$. Then, we can estimate heuristically the radius of convergence of $\Psi(z)$ from the ratio of consecutive nonzero coefficients. For instance,

$$\left(\frac{\Psi_{47}}{\Psi_{50}}\right)^{1/3} \doteq 1.40218\,21053\,2545\underline{6},$$

agrees with ρ till the *fifteenth decimal place.* Such close agreements between asymptotic and exact forms are usually indicative of dominant polar singularities, a fact that will be established in the next section.

2. The global structure of the $\mathcal{T}_{2,3}$ model

In this section, we elucidate the global structure of the Ψ function, first around the disc of convergence $|z| < \rho$, then over the whole complex plane. The elliptic connection eventually results from these considerations.

The fundamental function Ψ being defined by the parametrization[4] $\Psi(I(u)) = J(u)$, it is fully determined near $u = 0$, that is, near $z = 0$. There, one can write $\Psi = J \circ I^{(-1)}$, with $I^{(-1)}$ the functional inverse of u. Since J is analytic at all points not above a sixth root of unity, the core of the treatment consists in inverting $I(u)$ in wider regions, that is, making explicit the conformal mapping properties of $I(u)$.

In Subsection 2.1, we first establish some basic conformal mapping properties of I. Then, using simple rotational symmetries, we shall determine the behaviour associated with the dominant singularities of Ψ (Subsection 2.2) For such a local analysis, we may restrict the domain of variation of u to star-domains obtained by cutting the complex plane along the rays ζt with $t \geq 1$, where ζ is any sixth root of unity ($\zeta^6 = 1$). The complete ramified structure of $\delta(u) = (1 - u^6)^{1/6}$ intervenes critically when the *global* singularity structure of Ψ needs to be investigated, leading to the elliptic function solution in Subsection 2.3.

2.1. **The fundamental triangle.** Recall the parametrization of the Ψ function:

$$\Psi(I(u)) = J(u), \tag{33}$$

with

$$I(u) = \int_0^u \frac{t\,dt}{(1 - t^6)^{5/6}}, \quad J(u) = \int_0^u \frac{t^3\,dt}{(1 - t^6)^{7/6}}. \tag{34}$$

---

[4]The incomplete Beta integrals that make up Ψ are related to hypergeometric functions as well as to the Schwarz–Christoffel integrals of conformal mapping theory. (For the latter aspects, see, e.g., Nehari's book [41] and Exercise 4, p. 196; see also his Chapter V.)

Observe that, near 0, I and J are even analytic function of u with $I(u) \sim u^2/2$ and $J(u) \sim u^4/4$, so that Ψ is locally well defined and analytic in a sufficiently small neighbourhood of the origin, where it satisfies $\Psi(z) \sim z^2$. (One could have defined a new reduced parametrization by setting $u = v^2$; equivalently—we opt for this solution here—it suffices to restrict consideration to values of u ranging in the upper half plane, $\Im(u) \geq 0$ or any half-plane through 0.)

In order to gain insight into the nature of the singularity at ρ, we next examine the fundamental parameterization, but now at $u = 1$, since this is the value that gives rise to singularity. A crude analysis shows that when u approaches 1, one has[5] $I(1) - I(u) \asymp (1-u)^{1/6}$, while $J(u) \asymp (1-u)^{-1/6}$. Thus, $\rho - z$ and $\Psi(z)$ are approximately inverse of one another. In fact, choose the parameter $\tau := (1-u)^{1/6}$, that is, $1-u = \tau^6$. This is a "local uniformizing parameter" for $\delta(u)$ as it eliminates the multiple determinations of the sixth root: as τ dscribes a full circle around 0, u circles six time around 1, and all the determinations of $\delta(u)$ are covered in succession (see below for more). Local expansions then show that, in terms of τ,

$$\begin{aligned} I(u) &= \rho - \sqrt[6]{6}\left(\tau + \tfrac{13}{84}\tau^7 - \tfrac{25}{3744}\tau^{13} + \cdots\right) \\ J(u) &= \tfrac{1}{\sqrt[6]{6}}\left(\tau^{-1} + \tfrac{1}{60}\tau^5 + \tfrac{167}{1056}\tau^{11} + \cdots\right). \end{aligned} \tag{35}$$

Thus, one has, at least when $z \to \rho$ from the left,

$$\Psi(z) \underset{z\to\rho}{\sim} \frac{1}{\rho - z}, \tag{36}$$

which, upon eliminating τ, implies the expansion of $\Psi(z)$ as $z \to \rho$:

$$\Psi(z) = Z^{-1} + \frac{1}{35}Z^5 - \frac{1}{7007}Z^{11} + \cdots, \qquad Z := \rho - z, \tag{37}$$

Next, since there is an obvious periodicity of the coefficients of Ψ modulo 3 in (30), it is legitimate to expect the dominant singularities (the ones of smallest modulus) to be at ρ, $\rho\omega$, and $\rho\omega^2$, for ω a primitive cube root of unity. Indeed, by simple changes of variables, one finds

$$\zeta^6 = 1 \; : \qquad I(u\zeta) = \zeta^{-2} I(u), \quad J(u\zeta) = \zeta^{-4} J(u).$$

Thus, if the point (z, y) lies on the curve $y = \Psi(z)$, then so does the point $(\zeta^{-2}z, \zeta^{-4}y)$. In other words, one has

$$\Psi(\zeta^{-2}z) = \zeta^{-4}\Psi(z),$$

or, in terms of cube roots of unity:

$$\omega^3 = 1 \; : \qquad \Psi(\omega z) = \omega^2 \Psi(z). \tag{38}$$

Put otherwise, $\frac{\Psi(z)}{z^2}$ is a function of z^3 alone. Consequently, one has, at least in the sense of directional limits at this stage:

$$\Psi(z) \underset{z\to\rho\omega}{\sim} \frac{1}{\rho\omega - z}, \qquad \Psi(z) \underset{z\to\rho\omega^2}{\sim} \frac{1}{\rho\omega^2 - z}.$$

Thus Ψ must be singular at the three points ρ, $\rho\omega$, and $\rho\omega^2$. We are going to see that there are no other singularities of $\Psi(z)$ on $|z| = \rho$. The proof of this fact will occupy the rest of this subsection and the next one.

[5]The symbol $\asymp$ is used to denote the fact that two functions are of the same asymptotic order—a notation from Hardy's tract [27].

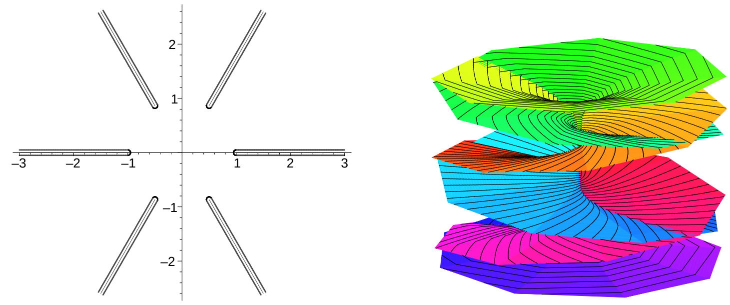

FIGURE 2. The region R_0 (left) and a rendering of the six-sheeted Riemann surface $\mathfrak{R}$ of $\delta(u)$ for u near 1 (right).

In order to attain a global understanding of $\Psi(z)$, one must consider all the possible values assumed by $I(u)$ and $J(u)$. Since the integrands involve the function

$$\delta(t) = (1 - t^6)^{1/6}$$

that is multivalued, this implies the necessity of taking into account all legal integration paths from 0 to u in the complex plane[6] within (34). Because of the branched character of δ at the sixth roots of unity, such paths must be considered in relation to these special points. In other words, we must consider u as ranging over the Riemann surface of δ. (We refer globally to Siegel's splendid expository notes [45, 46] for background on Riemann surfaces, algebraic functions, and Abelian integrals.)

We set $\zeta = e^{2i\pi/6}$ and define the region R_0 as the complex plane slit along the six rays $\zeta^j t$ with $t \in [1, +\infty[$, where $j = 0, \ldots, 5$; see Figure 2. On R_0, we adopt the principal determination of the sixth root for δ, namely, $\delta(t) \sim 1$ as $t \sim 0$, with an extension of the determination by continuity. Next, we introduce six copies of $R_0, \ldots, R_5$ of R_0 where by convention $\delta(t) \sim \zeta^j$ when $t \sim 0$ in R_j, with δ being also extended by continuity on R_j. It proves convenient to consider the rays $\zeta^j t$ with $t \geq 1$ as composed of two infinitesimally closed half-lines A_j, B_j of angles $j\frac{\pi}{3} - 0$ and $j\frac{\pi}{3} + 0$. A rotation of $+2\pi$ around any ζ^j then corresponds to changing the determination of δ by a factor of $e^{2i\pi/6} = \zeta$. In other words, we can identify B_j and $A_{j+1 \bmod 6}$ and accordingly "glue" together the six regions into a single surface. What this effects is of course nothing but the classical construction of the Riemann surface $\mathfrak{R}$ of δ by means of copies of the cut complex plane: δ becomes single-valued and continuous on $\mathfrak{R}$; see again [45, 46] for an exposition of the ideas.

Because of the already evoked phenomenon of "double parameterization" (u and $-u$ give rise to the same point by evenness of $I(u)$ and $J(u)$), we may freely restrict attention to u in the halfplane $\Im(u) \geq 0$ or, equivalently, identify points u and $-u$.

[6]For instance, in order to construct the complete curve $y = \exp(z)$ from the parameterization

$$z = \int_1^u \frac{dt}{t}, \qquad y = u,$$

one needs to take into account all paths from 1 to u that avoid 0, including the ones that wind several times around 0.

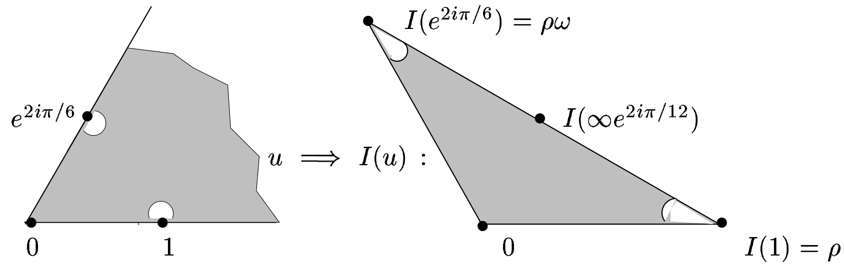

FIGURE 3. The "elementary triangle" T_0 (right) is the image of the basic sector S_0 (left) via the mapping $u \mapsto I(u)$.

To this effect, we define

$$\mathcal{H} := \left\{ z \;\middle|\; (\Im(z) > 0) \vee ((\Im(z) = 0) \wedge (\Re(z) \geq 0)) \right\}. \tag{39}$$

The first result describes an essential mapping property of $I(u)$.

Lemma 2. *The function $I(u)$ maps the interior of $(R_0 \cap \mathcal{H})$ conformally (i.e., in a one-to-one analytic manner) onto the interior of the equilateral triangle T with vertices $\rho, \rho\omega, \rho\omega^2$, where $\omega := e^{2i\pi/3}$.*

Proof. The proof proceeds by stages illustrated by Figures 3 and 4.

(i) We first examine the image by $I(u)$ of the sector of R_0 defined by

$$S_0 : \qquad z = re^{i\theta}, \quad 0 < \theta < \frac{\pi}{3}, \quad r \geq 0, \tag{40}$$

the determination of δ being by convention the principal one obtained from $\delta(0) = 1$; see Figure 3. We start by considering the boundary. As u increases from 0 to 1, the quantity $I(u)$ increases from 0 to

$$\rho = \int_0^1 \frac{t\,dt}{(1-t^6)^{5/6}} = \frac{1}{6}\frac{\Gamma(\frac{1}{3})\Gamma(\frac{1}{6})}{\Gamma(\frac{1}{2})}.$$

Then, since $I(u) - I(1)$ is of the order of $(u-1)^{1/6}$, the image of an infinitesimal semicircle centred at 1 in the upper half-plane is one-sixth of a semicircular arc centred at 1 and also situated in the upper half-plane. Then, for u larger than 1, one has (on R_0, with the principal determination followed by continuity),

$$I(u) - I(1) = e^{5i\pi/6} \int_1^u \frac{t\,dt}{(t^6-1)^{5/6}}.$$

Thus, as u runs over $[1, +\infty[$, the value of $I(u)$ varies from ρ to

$$\rho + e^{5i\pi/6} \int_1^\infty \frac{t\,dt}{(t^6-1)^{1/6}}.$$

(In other words, the image $I(u)$ comes back at an angle of $5\pi/6$.) The latter integral is readily evaluated as a complete Beta integral via the change of variables $t = w^{-1}$,

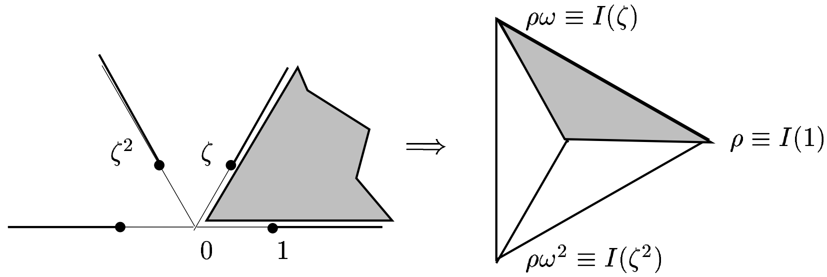

FIGURE 4. The "fundamental triangle" T (right) is the image of the slit upper half-plane $(R_0 \cap \mathcal{H})$ (left) via the mapping $u \mapsto I(u)$.

and one finds that, on S_0,

$$\begin{aligned} I(+\infty) &= \rho - e^{i\pi/6} \int_0^1 \frac{w^2\, dw}{(1-w^6)^{5/6}} \\ &= \rho - e^{i\pi/6} \frac{1}{6} B(\frac{1}{6}, \frac{1}{2}) \\ &= \rho(1 - e^{i\pi/6} \frac{\sqrt{3}}{2}). \end{aligned}$$

In summary, as u varies from 0 to $+\infty$, passing through 1, $I(u)$ describes first the segment from $[0, \rho]$, then the segment $[\rho, \rho(1+\omega)/2]$.

Let $\zeta = e^{2i\pi/6}$ be a sixth root of unity. Since $I(\zeta u) = \zeta^2 I(u)$, the image by u of the segment joining 0 to ζ is the segment with end points 0 and $\rho\omega$ where $\omega = \zeta^2$. An argument similar to the previous one shows that the image of the ray from ζ to $\zeta\infty$ (in S_0, one should imagine the contour as passing slightly "under" ζ) is now the segment with end points $\rho\omega$ and $\rho(1+\omega)/2$. In summary, the image of the two rays $(0, +\infty)$ and $(0, \zeta\infty)$ is the boundary of the triangle T_0 defined by its vertices $0, \rho, \rho\omega$; see Figure 3.

(*ii*) Next, we prove that $I(u)$ assumes each value in T_0 once and only once, using basic facts of conformal mapping theory. Let β be an arbitrary number interior to T_0. The number $\nu(\beta)$ of times that $I(u)$ assumes the value $\beta \in T_0$ for u interior to S_0 is by the residue theorem

$$\nu(\beta) = \frac{1}{2i\pi} \int_{\partial S_0} \frac{I'(u)}{I(u) - \beta}\, du,$$

where ∂X represents the boundary of a region X oriented positively. Then, the change of variables $I(u) = x$ gives

$$\nu(\beta) = \frac{1}{2i\pi} \int_{\partial T_0} \frac{dx}{x - \beta} = 1,$$

where the reduction to the value 1 is due to the fact that β is by assumption interior to T_0.

(*iii*) Finally, the basic formula $I(\zeta u) = \omega I(u)$ shows that the images by I of the rotated sectors ζS_0 and $\zeta^2 S_0$ are the rotated triangles ωT_0 and $\omega^2 T_0$. Altogether the images by $I(u)$ of $S_0, \zeta S_0, \zeta^2 S_0$ cover the equilateral triangle T, as was to be proved. □

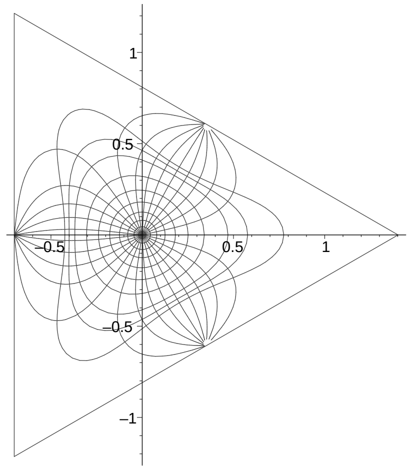

FIGURE 5. Another view of the image of $(R_0 \cap \mathcal{H})$ by $I(u)$ giving the fundamental triangle T: a representation of the images of rays emanating from 0 and of circles centred at 0

Figures 4 and 5 give a rendering of the mapping property stated in Lemma 2. Such properties are not too unexpected given the closeness of $I(u)$ with the classical Schwarz–Christoffel functions of conformal map theory [41, Sec. V.6].

2.2. **Analytic continuation beyond the dominant poles.** For the continuation of the discussion, it proves convenient to let z range in a region congruent to $\mathcal{H}$ as defined in (39), namely

$$\mathcal{H}' = \mathcal{H}e^{2i\pi/12},$$

as it makes it easier to discuss contours that loop freely several times around the points $\zeta^0, \zeta^1, \zeta^2$. We state:

Proposition 1. *There are no singularities of $\Psi(z)$ on $|z| = \rho$ other than ρ, $\rho\omega$, $\rho\omega^2$ that are simple poles. Precisely, let*

$$S(z) = \frac{1}{\rho - z} + \frac{1}{\rho\omega - z} + \frac{1}{\rho\omega^2 - z} = \frac{3z^2}{\rho^3 - z^3}.$$

The function

$$\Psi(z) - S(z)$$

is analytic in a disc $|z| < R$ for some R satisfying $R > \rho$. (One can take $R = 2\rho$.)

Proof. First, near $\zeta^0 = 1$, the function δ satisfies locally

$$\frac{t}{\delta(t)^5} \sim 6^{-5/6}(1-t)^{-5/6},$$

for t near 1. Consequently, one has on R_0

$$I(1) - I(u) \sim 6^{1/6}(1-u)^{1/6},$$

so that, locally near 1, $I(u)$ divides angles by a factor of 6. There results from this observation that a path in the complex plane that starts from 0, follows $(0,1)$ from below, then winds around 1 once, and returns to 0 above $(0,1)$ will have as image

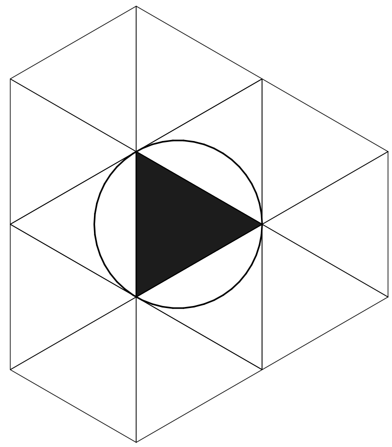

FIGURE 6. Rotated copies of the fundamental triangle around $\rho, \rho\omega, \rho\omega^2$ shown against the circle of convergence of $\Psi(z)$.

by $I(u)$ the union of the segment $(0, \rho)$ and of the segment $(\rho, \rho(1+\zeta))$. Generally, the image of a contour that is formed similarly, but winds r times around 1 will be composed of the segment $(0, \rho)$ and the segment connecting ρ to $\rho + \rho\zeta^r$. In other words, one reaches in this way the centres of the six triangles obtained from the fundamental triangle T by the six rotations around the vertex ρ having angles $2r\pi/6$ (with $0 \le r < 6$). .

The same argument as the one employed in the proof of Lemma 2 then shows that all the points in the interior of the rotated versions of T are accessible as values of $I(u)$. Similarly, paths that wind around ζ or ζ^2 give access to all the rotated versions of T around its vertices ω and ω^2.

In particular, the disc $|z| < 2\rho$ is entirely covered by values of $I(u)$ provided the integration path leading to u is constrained to wind (at most) around *one* of the points $\zeta^0, \zeta^1, \zeta^2$. At any such $z = I(u)$ that is not obtained by a value of the parameter u above $\zeta^0, \zeta^1, \zeta^2$ (i.e., $z \ne \rho, \rho\omega, \rho\omega^2$), $J(u)$ is an analytic function of u while $I(u)$ is locally invertible. The function Ψ is thus analytic (holomorphic) in the disc $|z| < 2\rho$ punctured at the points $\rho, \rho\omega, \rho\omega^2$. See Figure 6 for a display of such rotated copies. Following determinations of $\delta(u)$ then shows that the expansions (35) based on a local uniformizing parameter τ remain valid when τ lies in a complex neighbourhood of 0. Consequently, one has (by inversion of the first relation in (35) followed by substitution into the second relation) the convergent expansion (37) also for z in a *complex* neighbourhood of ρ, $z \ne \rho$. The point ρ as well as its conjugates $\rho\omega, \rho\omega^2$ are thus simple poles and the proposition is established. □

We observe that Proposition 1 suffices to establish the probabilistic estimates concerning the limit distributions, speed of convergence, and large deviations that are to be developed in Section 3. Its extension to the case of all 2×2 urns with subtraction constitutes Theorem 4 of Section 4.

2.3. **The elliptic connection.** An *elliptic* function is a function that is meromorphic in the whole complex plane and is doubly periodic. Amongst the many different ways to develop the corresponding theory, perhaps the simplest is the one originally proposed by Weierstraß, where elliptic functions are defined as sums of rational functions taken over lattices. (Accessible introductions appear in the books by Whittaker & Watson [51] and Chandrasekharan [12].)

Definition 1. *A* lattice Λ *with generators* $\xi, \eta \in \mathbb{C}$ *is defined as the set of complex numbers*

$$\Lambda(\xi, \eta) = \{n_1\xi + n_2\eta \mid n_1, n_2 \in \mathbb{Z}\}.$$

The Weierstraß zeta function *relative to* Λ *is classically defined as*

$$\zeta(z; \Lambda) := \frac{1}{z} + \sum_{w \in \Lambda \setminus \{0\}} \left(\frac{1}{z-w} + \frac{1}{w} + \frac{z}{w^2} \right), \tag{41}$$

and the Weierstraß $\wp$ *function is*

$$\wp(z; \Lambda) = -\zeta'(z; \Lambda) = \frac{1}{z^2} + \sum_{w \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right). \tag{42}$$

(The Weierstrass $\wp$-function is doubly periodic, whereas Λ is not but admits simple transformation formulæ under translations by elements of the lattice.)

We shall make use here of the "hexagonal" lattice Λ defined as the lattice generated by $e^{i\pi/6}, e^{-i\pi/6}$,

$$\Lambda_{\text{hex}} := \left\{ n_1 e^{i\pi/6} + n_2 e^{-i\pi/6} \mid n_1, n_2 \in \mathbb{Z} \right\}. \tag{43}$$

and its associated Weierstraß zeta function, $\zeta(z; \Lambda_{\text{hex}})$. We state:

Theorem 2. *The* Ψ*-function of the* $\mathcal{T}_{2,3}$ *model initialized with 2 balls of the first type* ($a_0 = t_0 = 2$) *is exactly*

$$\Psi(z) = \frac{1}{\rho\sqrt{3}} \left(-\zeta\left(\frac{z-\rho}{\rho\sqrt{3}} \right) + \zeta\left(-\frac{1}{\sqrt{3}} \right) \right), \qquad \text{with} \quad \rho := \frac{1}{6} \frac{\Gamma(\frac{1}{3})\Gamma(\frac{1}{6})}{\Gamma(\frac{1}{2})}, \tag{44}$$

where $\zeta(z) := \zeta(z; \Lambda_{\text{hex}})$ *is the Weierstraß zeta function of the hexagonal lattice. In particular the bivariate generating function of the model is expressible in terms of elliptic functions.*

Thus, the Ψ function is nothing but a renormalization of the ζ function such that: (i) the poles are at points of a lattice obtained from Λ after rescaling by a factor equal to $\rho\sqrt{3}$ followed by a translation of ρ; (ii) the residues of Ψ at its poles are *all* exactly equal to -1. (This is in full agreement with earlier local analyses at ρ and $\rho e^{\pm 2i\pi/3}$.)

Proof. Consider the whole complex plane tiled by non-overlapping copies of the hexagon of center ρ, radius $\rho\sqrt{3}$, having vertices at the points $\rho + i\zeta^j\rho\sqrt{3}$. We claim that any complex point z is reachable as a value $I(\gamma(u))$, where the notation $I(\gamma(u))$ indicates that the integral defining I is to be taken along a path $\gamma(u)$ that starts at 0 and ends at u. The principles are the same as before and the algorithm is as follows. Assume for simplicity that z is the center of one of the equilateral triangles in which the hexagonal tiling decomposes. The straight line L_0 from 0 to z can be first slightly deformed into a curve L_1 that avoids all the vertices of the tiling. This L_1 can then be transformed into a polygonal line L_2 that connects centers of successive equilateral triangles. Finally, each segment of L_2 can be changed into

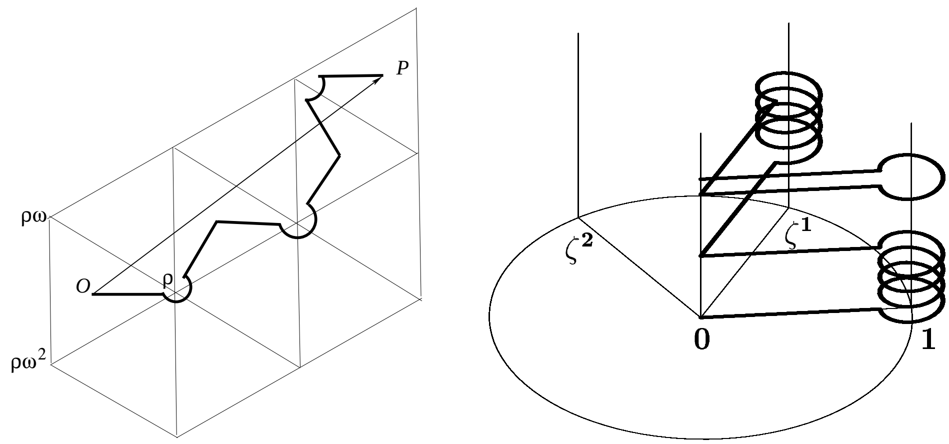

FIGURE 7. A standard path in the z-plane from 0 to $P \equiv z$ and the contour γ above the u-plane that realizes it via $u \mapsto z = I(\gamma(u))$.

a pair of segments going through one of the vertices of the lattice and forming an angle a positive multiple of $\pi/3$. The resulting polygonal line, L_3, will be called the *standard z–path*. See Figure 7 for a graphic rendering.

The contour γ, called the *standard u–path*, is then obtained from the standard z–path L_3 by first applying a contraction by a factor ρ, then applying the following rules:

— turn by an angle of 6θ whenever L_3 turns at an angle of θ (where θ is a multiple of $\pi/3$) around a vertex of the lattice,
— turn by an angle $\theta/2$ (where θ is a multiple of $2\pi/3$) whenever L_3 turns by θ around the centre of one of the equilateral triangles.

(The construction is easily modified to accommodate points that are not centres of triangles of the tiling.)

For any z in $\mathbb{C}$ that is not a vertex of the tiling, the algorithm determines constructively a path $\gamma(u)$. By design, along such a path, one has $I(\gamma(u)) = z$. Indeed, the standard u-path is precisely such that it "undoes" the effect of $I(u)$ on angles at points either vertices of the tiling or centres of the triangles while the variation of $I(u)$ along a segment from a point u_0 above 0 to a point u_1 above some ζ^j is precisely of modulus ρ and thus gives rise to a segment with the "right" length. See once more Figure 7. In this way, we find that $I(\gamma(u))$ reaches any point of the complex plane that is not a vertex of the tiling, and at the final u point, $J(u)$ is locally analytic. Thus, $\psi(z)$ can be at least continued to the complex plane punctured at vertices of the tiling.

When $z = w$ is one vertex of the lattice, then it is approached from a certain direction by a path $\gamma(u)$, where u is near ζ^0, ζ^1, or ζ^2. Then, along the path a certain determination $\delta^\circ(u)$ of δ is in force, where all determinations are of the form $\zeta^r\delta(u)$ with $0 \le r < 6$. Then, the very same determination δ° must be adopted in $J(u) = J(\gamma(u))$ that tends to infinity as $I(\gamma(u))$ approaches w. A local analysis entirely analogous to the one conducted for the three dominant poles shows that Ψ has a pole at w, that the pole is simple, and the corresponding residue consistently has the same value, namely -1.

The analytic continuation of $\Psi(z)$ along such paths γ has therefore the same (simple) poles and residues as the right side of Equation (44), namely the function

$$\widetilde{\Psi}(z) := \frac{1}{\rho\sqrt{3}}\left(-\zeta\left(\frac{z-\rho}{\rho\sqrt{3}}\right)+\zeta\left(-\frac{1}{\sqrt{3}}\right)\right).$$

Consequently, the difference $\Psi(z) - \widetilde{\Psi}(z)$ is an entire function. That this entire function reduces to 0 results plainly from Liouville's theorem, as we now argue.

Draw discs of some sufficiently small but fixed radius around the six roots of unity in the u plane and consider these are excluded regions. Then the image by $I(\gamma(u))$ as $\gamma(u)$ varies avoids the plan stripped of small ovals around the lattice points Λ. Since the integrand of J remains uniformly bounded from above in the stripped plane, one has,

$$|J(\gamma(u))| \le c_1|\gamma(u)|,$$

for some $c_1 > 0$, where $|\gamma(u)|$ is the length of γ. (The assumption that small discs around singularities are avoided is of course essential.) On the other hand, with the standard construction of a path $\gamma(u)$ described above and such that $I(\gamma(u)) = z$, one has also, for some $c_2, c_3 > 0$ and $|z|$ large enough,

$$c_2|z| < |\gamma(u)| < c_3|z|.$$

Thus, on the complex plane with "holes", $\Psi(z)$ is of a growth at most linear in $|z|$. A similar property holds for the Weierstrass function $\zeta(z)$ and its normalization $\widetilde{\Psi}(z)$: this can be seen directly from the definition or, alternatively, it is a consequence of the fact that ζ' is doubly periodic. There results that, over the plane with holes,

$$|\Psi(z) - \widetilde{\Psi}(z)| < c_4|z| + c_5,$$

for some c_4, c_5. In particular, provided the ovals are of a small enough diameter (say $< \rho/10$), the linear growth holds on an infinity of larger and larger circles centred at the origin. Then, by virtue of Liouville's theorem, (an entire function polynomially bounded in modulus is a polynomial) one must have exactly

$$\Psi(z) - \widetilde{\Psi}(z) = d_1 z + d_2,$$

for some complex constants d_1, d_2. These constants are actually equal to 0 as is seen from comparing the expansions of $\Psi(z)$ and $\widetilde{\Psi}(z)$ at 0, where $\Psi(z) = O(z^2)$ and $\widetilde{\Psi}(z) = O(z^2)$. This completes the proof of the main theorem. □

3. Probabilistic consequences of the analytic model for $\mathcal{T}_{2,3}$

We are now in a position to exploit fully the analytic solutions expressed by Theorems 1 and 2. First curious exact representations as sums over lattice points result for the probability generating functions describing the urn composition (Section 3.1). In particular, the representations imply a "quasi-powers approximation", which means that the analytic situation parallels that of a sum of independent random variables. The known Gaussian limit law results from there, together with an estimate of the rate of convergence. Surprisingly perhaps, a very precise form of all moments can be obtained in terms of a family of polynomials of "binomial type" [44]; see Section 3.2. Finally, the large deviation rate function appears to be well characterized and computable in terms of the basic Abelian integral $I(u)$; se Section 3.3.

3.1. **Exact representations and Gaussian laws.** The lattice structure that underlies the Weierstraß function is directly reflected at the level of coefficients. The resulting form below is naturally a very strong form, as it is an *exact* description of the probability generating function at time n.

Corollary 1. *For the $\mathcal{T}_{2,3}$ model, the probability generating function $p_n(u) = \mathbb{E}(u^{X_n})$ admits an* exact *formula valid for all $n \geq 2$,*

$$p_n(u) = \sum_{n_1,n_2=-\infty}^{+\infty} \left(K(u) + \frac{\rho\sqrt{3}}{\delta(u)}(n_1 e^{i\pi/6} + n_2 e^{-i\pi/6}) \right)^{-n-1}, \tag{45}$$

where

$$K(u) := \frac{1}{\delta(u)} \int_u^1 \frac{t}{\delta(t)^5}\, dt, \qquad \delta(u) = (1-u^6)^{1/6}.$$

Proof. Set $\alpha := \rho\sqrt{3}$. From Theorem 2, we need to extract $[z^n]\delta\Psi(\delta z + I)$, where Ψ is a variant of the Weierstraß ζ function that admits a decomposition as a sum of rational fractions and the sum ranges over elements of the lattice Λ. We then have, upon substituting the sum expression for Ψ and extracting coefficients termwise:

$$p_n(u) = \frac{\delta}{\alpha} \sum_{w \in \Lambda} \left([z^n] \frac{1}{w - \frac{z\delta + I - \rho}{\alpha}} \right).$$

(The process is readily justified by Cauchy's integral formula for coefficients upon integrating along a large contour that avoids poles.) Then, after a simple calculation, one gets

$$p_n(u) = \sum_{w \in \Lambda^\star} \left([z^n] \frac{1}{w - z} \right),$$

where $\Lambda^\star$ is a translated and scaled verson of Λ:

$$\Lambda^\star = \frac{\alpha}{\delta(u)} \Lambda + K(u).$$

The result follows. $\square$

Next, we summarize the basic technology used to derive a Gaussian limit by the following statement, a simplified form of what is often referred to as the "Quasi-Powers Theorem" and is to be found in works of Bender and Hwang; see for instance [6, 29]. Throughout this article, we use $\mathbb{E}$ and $\mathbb{V}$ to denote the expectation and variance operators.

Lemma 3 (Quasi-Powers Theorem)**.** *Let $q_n(u) = \mathbb{E}(u^{X_n})$ be a family of probability generating functions relative to discrete random variables X_n. Assume that there exist two functions $A(u), B(u)$ analytic in a neighbourhood $\mathcal{V}$ of $u = 1$, such that, in this neighbourhood the* quasi-power *approximation*

$$q_n(u) = A(u)B(u)^n(1 + \epsilon_n(u)) \qquad \text{as } n \to \infty \tag{46}$$

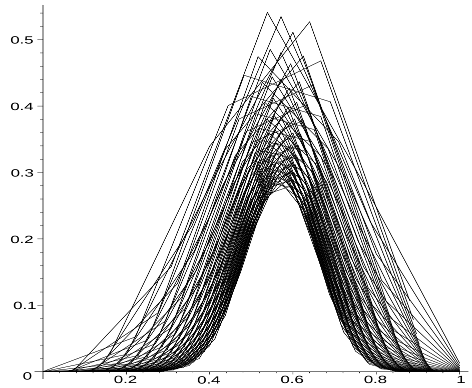

FIGURE 8. A Sedgewick plot of $\{\mathbb{P}(X_n = k)\}_{k=0}^{n-1}$ for $n = 24\,..\,96$ (the horizontal axis is normalized to $n+1$).

holds, where $|\epsilon_n(u)| = O(n^{-1/2})$ *uniformly with respect to* u*, i.e.,* $\sup_{u\in\mathcal{V}} |\epsilon_n(u)| = O(n^{-1/2})$*. Assume also the* variability condition

$$\sigma^2 \neq 0, \qquad \textit{where} \quad \sigma^2 := \lim_{n\to\infty} \frac{\mathbb{V}X_n}{n} \equiv B''(1) + B'(1) - B'(1)^2. \tag{47}$$

(Equivalence between the two forms of σ^2 *is granted under condition* (46)*.) Then, the random variables* X_n *converge in law to a Gaussian limit, with speed of convergence* $O(n^{-1/2})$*: for any* x*, one has*

$$\mathbb{P}\left(\frac{X_n - \mathbb{E}(X_n)}{\sqrt{\mathbb{V}X_n}} \leq x\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy + O\left(\frac{1}{\sqrt{n}}\right). \tag{48}$$

Proof (Sketch; see [6, 29] *for details).* The characteristic function $q_n(e^{it})$ of the X_n is by assumption closely approximated by an nth power. The variable X_n is next centred around its mean and scaled by its standard deviation in the usual way. A calculation similar to the usual case of independent random variables (e.g., [10, p. 367]) then shows the standardized version of $q_n(e^{it})$ to converge to $e^{-t^2/2}$, which is the characteristic function of a Gaussian law. The speed of convergence estimate finally results from the Berry-Esseen inequality [36]. □

Corollary 2 (Gaussian limit)**.** *For the* $\mathcal{T}_{2,3}$ *model, the random variable* X_n *representing the number of balls of the first type at time* n *is asymptotically Gaussian with speed of convergence to the limit* $O(n^{-1/2})$*, as expressed by* (48)*.*

Proof. Choose first a small neighbourhood of 1, say, $|u-1| < \frac{1}{100}$ to fix ideas. The quantity $K(u)$ is analytic at $u = 1$. Though $\delta(u)$ is not analytic, all its determinations are close to 0, and they remain small when u is confined to this small neighbourhood of 1. This means that the points of the lattice over which the sum in (45) is taken are far away from $K(u)$ for all n_1, n_2 satisfying $n_1^2 + n_2^2 \neq 0$, so that their global contribution is small. A trite calculation then shows that

$$p_n(u) = K(u)^{-n-1}(1 + O(2^{-n})).$$

This suffices to guarantee the conditions of the Quasi-Powers Theorem (Lemma 3 above), with $A(u) = B(u) = K(u)^{-1}$. □

The finite distributions are displayed in Figure 8. The random variable X_n superficially resembles a sum of independent random variables since its probability generating function is essentially an nth power of the fixed function $K(u)^{-1}$. It is however of interest to observe that the function $K(u)^{-1}$, though analytic at 0 and satisfying $K(1) = 1$ is *not* a probability generating function, as its Taylor coefficients of index $6, 12, 18, \dots$ turn out to be negative:

$$K(u)^{-1} \doteq 0.713 + 0.254u^2 + 0.090u^4 - 0.086u^6 + 0.022u^8 + \cdots .$$

As this illustrates, the purely analytic conditions of the Quasi-Powers Theorem prove not to be strictly equivalent to the usual probabilistic conditions of the Central Limit Theorem (although the underlying analytic machineries are quite similar).

3.2. **The shape of moments.** An interesting consequence concerns the moments of the distribution. Let X_n be the random variable of interest. Bagchi and Pal, Mahmoud, Panholzer and Prodinger have determined the exact form of the first two moments, while Bagchi and Pal have obtained further asymptotic information on the moments of higher order. This involves a certain amount of calculational effort with recurrences. In fact, globally, the moments have an amazingly simple form.

The usual notation for falling factorials [26] is employed here, namely

$$a^{\underline{r}} = a(a-1)\cdots(a-r+1). \tag{49}$$

Corollary 3 (Moments). *For the $\mathcal{T}_{2,3}$ model, exact polynomial forms for moments of* any *order are available: the factorial moments satisfy*

$$\mathbb{E}((X_n)^{\underline{r}}) = P_r(n+1), \qquad n \geq 6r,$$

where the P_r are polynomials generated by

$$e^{vL(h)} = \sum_{r=0}^{\infty} \frac{h^r}{r!} P_r(v) \quad \text{and} \quad L(h) = -\log K(1+h). \tag{50}$$

Using a symbolic manipulation system, the polynomials are easily computed from the expansion of K at $u = 1$. To wit:

$$K(1+h) = 1 - \frac{4}{7}\,h + \frac{10}{91}\,h^2 + \frac{300}{1729}\,h^3 - \frac{1689}{8645}\,h^4 + \cdots .$$

One then finds mechanically

$$P_1(\nu) = \frac{4\nu}{7},\ P_2(\nu) = \frac{4\nu}{637}\,(52\,\nu + 17)\,,\ P_3(\nu) = \frac{8\nu}{84721}\left(1976\,\nu^2 + 1938\,\nu - 11063\right).$$

In particular, the mean and variance of X_n are:

$$\mathbb{E}(X_n) = \frac{4}{7}(n+1), \qquad \mathbb{V}(X_n) = \frac{432}{637}(n+1)^2.$$

Proof. Take the fundamental PDE, isolate $G'_u(z,u)$, and repeatedly differentiate with respect to u, then set $u = 1$. This provides a triangular system from which one can "pump" in succession the generating functions of moments of order $1, 2, 3, \dots$. One then verifies by induction that the ordinary generating function of the moments of order r is of the form

$$\sum_n \mathbb{E}(X_n^{\underline{r}}) z^n = \frac{\widetilde{P}_r(z)}{(1-z)^{r+1}} + \widetilde{Q}_r(z)$$

where $\widetilde{P}_r, \widetilde{Q}_r$ are polynomials and

$$\deg(\widetilde{P}_r(z)) \leq r-1, \qquad \deg(\widetilde{Q}_r(z)) \leq 6r-1.$$

As an illustration, the values of $\widetilde{P}_r(z)/(1-z)^{r+1}$ are for $r=1,2,3$

$$\frac{4}{7}\frac{1}{(1-z)^2}, \ \frac{4}{637}\frac{69+35\,z}{(1-z)^3}, \ -\frac{24}{84721}\frac{2383-10010\,z+3675\,z^2}{(1-z)^4},$$

This argument grants us *nonconstructively* the existence of a polynomial representation for each moment as soon as n is large enough.

There remains to identify the particular class of polynomials involved. Start from the fact that

$$p_n(u) = K(u)^{-n-1} + \text{exponentially small terms in } n.$$

Since the factorial moment of order r satisfies

$$\mathbb{E}(X_n^{\underline{r}}) = \left(\partial_u^r p_n(u)\right)_{u=1} = [(u-1)^r]\,p_n(u),$$

it can be obtained, up to exponentially small error terms, by expanding $K(u)^{-n-1}$ around $u=1$. Retaining only the polynomial part (in n),

$$[(u-1)^r]\ K(u)^{-n-1} = [(u-1)^r]\,e^{-(n+1)\log K(u)} = [h^r]\ e^{-(n+1)\log K(1+h)},$$

we get what the statement asserts. □

A similar construction gives the standard "power" moments: it suffices to take

$$-\log K(e^h)$$

as a replacement to $-\log K(1+h)$ in order to define the right family of polynomials by its generating function. The polynomials P_r relative to the factorial moments as well as their counterparts for the standard power moments are of what Rota has named "binomial type" [44]. Obvious identities like

$$e^{(v+w)L} = e^{vL}\cdot e^{wL},$$

then translate into convolution formulæ, e.g.,

$$P_r(v+w) = \sum_j \binom{r}{j} P_j(v)P_{r-j}(w),$$

which in turn imply curious convolution identities for moments of the urn model.

3.3. **Large deviations.** An immediate consequence of the analysis of the polar singularities of Ψ is a quantification of *extreme large deviations*:

Corollary 4 (Extreme large deviations). *The probability that, in the $\mathcal{T}_{2,3}$ model, all balls are of the second colour satisfies*

$$[z^{3n+2}]\Psi(z) \sim 3\rho^{-3n-3}\left(1+O(A^{-n})\right),$$

for any $A<8$.

Proof. The function $\Psi(z)$ is exactly the BGF of the urn taken at $u=0$. Thus $[z^n]\Psi(z)$ is the probability for the urn not to contain any ball of the first type. The property then results immediately from the fact that $\Psi(z)-S(z)$ (in the notations of Lemma 1) is analytic in $|z|<2\rho$. □

We next turn to large deviations, for which the book of den Hollander [15] can serve as a smooth introduction. It is known from the works of Hwang [28] that a quasi-power approximation (in the sense of Lemma 3) for a family of PGFs leads to very precise "moderate deviation" estimates valid in some range not too far from the center of the distribution. We state:

Corollary 5 (Large deviations). *Let ξ be a number of the open interval $(0, \frac{4}{7})$. One has*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(X_n \le \xi \cdot n\right) = -\mathcal{R}(\xi), \tag{51}$$

where the rate function $\mathcal{R}$ is determined from $K(u)$ of Corollary 1 by

$$\mathcal{R}(\xi) = \max_{\lambda \in (0, \frac{4}{7})} \log K(\lambda)\lambda^{\xi}. \tag{52}$$

Equivalently:

$$\mathcal{R}(\xi) = \log\left(\lambda_0^{\xi} K(\lambda_0)\right) \qquad \text{where} \qquad \frac{\lambda_0 K'(\lambda_0)}{K(\lambda_0)} + \xi = 0, \quad \lambda_0 \in \left(0, \frac{4}{7}\right). \tag{53}$$

In other words, the probability of departing significantly from the mean value is exponentially small, the rate of exponential decay being explicitly related to the already encountered Abelian integrals.

Proof. Notice first that $\mathbb{E}(X_n) = \frac{4}{7}(n+1)$, so that (51) quantifies the left part of the distribution as approximately given by $e^{-n\mathcal{R}(\xi)}$. Theorem 2.1 in [28] asserts that quasi-powers approximations near $u = 1$ systematically entail "moderate" large deviation estimates expressed in terms of large values of the Gaussian error function. The proof below recycles most of the technology of [28], though the range is different. For completeness, we outline the adapted reasoning that underlies the lemma as it also serves to justify the conditions in (53) above. The basic ingredient is Cramér's technique of "shifting the mean" conjugated with upper bounds of the saddle point (equivalently, Chernoff) type as well as lower bounds based on the quasipowers theorem in a shifted region.

First, one has

$$\mathbb{P}(X_n \le \xi n) = [u^k]\, \frac{p_n(u)}{1-u},$$

since multiplication by $(1-u)^{-1}$ sums coefficients of generating functions. Next, for any $f(u)$ analytic at 0 having nonnegative Taylor coefficients, the easy inequality $[u^k]f(u) \le f(\lambda)\lambda^{-k}$ holds provided the positive quantity λ is taken inside the disc of convergence of $f(u)$. There results from these two observations the majorization

$$\mathbb{P}(X_n \le \xi n) \le \frac{p_n(\lambda)}{(1-\lambda)\lambda^{\lfloor \xi n \rfloor}}, \tag{54}$$

valid for any $\lambda \in (0,1)$.

In order to derive an *upper bound* on large deviations, it suffices to choose the best possible value of λ in (54). Now, for fixed positive $\lambda \in (0,1)$, the function $G(z,\lambda)$ has a dominant pole at the point ζ such that $\zeta\delta(\lambda) + I(\lambda) = \rho$, that is, at $\zeta = K(\lambda)^{-1}$. This pole is simple and it uniquely dominates. Therefore one has

$$p_n(\lambda) \underset{n\to\infty}{\sim} C_\lambda \cdot K(\lambda)^{-n}, \tag{55}$$

for some constant C (depending smoothly on λ). When ξ lies in any fixed compact subinterval of $(0, \frac{4}{7})$, the upper bound (54) can then be rewritten as

$$\mathbb{P}(X_n \leq \xi n) \leq \overline{C} K(\lambda)^{-n} \lambda^{-\xi n}$$

for some constant $\overline{C}$. This is a form amenable to optimization. Let λ_0 be such that $K(\lambda)^{-1}\lambda^{-\xi}$ attains its minimum[7] over $(0,1)$ at λ_0. The value of λ_0 is obtained by cancelling the derivative of $K(u)^{-1}\lambda^{-\xi}$ and is thus a root of the second equation in (53). Up to factors that are subexponential in n, the upper bound in (55) is of the form $e^{-n\mathcal{R}(\xi)}$, with $\mathcal{R}(\xi)$ as given by (53) and (52). We have thus established "one half" of (51) , namely,

$$\frac{1}{n} \log \mathbb{P}\left(X_n \leq \xi n\right) \leq -\mathcal{R}(\xi) + o(1),$$

with $\mathcal{R}(\xi)$ determined by (53).

There finally remains to argue that the upper bound is tight, that is, derive a *lower bound* on the probability values. This results from Cramér's technique of shifting the mean. The shifted law $r_{n,k} = [u^k]r_n(u)$ defined by the probability generating function

$$r_n(u) := \frac{p_n(\lambda_0 u)}{p_n(\lambda_0)}$$

satisfies a standard quasi-powers approximation and is itself amenable to Lemma 3. Assume first that the variability condition (47) holds for the shifted law given by $r_n(u)$. In that case the sum of probabilities $\sum_{\xi n - \sqrt{n} < k \leq \xi n} r_{n,k}$ of the shifted law tends to a nonzero constant as it is approximated by a Gaussian integral. By construction, the $r_{n,k}$ are the $p_{n,k} \equiv [u^k]p_n(u)$ multiplied by a quantity λ_0^k which varies between $e^{-O(\sqrt{n})}\lambda_0^{\xi n}$ and $O(1)\lambda_0^{\xi n}$. Thus, the corresponding sum $\sum_{\xi n - \sqrt{n} < k \leq \xi n} p_{n,k}$ is, up to subexponential factors (themselves of the form $e^{-O(\sqrt{n})}$), of the type $e^{-n\mathcal{R}(\xi)}$. This implies a lower bound, hence the "other half" of the equality in (51). Finally, if the variability condition at 0 is not satisfied, then an even stronger type of concentration holds for the shifted distribution $r_{n,k}$: in that case, the variance of the shifted distribution is $o(n)$, which, by Chebyshev's inequality, entails the stated lower bound on the sum of the $r_{n,k}$, hence the lower bound on partial sums of the $p_{n,k}$. □

The large deviation function appears as contour of plots of the suitably scaled finite distribution[8]; see Figure 9. The right side of the large deviation function corresponding to $\xi > \frac{4}{7}$ can be obtained by similar means using values of $u > 1$.

---

[7]Let $P(u)$ be the probability generating function of a nonnegative nondegenerate random variable and $R(u;\lambda) := P(\lambda u)/P(\lambda)$ the PGF of an RV $Y(\lambda)$. The expectation and variance of Y are given by

$$\mathbb{E}(Y(\lambda)) = \frac{\lambda P'(\lambda)}{P(\lambda)}, \qquad \mathbb{V}(Y(\lambda)) = \lambda \frac{d}{d\lambda}\left(\frac{\lambda P'(\lambda)}{P(\lambda)}\right).$$

There results that $\lambda P'(\lambda)/P(\lambda)$ is always increasing, no matter what X is. Taking $P(u) = p_n(u)$ and $R(u;\lambda) = r_n(u)$ and passing to the asymptotic limit $n \to \infty$, one gets that $-\lambda K'(\lambda)/K(\lambda)$ is increasing for $\lambda \in (0,1)$, since its derivative is necessarily nonnegative and can be 0 only at isolated points.

[8]We owe to Robert Sedgewick the idea of graphically exhibiting limit distributions and large deviation rates by means of such suitably scaled plots.

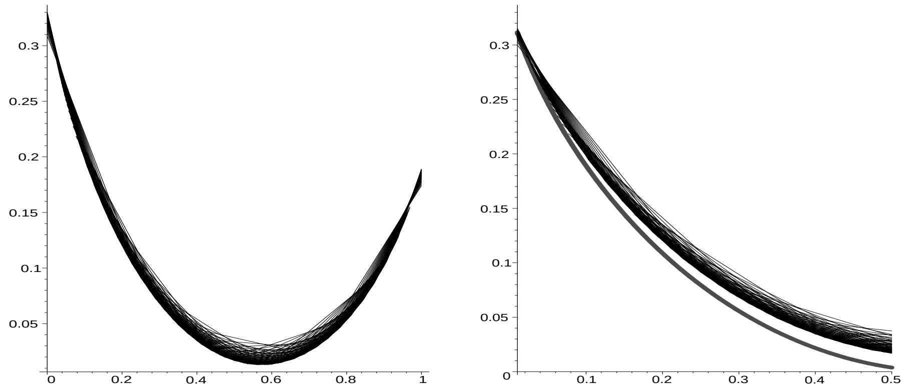

FIGURE 9. Left: a Sedgewick plot of $\{-\frac{1}{n}\log \mathbb{P}(X_n = k)\}_{k=0}^{n+1}$ for $n = 24\,..\,96$ (the horizontal axis is normalized to $n+1$); right: a comparison against the large deviation rate (thick line) as computed from Corollary 5.

We do not address the question of the existence of local limit laws in this article. In principle, such local laws are accessible via saddle point integrals as soon as certain technical growth conditions of $K(u)$ along the unit circle have been verified. Since we only have numerical evidence at the moment but not a general argument applicable to all balanced urns, we shall abstain from discussing this point here.

4. Analytic solution of the general case

We now take up the *general case* of a balanced urn model with two types of balls and negative diagonal entries. The matrix is then of the form

$$M = \begin{pmatrix} -a & a+s \\ b+s & -b \end{pmatrix}, \qquad a, b > 0, \tag{56}$$

with $s > 0$ the balance. We refer to such models as *urns with subtraction.* The urn is initialized at time 0 with a_0 balls of the first type ("black") and b_0 balls of the second type ("white"). With $t_0 = a_0 + b_0$, the initial size, the size of the urn at time n is exactly the deterministic quantity $t_0 + ns$. (The conventions adopted from now on differ marginally from those of the $\mathcal{T}_{2,3}$ models where it appeared to be convenient to have the urn starting its evolution from time 1 rather than time 0.) In order for the urn not to be blocked by an infeasible request, the usual "tenability" conditions from [4, 23] for urns with subtraction are assumed:

$$\begin{cases} (\mathbf{T}_0): & a \text{ divides } a_0 \text{ and } b \text{ divides } b_0; \\ (\mathbf{T}_1): & a \text{ divides } b+s \text{ and } b \text{ divides } a+s. \end{cases} \tag{57}$$

We shall see soon that all such models are "solvable by quadrature" in the spirit of Theorem 1 above. In other words, only elementary algebraic functions, composition and inversion, as well as integration are involved in the solution, as is expressed by the general statement of Theorem 3 below. Solutions in terms of elliptic functions are no longer available in the general case. Nonetheless, what survives is a complete characterization of dominant singularities, as summarized by Theorem 4 below, which constitutes an extension of our earlier Proposition 1 and a weaker form of Theorem 2. Based on dominant singularities, the Gaussian limit law results, together with speed of convergence estimates, structural information on the shape of moments, and an explicit determination of the large deviation rate function. In Section 5, we shall also see a number of cases where the calculations can be made fully explicit.

4.1. **Algebraic Approach.** Based on formal operator calculus, there is an elegant symbolic approach to the derivation of PDE's for urn models. Its interest lies in the fact that it establishes a transparent connection between the combinatorial structure of a model and the PDE that expresses it.

Consider an urn that starts with t_0 balls in total at time 0; assume that a_0 of these are of the first type, the matrix being given by (56). The tenability conditions $(\mathbf{T}_0, \mathbf{T}_1)$ of (57) are assumed. The evolution is dictated by the rules described in the introduction. The *combinatorial model* considers all balls involved in the game to be distinguished by distinct integer stamps: urns present at time 0 are stamped, say, $1, \dots, a_0$ for type I and $a_0+1, \dots, t_0$ for type II. New balls are stamped with "new" numbers: the balls that are taken away from the urn are the ball chosen as well as others taken according to a deterministic choice, e.g., by starting from smallest numbers. For instance, the urn $\left(\begin{smallmatrix}-1 & 2\\ 3 & -2\end{smallmatrix}\right)$ initialized with two balls of type I stamped with 1 and 2 may give rise to an evolution history starting as

Time:	0	1	2	3	
	choose 2	choose 3	choose 5	choose 1	
Urn:	$\overbrace{1_I, 2_I}$,	$\overbrace{1_I, 3_{II}, 4_{II}}$,	$\overbrace{1_I, 5_I, 6_I, 7_I}$,	$\overbrace{1_I, 6_I, 7_I, 8_{II}, 9_{II}}$,	$\cdots$

with subscripts indicating colours/types of the corresponding balls.

One needs to relate the combinatorial model underlying the operator method and the probabilistic model. We let $p_n(u)$ denote the PGF (probability generating function) of the number of balls of the first type at time n, which is precisely the quantity of interest and is denoted by X_n. Let $f_n(u)$ be the *counting generating function* of the evolution histories of length n, where u marks the number of balls of the first type: the coefficient $[u^k]f_n(u)$ is then the number of histories comprising n transformations of the urn and such that the resulting number X_n of balls of type I equals k. We have $f_0(u) = p_0(u) = u^{a_0}$ and

$$p_n(u) = \frac{f_n(u)}{t_0(t_0+s)\cdots(t_0+(n-1)s)}, \tag{58}$$

since the total number of possible histories of length n is

$$t_0(t_0+s)\cdots(t_0+(n-1)s) = n!s^n \binom{n+t_0/s-1}{n}, \tag{59}$$

as results from multiplication of n elementary choices. Introduce finally the exponential generating function of the $f_n(u)$, so that

$$H(z,u) := \sum_{n\geq 0} f_n(u)\frac{z^n}{n!},$$

is a bivariate generating function, and, by design, knowledge of H determines the $p_n(u)$ by (58). (As $u \to 1$, the BGF $H(z,u)$ degenerates into a simple algebraic function

$$H(z,1) = (1-sz)^{-t_0/s}, \tag{60}$$

since it then only counts histories in accordance with (59). Thus, $H(z,u)$ is *a priori* a "deformation" of a simple algebraic function.)

For u a variable, we let $\partial_u \equiv \frac{\partial}{\partial u}$ be the corresponding partial differential operator. It is notationally convenient to make use of the modified operator

$$\theta_u = u\partial_u \quad \text{so that} \quad \theta_u f = u\frac{\partial f}{\partial u}.$$

Differential operators are well-known to correspond combinatorially to a "pointing" operation. For instance, one has

$$\partial_u u^a = a u^{a-1}, \qquad \theta_u u^a = a u^a,$$

so that ∂_u may be interpreted as "select a u-element in all possible ways and remove it" while θ_u means "select a u-element in all possible ways and keep it". There are many instances in the combinatorics literature of such a usage of differential operators; see, e.g., [21], [24, p. 45], or [9, Sec. 2.1].

Consider now an urn model defined by a matrix M of the form (56). Let a particular urn configuration with λ white balls and μ black balls be represented by the monomial $\mathfrak{m}_{\lambda,\mu} = u^\lambda v^\mu$. Introduce the partial differential operator (associated to M)

$$\Gamma = u^{-a} v^{s+a}\theta_u + u^{s+b} v^{-b}\theta_v. \tag{61}$$

Then, the application of Γ to $\mathfrak{m}_{\lambda,\mu}$ describes all the possible successors of the urn represented by $\mathfrak{m}_{\lambda,\mu} = u^\lambda v^\mu$ when one step of ball replacement is performed.

Start with an urn of initial type $\langle a_0, b_0\rangle$, with $t_0 := a_0 + b_0$. Let $h_n(u,v)$ be the polynomial enumerating all possible evolutions of the urn in n steps. (In particular, $h_n(u,1) = f_n(u)$.) Then, one has

$$h_n(u,v) = \Gamma^n \circ \left(u^{a_0} v^{b_0}\right).$$

We opt for exponential generating functions and define

$$\widehat{H}(z;u,v) = \sum_{n\geq 0} h_n(u,v)\frac{z^n}{n!}.$$

One has symbolically:

$$\widehat{H}(z,u,v) = e^{z\Gamma} \circ \left(u^{a_0} v^{b_0}\right),$$

where the exponential of operators is defined in the usual way:

$$e^{z\Gamma} \circ g := \sum_{n\geq 0} \frac{z^n}{n!}\Gamma^n g.$$

Then, the definition of the exponential implies immediately the differential relation

$$\partial_z \left(e^{z\Gamma} f\right) = \Gamma e^{z\Gamma} f.$$

In other words, $\widehat{H}$ satisfies the PDE

$$\partial_z \widehat{H} = \Gamma \circ \widehat{H}. \tag{62}$$

This last equation is almost the PDE we are looking for but not quite (it has a supplementary variable, v). Given the regularity conditions imposed in the choice of the matrix M, where row sums are equal to the constant s, the urn population increases by exactly s at each step. Accordingly, $\widehat{H}$ involves three variables, u, v, z, but their exponents in $\widehat{H}$ are bound by a homogeneity condition, each monomial generated being of the form $u^\lambda v^\mu z^n$ with $\lambda+\mu = sn+t_0$. In other words, each monomial $\mathfrak{m}$ composing H satisfies

$$(\theta_u+\theta_v - s\theta_z)\mathfrak{m} = t_0\mathfrak{m}, \tag{63}$$

and the relation extends by linearity to $\widehat{H}$ itself.

In summary, a system of two equations now determines $\widehat{H}$ (with $\theta_u \equiv u\partial_u$):

$$\begin{cases} \partial_z \widehat{H} = \Gamma \circ \widehat{H} \\ (\theta_u+\theta_v-s\theta_z)\widehat{H} = t_0\widehat{H}. \end{cases} \tag{64}$$

One can then eliminate the explicit differential dependency on v (the operator ∂_v), and get from (61)

$$\partial_z\widehat{H} = u^{-a}v^{1+a}\theta_u\widehat{H} + u^{1+b}v^{1-b}\left(s\theta_z\widehat{H} - \theta_u\widehat{H} - t_0\widehat{H}\right).$$

At this stage it becomes possible to set $v=1$, i.e., completely eliminate the redundant variable v itself. In this way one has the *fundamental PDE*

$$\left[(1-szu^{b+s})\partial_z + (u^{b+s+1}-u^{1-a})\partial_u - t_0u^{b+s}I\right]\circ H(z,u) = 0. \tag{65}$$

where $H \equiv H(z,u)$ is $\widehat{H}(z;u,1)$ in earlier notations. This is seen to agree with what we had otherwise for the $\mathcal{T}_{2,3}$ urn ($a=2$, $b=3$), in which case $H(z,u) = \partial_z G(z,u)$ due to the slight difference in normalizations (time starts at 1 for G but at 0 for H).

The main result is then:

Theorem 3. *The probability generating function at time n of the urn specified by*

$$\textit{matrix:}\quad \begin{pmatrix} -a & a+s \\ b+s & -b \end{pmatrix}, \qquad \textit{initial conditions:}\quad a_0, b_0$$

assuming it is tenable, is, with $t_0 := a_0+b_0$,

$$p_n(u) = \frac{\Gamma(n+1)\Gamma\left(\frac{t_0}{s}\right)}{s^n\Gamma\left(n+\frac{t_0}{s}\right)}[z^n]H(z,u).$$

There $H(z,u)$ is given by

$$H(z,u) = \delta(u)^{t_0}\psi(z\delta(u)^s + I(u)),$$

where $h := a+b+s$,

$$\delta(u) := (1-u^h)^{1/h}, \qquad I(u) := \int_0^u \frac{t^{a-1}}{\delta(t)^{a+b}}\,dt$$

and the function ψ is defined implicitly by

$$\psi(I(u)) = \frac{u^{a_0}}{\delta(u)^{t_0}}.$$

This result holds for $-a,-b<0$ but it can be readily adapted to cope with other cases as well (see Section 5 for some characteristic instances). It can be viewed as an explicitation and a vast extension of remarks made earlier by Bernard Friedman [22] regarding a parameterization of the special class of urns with matrices $\left(\begin{smallmatrix} a & b \\ b & a\end{smallmatrix}\right)$. It transpires from the statement of Theorem 3 that the major determinant of the urn's analytic behaviour is the quantity $h=s+a+b$ that involves the balance as well as the values of the two diagonal entries.

Proof. In accordance with the general strategy summarized in Figure 1, one looks for first integrals of the ordinary differential system

$$\frac{dz}{1-sz^{b+s}} = \frac{du}{u^{s+b+1}-u^{1-a}} = \frac{dw}{t_0 u^{b+s} w}.$$

The equation binding w and u is the easiest to solve as it allows for separation of variables,

$$\frac{dw}{w} = t_0 \frac{u^{h-1}}{u^h - 1}\, du,$$

so that a first integral is, in the notations of the statement,

$$w\delta(u)^{-t_0} = C_1. \tag{66}$$

The equation binding z and u is similar but inhomogeneous:

$$\frac{dz}{du} = -sz\frac{u^{h-1}}{u^h-1} + \frac{u^{a-1}}{u^h-1}.$$

The homogeneous equation is solved by separation of variables as $z = \xi\cdot(1-u^h)^{-s/h}$. By the variation-of-constant technique, one finds

$$z = \xi(u)(1-u^h)^{-s/h}, \qquad \xi(u) = -\int^u \frac{t^{a-1}}{(1-t^h)^{(a+b)/h}}\, dt,$$

so that a first integral is

$$z\delta(u)^s + I(u) = C_2. \tag{67}$$

The general solution to the problem is obtained by coupling the two first integrals (66) and (67), namely

$$\Phi(H(z,u)\delta(u)^{-t_0}, z\delta(u)^s + I(u)) = 0,$$

for an arbitrary bivariate function Φ. Solving for H puts the solution in the form

$$H(z,u) = \delta(u)^{t_0}\psi\left(z\delta(u)^s + I(u)\right), \tag{68}$$

for an arbitrary univariate function ψ. The initial condition $H(0,u) = u^{a_0}$ finally identifies ψ as defined implicitly through inversion of $I(u)$, $\psi(I(u)) = u^{a_0}/\delta(u)^{t_0}$.

We observe next that $\psi(z)$ is analytic at 0. Indeed the tenability conditions of (57) imply that a must divide a_0 and a must divide $b+s$, hence a divides $h = a+b+s$. In particular, the general form of the parameterization of ψ near 0 is $\psi(u^a) \asymp u^{a_0}$, that is $\psi(z) \asymp u^{a_0/a}$, which is compatible with analyticity. In fact, the expansions involved are of the form

$$\psi\left(u^a\left(\sum_j \lambda_{j\geq 0} u^{jh}\right)\right) = u^{a_0}\left(\sum_{j\geq 0} \mu_j u^{jh}\right),$$

for some real coefficients λ_j, μ_j and u ranging in a small enough complex neighbourhood of 0. Examination of the exponents involved in the inversion shows that $\psi(z)$ can be expanded as a power series in z, and analyticity of ψ at 0 results.

The formal derivation of the solution to the PDE was performed regardless of analytic details. Once found, the proposed solution (68), which, by previous considerations, appears to be analytic for (z,u) in a neighbourhood of $(0,0)$, is easily checked to be valid by back substitution into the original PDE. □

The operator approach leading to the main PDE (65) is of course merely a convenient way to carry out calculations and it could well be dispensed with. Its merit however is to establish transparent connections between the urn model and the PDE that expresses it—for instance it applies equally well to balanced models involving more than two types of balls. Perhaps more importantly, it yields naturally the "right" kind of normalization which tends to make equations simpler.

Sensitivity to initial conditions. When the initial state of the urn is changed, the functions involved still live in the same general class. Indeed, the ψ function corresponding to an initial urn of type (a_0, b_0) factorizes as

$$\psi(z) = \psi_I(z)^{a_0/a} \cdot \psi_{II}(z)^{b_0/b}, \tag{69}$$

where ψ_I, ψ_{II} are determined implicitly by

$$\psi_I(I(u)) = \left(\frac{u}{\delta(u)}\right)^a, \qquad \psi_{II}(I(u)) = \left(\frac{1}{\delta(u)}\right)^b, \tag{70}$$

corresponding to an urn initialized with $t_0 = a_0 = a$ and $t_0 = b_0 = b$, respectively. The analytic treatment given below extends to both functions ψ_I, ψ_{II}, and it is seen that the main determinant of the category of special functions encountered is the index h of the Fermat curve and the integral $I(u)$. For instance, if the $\mathcal{T}_{2,3}$ urn is initialized with $a_0 = 2r$ black balls and $b_0 = 3s$ white balls, the ψ function is of the form $\wp^r \wp'^s$, where $\wp$, the Weierstrass function is as of Definition 1. Equations (69) and (70) thus give us flexibility for the choice of the initial conditions, as is done in the various examples of Section 5 below.

In the case where a and b are each at least -1, balls have a "descendance" and the evolution of descendants are combinatorially independent. Accordingly, the factorization (70) can be viewed as expressing the fact that the histories of all the initial balls can be freely shuffled. (It is known that shuffle products correspond to products of exponential generating functions.) A parallel decomposition underlies the probabilistic reduction of this class of urn models to multitype branching processes in [3], at least in the case where no diagonal entry is below -1, so that the disappearances of balls are not coupled.

4.2. **Analytic aspects.** The earlier discussion of the 2–3 urn, as regards the basic conformal mapping argument and the dominant singularities, can be copied almost verbatim. For notational simplicity, we shall usually adopt the initial conditions $a_0 = t_0 = a$, that is, the urn is initialized with exactly a balls of type I: by Equations (69) and (70), no essential loss of generality is implied by such a choice.

We have already defined the quantity $h := a + b + s$. The function $v = \delta(u)$ corresponds to the Fermat curve,

$$u^h + v^h = 1,$$

which has topological genus $g = (h-1)(h-2)/2$. Following a classical terminology, the integral $I(u) \equiv \int u^{a-1} v^{-a-b}$ is an Abelian integral over this curve. The diagram

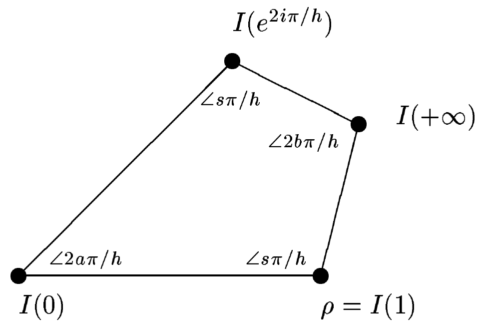

FIGURE 10. The elementary kite.

that summarizes the definition of ψ is now

$$\begin{array}{ccc} & u & \\ \swarrow & & \searrow \\ z = I(u) & \xrightarrow{\psi} & \psi(z) = J(u), \end{array} \qquad J(u) \equiv \frac{u^a}{\delta(u)^a}.$$

The major characteristics of an urn model are determined by the nature of the map $u \mapsto I(u)$ in the complex plane.

As observed in the proof of Theorem 3, the function ψ is analytic at 0 and it satisfies $\psi(z) \asymp z^{a_0/a}$ there. Also, the nature of the parameterization near 0, where $I(u) \asymp u^a$ implies that $I(u)$ effects an a-fold covering a neighbourhood of the origin and that $\psi(z)$ is of the form

$$\psi(z) = z^{a_0/a} \widehat{\psi}(z^{h/a}), \tag{71}$$

for some $\widehat{\psi}$ analytic at the origin. In other words, in order to define ψ parametrically by means of u it is enough to let u range in a sector $\mathcal{H}$ of angle $2\pi/(h/a)$ at the origin, and from now on, we shall do so. As already noted, the tenability conditions precisely imply that a divides h. The sector $\mathcal{H}$ is the analogue of the half-plane of (39).

Consider first the complex plane with h rays emanating from 0 and having directions given by all the hth roots of unity. The sector S_j is defined parametrically as

$$S_j := \left\{ z, \quad z = Re^{i\theta},\ 0 < R < \infty,\ \frac{2j\pi}{h} < \theta < \frac{2(j+1)\pi}{h} \right\}.$$

The image of S_0 by $I(u)$ is a quadrilateral, with vertices at the points

$$0, \quad I(1), \quad I(+\infty), \quad I(e^{2i\pi/h}).$$

We call it the *elementary kite* (Figure 10). One has

$$I(1) = \int_0^1 \frac{t^{a-1}}{(1-t^h)^{(a+b)/h}} = \frac{1}{h} B(\frac{a}{h}, \frac{s}{h}) = \frac{1}{h} \frac{\Gamma(\frac{a}{h})\Gamma(\frac{s}{h})}{\Gamma(\frac{a+s}{h})}.$$

We henceforth denote this quantity by ρ. The integral along the positive half-ray (going *over* the point 1 in the upper half-plane) is similarly expressed in terms of a Beta integral via $I(+\infty) - I(1)$.

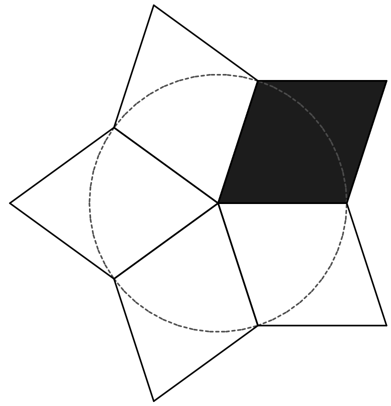

FIGURE 11. The fundamental polygon associated with the urn $(-1,4,4,-1)$.

The mapping properties at the four vertices of the kite are easily determined: (i) at 0, $I(u)$ multiplies angles by a, so that the angle of the kite at 0 is $\frac{2\pi a}{h}$; (ii) at 1, $I(u)$ multiplies angles by $\frac{s}{h}$, so that the angle of the kite at vertex $I(u)$ is $\frac{\pi s}{h}$ (and similarly for vertex $I(e^{2i\pi/h})$); (iii) at infinity, $I(u)$ multiplies angles by b, so that the angle at $I(+\infty)$ is $\frac{2\pi b}{h}$. Thus, the geometry of the elementary kite is entirely determined (Figure 10).

Let $\zeta := e^{2i\pi/h}$. The image of sector S_j is simply obtained as the image of S_0 by $I(u\zeta^j)$, which, by a linear change of variables (since $I(u\zeta^j) = \zeta^{-ja}I(u)$) is the image of the elementary kite under a rotation of angle $-\frac{2ja\pi}{h}$; see Figure 11 for a particular instance. Because of (71) and the accompanying remarks, it is sufficient to consider $0 \le j < \frac{h}{a}$.

Definition 2. *The* fundamental polygon *of an urn model is the (closure of) the union of h/a regularly rotated versions of the elementary kite about the origin.*

We state:

Theorem 4. *Consider a balanced 2×2 urn with subtraction as in the previous theorem and let it be initialized with $a_0 = a$, $b_0 = 0$. The corresponding function ψ is analytic for z in the fundamental polygon of Definition 2. Furthermore it is analytic in $|z| < \rho$, where*

$$\rho = I(1) = \int_0^1 \frac{t^{a-1}}{(1-t^h)^{(a+b)/h}} = \frac{1}{h}B(\frac{a}{h},\frac{s}{h}) = \frac{1}{h}\frac{\Gamma(\frac{a}{h})\Gamma(\frac{s}{h})}{\Gamma(\frac{a+s}{h})}.$$

On $|z| = \rho$, the function ψ is singular at ρ and at the points $\rho\omega^j$ where $\omega = \exp(2i\pi\frac{a}{h})$ is an (h/a)th root of unity, regular at the other points. Its singular expansion az $z \to \rho$ is of the form

$$\psi(z) = s^{-t_0/s}(\rho - z)^{-t_0/s}\mathcal{A}\left((\rho-z)^{h/s}\right), \tag{72}$$

with $\mathcal{A}$ analytic at 0, $\mathcal{A}(0) = 1$, $\mathcal{A}'(0) \neq 0$. (Principal determinations as $z \to \rho^-$ are assumed.)

At the points $z = \rho\omega^j$, the singular expansion is determined from the expansion at $z = \rho$ by the fact that $\psi(z)z^{-a_0/a}$ is invariant under the mapping $z \mapsto \omega z$.

The expansion (72) gives $\psi(z)$ as the product of a main singular part of the form $(\rho - z)^{-t_0/s}$ multiplied by a Puiseux series, that is, a series in fractional powers of $(\rho - z)$. We shall occasionally refer to the quantity h/s as the *Puiseux exponent* of ψ. It plays a special rôle in the discussion of elliptic urns in Subsection 5.3. As should be clear from the proof, similar statements can cover all initial conditions.

Proof. We examine the behaviour of ψ near $\rho = I(1)$, corresponding to u in the vicinity of u (say, $u \to 1^-$, to fix ideas). The expansion can be constructed by means of a local uniformizing parameter, here, $1 - u = \tau^h$. Write

$$\delta(y) = \Delta(y)(1-y)^{1/h},$$

so that $\Delta(1-y)$ is analytic at $y = 0$. By the change of variables $u \mapsto 1 - \tau^h$, one finds

$$\begin{aligned} I(1) - I(u) &= h\int_0^\tau (1-y^h)^{a-1}\Delta(1-y^h)^{-a-b}y^{s-1}\,dy, \\ &= \frac{1}{s}\left(h^{1/h}\tau\right)^s\left(1 + \frac{(h(b-a+2)-a-b)s}{2h(s+1)}\tau^h + \cdots\right) \\ J(u) &= \frac{(1-\tau^h)^{a_0}}{\Delta(1-\tau^h)^{t_0}} \\ &= \left(h^{1/h}\tau\right)^{-t_0}\left(1 - \frac{h(2a_0-t_0)+t_0}{2h}\tau^h + \cdots\right) \end{aligned}$$

where now $\tau \to 0$ corresponds to $u \to 1$ (the series expansions proceed by powers of τ^h). Thus the parameterization is of the form

$$\rho - z = \frac{1}{s}\left(h^{1/h}\tau\right)^s U(\tau^h), \qquad \psi(z) = \left(h^{1/h}\tau\right)^{-t_0} V(\tau^h),$$

where U, V are analytic at 0 and $U(0) = V(0) = 1$. This shows that there exists a full expansion of the type (72), with $\mathcal{A}$ analytic at 0. In other words, the point ρ is a singularity of ψ that is a branch point with dominant singular exponent equal $-t_0/h$. Since the expansion (72) holds with an analytic $\mathcal{A}$, the function ψ is necessarily analytic in a small neighbourhood of ρ slit along the ray $(\rho, +\infty)$.

By symmetry, an expansion of a nature similar to (72) also holds at the conjugate points $\rho\omega^j$ where $\omega = e^{2i\pi/h}$ is an hth root of unity. Since $\psi(z)$ has nonnegative coefficients, it satisfies Pringsheim's theorem and is thus analytic in $|z| < \rho$. By the triangular inequality, we have $|I(ue^{i\theta})| \le I(u)$ for $u \in (0,1)$ and $\theta \in (-\pi, \pi)$. Since the nonzero terms composing the Taylor expansion of I at the origin are of the form $u^a + jh$, the inequality $|I(ue^{i\theta})| < I(u)$ is strict as soon as θ is not a multiple of $\frac{2\pi}{h}$ and $I(u)$ is invertible. From there, it results that ψ is analytic on $|z| = \rho$ except for the regularly spaced singularities quoted in the statement.

When u-paths starting from 0 on the Riemann surface of the Fermat curve are considered, the effect by $I(u)$ is to rotate the fundamental polygon along its external vertices. In the case of the $\mathcal{T}_{2,3}$ urn, we saw that this results in a regular tiling of the plane by equilateral triangles (the particular form of the fundamental polygon) and accordingly to explicit solutions by an elliptic function. In the general case, no tiling is likely to be generated (see Section 5.3 for a discussion of special cases still conducive to tilings). Rotations around the dominant singularities are however sufficient to guarantee analytic continuation of the ψ function in regions extending beyond its disc of convergence at 0, just like in the proof of Proposition 1. □

For instance, the urn $X = \begin{pmatrix} -1 & 4 \\ 4 & -1 \end{pmatrix}$ gives rise to the fundamental polygon displayed in Figure 11. One has $s = 3$, $h = 5$, and $\delta(u) = (1 - u^5)^{1/5}$, so that the fundamental polygon is a star with five branches. At the origin, we find

$$I(u) = u + \frac{1}{15}u^6 + \frac{7}{275}u^{11} + \cdots, \qquad \psi(z) = z + \frac{2}{15}z^6 + \frac{34}{825}z^{11} + \cdots.$$

There is an algebraic branch point at ρ where $\psi(\rho - x) \asymp (\rho - x)^{-1/3}$ and at the conjugate points $\rho\omega^j$ where $\omega^5 = 1$. The nature of the branch point ρ, where

$$\psi(z) \sim (3Z)^{-1/3}\left(1 - \frac{9}{40}(3Z)^{5/3} - \frac{1143}{10400}(3Z)^{10/3} + \cdots\right), \qquad Z := (\rho - z),$$

the Puiseux exponent (associated with $\mathcal{A}$) being $h/s = \frac{5}{3}$, precludes the possibility of a solution purely in terms of elliptic functions, since neither ψ nor any of its powers is a meromorphic function.

4.3. **Probabilistic consequences.** Singularity analysis [20] makes it possible to extract very precise information on coefficients of a generating function once the function is recognized to have isolated singularities on the boundary of its disc of convergence; see [42] for an insightful perspective. Under such conditions, assuming first unicity of the dominant singularity σ, an asymptotic estimate for a function F of the form

$$F(z) \underset{z\to\sigma}{\sim} c(1 - z/\sigma)^{-\alpha}$$

valid in a complex region beyond σ (a sector centred at σ of opening angle larger than π and including the disc of convergence) entails a matching estimate for the function's coefficients:

$$[z^n]F(z) \underset{n\to\infty}{\sim} c\sigma^{-n}\frac{n^\alpha - 1}{\Gamma(\alpha)}.$$

Full asymptotic expansions can be transferred from functions to coefficients in a similar way. Also, in the case of several dominant singularities, contributions to coefficients are to be composed additively. This technology[9] based on Hankel contours is of the complex Tauberian type. It applies immediately to the function $\psi(z)$ itself. Since $\psi(z) = H(z, 0)$, it provides for sharp estimates of the probabilities that all balls are of the same colour at epoch n, which corresponds to extreme large deviations.

Corollary 6. *For any balanced 2×2 urn with subtraction (i.e., negative diagonal entries), the probability that all balls at time n are of the same colour and of the first type is:*

$$\frac{h}{a}(s\rho)^{-n-t_0/s}\left(1 + O\left(\frac{1}{n^{h/s}}\right)\right).$$

for $n \equiv \frac{a_0}{a} (\mathrm{mod}\, \frac{h}{a})$.

Proof. The singularity at $z = \rho$ of $H(z, 0) = \psi(z)$ contributes to $[z^n]\psi(z)$ a term

$$(s\rho)^{-t_0/s}\rho^{-n}\frac{n^{t_0/s-1}}{\Gamma(t_0/s)}\left(1 + O\left(\frac{1}{n^{h/s}}\right)\right).$$

[9]Singularity analysis could be bypassed in the case of the $\mathcal{T}_{2,3}$ urn since then only polar singularities are present.

We have the periodicity expressed by (71). Thus, for n in a suitable congruence class, there are h/a similarly behaving singularities to be combined. The total number of histories of length n, which is, from (59) and (60), asymptotic to

$$n!s^n \frac{n^{t_0/s-1}}{\Gamma(t_0/s)}.$$

The result follows after normalization by the latter quantity. □

This corollary is seen to extend Corollary 4 for the 2–3 urn, although presently the error terms are usually not exponentially small, given the nonpolar singularity involved.

Next, we choose a small complex neighbourhood of $u = 1$, say $|u - 1| < \frac{1}{10}$, and keep u in this neighbourhood. We can rewrite the BGF $H(z, u)$ as

$$H(z,u) = \delta(u)^{t_0}\psi\Big(\rho - \delta(u)^s\,(z - K(u))\Big), \tag{73}$$

where

$$K(u) := \frac{1}{\delta(u)^s}\int_u^1 \frac{t^{a-1}}{\delta(u)^{a+b}}\,dt, \tag{74}$$

and $K(u)$ has a removable singularity at 1 with $K(1) = 1/s$. Treating u as a parameter, we find that, as a function of z, the quantity $H(z, u)$ has a singularity at $z = K(u)$ that gets smoothly displaced when u varies. Because of the nature of the singularity of ψ at ρ, the singular exponent remains equal to the constant $-t_0/s$. Thus, for some function $L(u)$ that is analytic at $u = 1$, one has

$$[z^n]H(z,u) = L(u)K(u)^{-n}n^{t_0/s-1}\left(1 + O\left(\frac{1}{n^{h/s}}\right)\right).$$

This has the shape of a *bona fide* quasipowers approximation for the probability generating function:

$$p_n(u) = \frac{[z^n]H(z,u)}{[z^n]H(z,1)} = \frac{L(u)}{L(1)}\left(\frac{K(u)}{K(1)}\right)^{-n}\left(1 + O\left(\frac{1}{n^{h/s}}\right)\right).$$

Observe that the error term is always $O(n^{-1/2})$. Thus, the Quasi-Powers Theorem (Lemma 3 above) applies.

Corollary 7. *For any balanced 2×2 urn with subtraction, the random variable X_n representing the number of balls of the first type at time n is asymptotically Gaussian with speed of convergence to the limit $O(n^{-1/2})$, as expressed by* (48).

This corollary is entirely analogous to Corollary 2 for the $\mathcal{T}_{2,3}$ urn. The fact that the limit distribution is Gaussian has been first observed by Bagchi and Pal in [4]. These authors applied the moment method and determined the main asymptotic orders of moments of the centred variable $X_n - \mathbb{E}(X_n)$. Their method does not however appear to give access to the speed of convergence as expressed above. This speed is itself neatly implied by the functional limit theorem of Gouet [23].

In general, the moments are computable systematically from the exact expressions of Theorem 3 by successive differentiation with respect to u upon setting $u = 1$ and making use of the singularities of ψ and its derivatives as expressed by Theorem 4. However, error terms due to the fact that the singularity is in general no longer polar preclude the possibility of as simple structure of moments as expressed in Corollary 3 for the $\mathcal{T}_{2,3}$ urn. We have at least the fact that all moments are all expressible in finite computable closed form.

Corollary 8. *For any balanced 2×2 urn with subtraction and any $r \geq 0$, the rth factorial moment of the distribution of X_n is of hypergeometric type: it is a linear combination of terms of the form*

$$\frac{\binom{n + t_0/s + \ell - kh/s - 1}{n}}{\binom{n + t_0/s - 1}{n}}, \qquad 0 \leq k, \ell \leq r.$$

For instance, such an explicit form is given by Kotz–Mahmoud–Robert [33], but only for the first moment, in the case of an urn model discussed in Subsection 5.5. Bagchi and Pal obtained similar looking expressions for a wide class of urns, but in the case of the first two moments only. The result is seen to be consistent with the forms already obtained for the $\mathcal{T}_{2,3}$ urn (Corollary 3) where $t_0/s = 2$ and $h/s = 6$. An explicit calculation is detailed in Equation (85) below.

Proof. The quantity

$$h_r(z) := \frac{1}{r!} \left(\partial_u^r H(z,u)\right)_{u=1} = [(u-1)^r]\, H(z,u)$$

is a generating function of the rth factorial moment of X_n in the sense that

$$\mathbb{E}(X_n^{\underline{r}}) = \frac{[z^n] h_r(z)}{[z^n] H(z,1)}$$

where the notation $X^{\underline{r}}$ is that of (49). In order to gain access to such moments, we make use of the singular expansion of ψ near ρ, as given in (72) of Theorem 4. We have

$$H(z,u) = \delta(u)^{t_0} \psi\Big(\rho - \delta(u)^s (K(u) - z)\Big),$$

which repeats (73), where $K(u)$ is determined by (74). The singular expansion (72) then provides

$$H(z,u) = s^{-t_0/s} (K(u) - z)^{-t_0/s} \mathcal{A}\left((1 - u^h)(K(u) - z)^{h/s}\right), \tag{75}$$

which is our starting point. This expansion is analytically valid when (say) $|z| < \frac{1}{2}$ provided u stays in a small enough neighbourhood of 1, where one may additionally postulate that $|K(u) - 1| < \frac{1}{10}$. Write $\mathcal{A}(w) = \sum_{k \geq 0} a_k w^k$. One then has

$$H(z,u) = \sum_{k \geq 0} a_k (1 - u^h)^k (K(u) - z)^{kh/s - t_0/s}. \tag{76}$$

Clearly, for the rth moment, it suffices to consider the sum in (76) with the index k restricted to values in the interval $[0, r]$, so that

$$h_r(z) = \frac{1}{r!} \sum_{k=0}^{r} a_k \left(\partial_u^r \left((1 - u^h)^k \left(K(u) - z\right)^{-t_0/s + kh/s}\right)\right)_{u=1}. \tag{77}$$

The function $K(u)$ is analytic at $u = 1$. Consequently, the quantity $(K(u) - z)^{-1}$ admits derivatives at $u = 1$ that are of the form

$$\frac{s}{1-sz}, \quad -\frac{s^2K'(1)}{(1-sz)^2}, \quad \frac{2s^3K'(1)^2}{(1-sz)^3} - \frac{s^2K''(1)}{(1-sz)^2},$$

and so on, with similar looking formulæ holding for powers. Thus $h_r(z)$ is invariably an algebraic function of a very special form, namely a finite linear combination of terms of the type

$$(1-sz)^{-t_0/s+kh/s-\ell}, \qquad 0 \le k, \ell \le r.$$

The statement then follows by coefficient extraction. □

These results are seen to be consistent with the estimates of Bagchi and Pal [4, pp. 395–397]. In particular, one gets mechanically:

$$\mathbb{E}(X_n) \sim \frac{s+b}{s+h} sn, \qquad \mathbb{V}(X_n) \sim \frac{sh^2(s+a)(s+b)}{(s+h)^2(s+2h)} n.$$

Finally, the large deviation rate is also fully characterized:

Corollary 9. *Consider any balanced* 2×2 *urn with subtraction. Let* ξ *be any number of the open interval* $(0, s\frac{s+b}{s+h})$*. One has*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(X_n \le \xi \cdot n) = -\mathcal{R}(\xi), \tag{78}$$

where the rate function $\mathcal{R}$ *is determined from* $K(u)$ *defined in* (74) *by*

$$\mathcal{R}(\xi) = \max_{\lambda \in (0, s\frac{s+b}{s+h})} \log\left(s\lambda_0^\alpha K(\lambda_0)\right). \tag{79}$$

Equivalently:

$$\mathcal{R}(\xi) = \log\left(s\lambda_0^\alpha K(\lambda_0)\right) \qquad \textit{where} \quad \frac{\lambda_0 K'(\lambda_0)}{K(\lambda_0)} + \xi = 0, \quad \lambda_0 \in (0, s\frac{s+b}{s+h}). \tag{80}$$

Proof. The proof is entirely analogous to that of Corollary 5, taking into account the fact that the PGF $p_n(u)$ is of the form

$$p_n(u) \approx \left(\frac{1}{sK(u)}\right)^n,$$

up to subexponential factors. □

The dual regime of large deviations on the right tail of the distribution is determined upon exchanging the rôles of quantities a and b.

5. Some exactly solvable models

This section does principally three things:

— It verifies that some very simple models are amenable to our general treatment. This includes drawing with or without replacement, and perhaps more surprisingly the usual coupon collector problem, of which a formulation explicitly in terms of urns is already to be found in Laplace's treatise [35, Vol. II] first published in 1812.

— It pushes the analysis of balanced urn models with subtraction (i.e., with negative entries on the diagonal), whenever it is possible to go further. First in Section 5.2, we rederive the analytic formulæ known for Ehrenfest's model in our framework—this is the unique model with balance 0. Next, we show in Section 5.3 that all the five models with balance equal to 1 or 2 as well as a sporadic urn of balance 3 admit of explicit elliptic function solutions, so that results entirely parallel to those of Section 3 hold for these models.

— It demonstrates that the approach based on bivariate generating functions, partial differential equations, and singularities extends to some models that do not involve subtraction. This includes the original Pólya-Eggenberger model and a subclass of Bernard Friedman's urn [22]; see Section 5.4. In particular, we exhibit the occurrence of *stable laws* inside certain urn models of a type considered by Kotz, Mahmoud, and Robert; see Section 5.5.

In all cases, the starting point is the PDE (65) which remains valid whatever values of the parameters a, b are, as the operator approach makes no assumptions on signs of data. Theorems 3 and 4 need to be mildly adapted in cases not involving subtraction of balls, as there may arise singularities at $u = 0$.

5.1. **Some simple urn models.** We briefly verify that the general treatment of this paper is applicable to some very simple urn models. Consider the three urns,

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}. \tag{81}$$

The first urn corresponds in the classical terminology to "drawing with replacement"; it has $h = 0$ and it can be solved by $\delta(u) = 1$, $I(u) = \log u$, and $\psi(z) = e^{(a_0+b_0)z}$, so that the bivariate generating function is

$$H(z,u) = u^{a_0} e^{(a_0+b_0)z},$$

which is easily checked to generate sequences of Bernoulli trials.

The second urn from (81) corresponds to "drawing without replacement". We have $a = b = 1$, $s = -1$, and $h = 1$. This gives $\delta(u) = 1 - u$, $I(u) = u/(1-u)$, $\psi(z) = z^{a_0}(1+z)^{b_0}$, which results in

$$H(z,u) = (z+u)^{a_0}(z+1)^{b_0},$$

whose combinatorial interpretation is obvious.

The third urn in (81) translates the problem which is in Laplace's words (see p. 311 of [35, Vol. II]): "Considérons une urne A renfermant un très grand nombre $[\cdots]$ de boules blanches et noires, et supposons qu'à chaque tirage on tire une boule de l'urne et qu'on la remplace par une boule noire." (Laplace then proceeds with a continuous model of this game.) This is precisely a way of describing the classical coupon collector problem. We have $s = 0$ and $h = 1$, so that $\delta(u) = 1 - u$ and ψ satisfies, with $a_0 = m$ the principal parameter:

$$\psi\left(\log\frac{1}{1-u}\right) = \frac{u^m}{1-u^m}, \quad \text{implying} \quad \psi(z) = (e^z - 1)^m.$$

The bivariate generating function is then

$$H(z,u) = (e^z - 1 + u)^m,$$

which gives back the usual solution of this model in terms of Stirling numbers.

5.2. **The Ehrenfest model.** This one corresponds to the urn

$$E_0 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

In other words, a ball changes colour upon being drawn. This model was proposed as a discrete model of heat transfer and related diffusion processes by Paul and Tatiana Ehrenfest in 1907 and it played a rôle in the discussion of an apparent contradiction between irreversibility and ergodic properties in thermodynamics; see Kac's account [31] for a vivid discussion. (The urn models balls randomly switching between two urns.) There are some interesting combinatorial aspects related to continued fractions, and in particular to

$$\int_0^\infty e^{-zu} \cosh^m u \, du = \cfrac{1}{1 - \cfrac{1 \cdot m \, z^2}{1 - \cfrac{2 \cdot (m-1) \, z^2}{1 - \cfrac{3 \cdot (m-2) \, z^2}{\ddots}}}},$$

which is due to Stieltjes; see [13, 19] and especially the paper of Goulden and Jackson [25].

The balance is $s = 0$, while $a = 1$ and $b = 1$. We fix here an integer parameter m. Start with $a_0 = t_0 = m$, for simplicity. One has $\delta(u) = (1 - u^2)^{1/2}$. The basic integral $I(u)$ is thus simply a hyperbolic integral (of genus 0 but with the integrand now having poles):

$$I(u) = \int_0^u \frac{dt}{1 - t^2} = \frac{1}{2} \log \frac{1+u}{1-u} = \operatorname{atanh}(u).$$

The function ψ is then defined implicitly by

$$\psi(\operatorname{atanh}(u)) = \left(\frac{u}{\sqrt{1-u^2}} \right)^m,$$

which is equivalent to

$$\psi(w) = \sinh^m w.$$

Then, the BGF solution to the model is

$$H(z,u) = (1-u^2)^{m/2} \sinh^m(z + \operatorname{atanh} u),$$

which simplifies to

$$H(z,u) = (\sinh z + u \cosh z)^m.$$

This last formula is in agreement with what is classically known and is susceptible to a direct combinatorial interpretation [13, 19, 25]. (The paper by Edelman and Kostlan [16] makes for pleasant collateral reading.) The functions obtained are here elementary since only integration on an algebraic curve of genus 0 (a conic) is needed.

5.3. **Urns with subtraction: the elliptic cases.** In this subsection, we list all the cases of urns with subtraction which, like the $\mathcal{T}_{2,3}$ model, lead to elliptic functions. We shall say that an urn is *elliptic* if, for some choice of initial conditions (a_0, b_0), the fundamental function ψ is a power of an elliptic function,

$$\psi(z) = \Pi(z; \Lambda)^{p/q}, \tag{82}$$

where Π is meromorphic and admits a period lattice Λ. The power will depend on the initial conditions, but by taking the initial configuration determined by a_0 and b_0 sufficiently large, one can always render the exponent integral: see the remarks on "sensitivity to initial conditions" as well as Equations (69) and (70). Obviously, for urns with subtraction, one need only consider models that are *arithmetically irreducible*. By this is meant that the matrix $M = \left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right)$ defining the urn has coprime entries: $\gcd(\alpha, \beta, \gamma, \delta) = 1$.

The key characters in this section are the following six urns:

$$A = \begin{pmatrix} -2 & 3 \\ 4 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 2 \\ 3 & -2 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix},$$

$$D = \begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} -1 & 3 \\ 5 & -3 \end{pmatrix}, \quad F = \begin{pmatrix} -1 & 4 \\ 5 & -2 \end{pmatrix}.$$

The urn A is of course the $\mathcal{T}_{2,3}$ model. As is easily checked, any urn of balance $s = 1$ is necessarily of type A, B, or C and any arithmetically irreducible urn of balance 2 can only be of type D or E. Thus, the first five cases exhaust all possible types of urns with subtraction having balance $s = 1, 2$. The urn F is one of the four possible irreducible urns having balance $s = 3$. We state:

Corollary 10. *All balanced 2×2 uns with balance $s = 1$ (cases A, B, C) are elliptic. All urns with balance $s = 2$ (cases D, E) are also elliptic. There exists one "sporadic" urn (case F) of balance $s = 3$ that is elliptic. These six models represent the only irreducible cases of urns with subtraction that are elliptic.*

Proof. For 2–3 trees, the matrix $\left(\begin{smallmatrix}-2 & 3 \\ 4 & -3\end{smallmatrix}\right)$ corresponds to $a = 2,\ b = 3,\ s = 1,\ h = 6$. As seen in Section 2, it provides a regular tiling of the plane by equilateral triangles. Its ψ function has, by virtue of Theorems 2 and 4 a *double pole* at ρ and all other points of the lattice. It is thus a Weierstraß $\wp$ function. (We are here operating at the level of derivatives and $\psi = \Psi'$ is the derivative of the function Ψ studied earlier in Section 2, which is itself a ζ function.)

Next, we turn to the other urns of balance 1. The corresponding mapping properties are represented in Figure 12. The model C leads to a lattice much similar to the 2–3 tree case. The model B has a basic kite which is a right triangle with vertices $0, \rho, i\rho$, so that the fundamental polygon is the square with vertices $\rho, i\rho, -\rho, -i\rho$. The fundamental square tiles the plane. Theorem 4 shows that in each case the ψ function has a pole since in all three cases the principal exponent at the singularity, t_0/s, as well as the Puiseux exponent h/s in Equation (72) are integers, given that $s = 1$. Double periodicity then results from arguments entirely similar to those developed in the proof of Theorem 2.

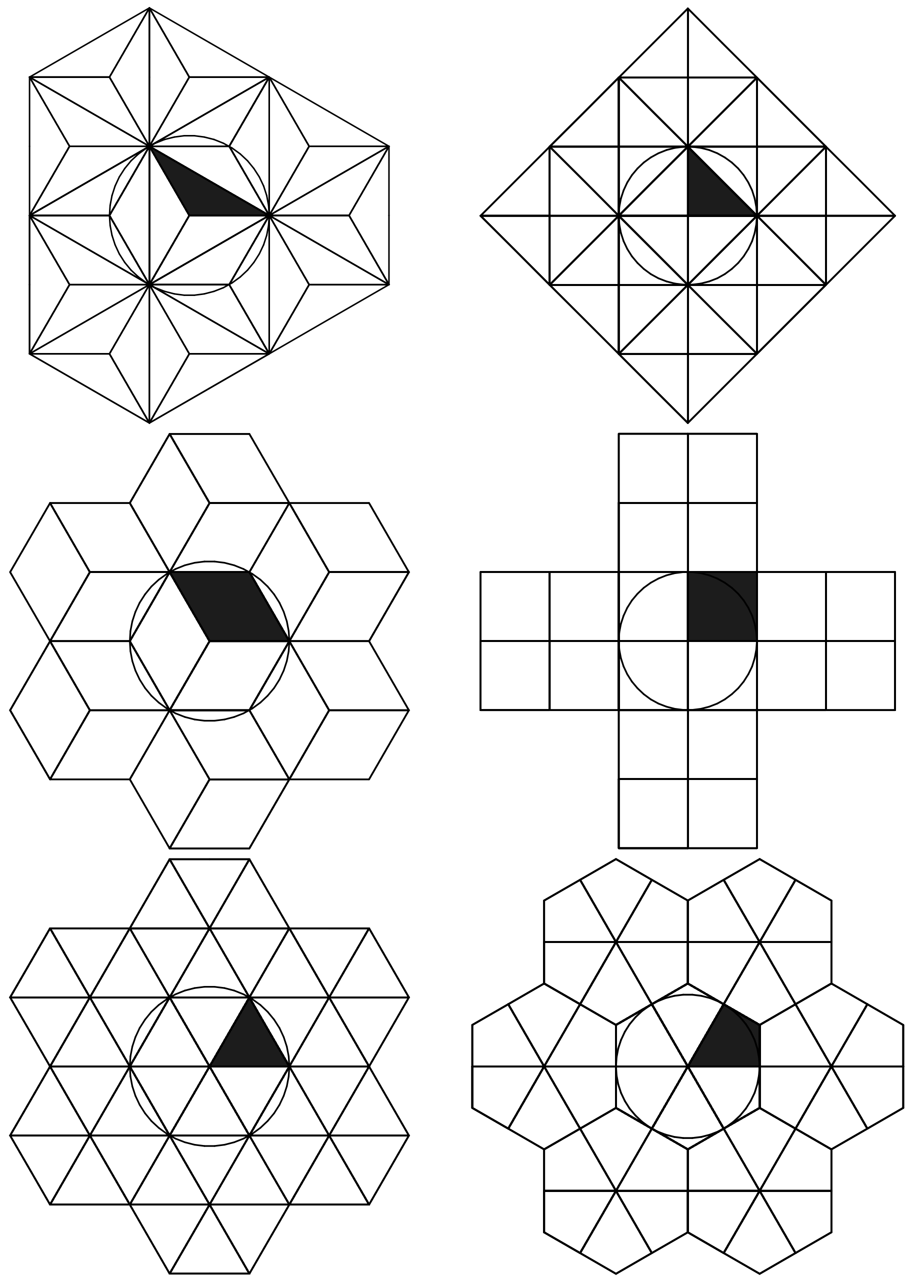

FIGURE 12. The six elliptic cases in order A, B, C, D, E, F: The diagrams formed by the fundamental polygon together with its rotated images. (The elementary kite is darkened.)

A similar discussion applies to the two urns of balance 2, namely D and E. In the case of D, it is now the function $I(u)$ that is directly a lemniscatic integral:

$$I(u) = \int_0^u \frac{dt}{\sqrt{1-t^4}}.$$

Finally, amongst the four urns of balance 3, namely

$$\begin{pmatrix} -1 & 4 \\ 4 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 4 \\ 5 & -2 \end{pmatrix}, \quad \begin{pmatrix} -1 & 4 \\ 7 & -4 \end{pmatrix}, \quad \begin{pmatrix} -2 & 5 \\ 8 & -5 \end{pmatrix},$$

one found to be elliptic has $s = 3$, $h = 6$, so that $h/s = 2$ is integral:

$$F = \begin{pmatrix} -1 & 4 \\ 5 & -2 \end{pmatrix}.$$

By the usual reasoning, the function ψ is elliptic when one starts with $a_0 = 3$ and $b_0 = 0$.

Note a necessary condition for an urn to be elliptic: in addition to the tenability conditions, s should divide h by virtue of Theorem 4 and Equation (72), that is, the Puiseux exponent h/s should be integral, given our definition (82) of an elliptic urn. (Otherwise, all powers of ψ inherently have branch points and hence cannot be meromorphic.) The other three cases of balance $s = 3$ (these include the "pentagonal" urn of Figure 11) correspond to a fractional value of the Puiseux exponent h/s in Equation (72) and therefore cannot be reduced to elliptic functions.

There now remains to prove that all elliptic urns have indeed been found. The condition that the Puiseux exponent h/s is integral, arithmetic irreducibility, and the tenability conditions taken together imply the existence of triples (x, y, z) of integers, representing up to possible permutation the values (a, b, s), such that

$$\gcd(x, y, z) = 1, \quad x \mid y + z, \quad y \mid z + x, \quad z \mid x + y. \tag{83}$$

Simple arithmetics[10] shows that the only values for which the system admits a solution are permutations of the basic types

$$(1, 1, 1), \quad (1, 1, 2), \quad (1, 2, 3).$$

In particular, one must have $s \leq 3$. This verifies that elliptic cases have indeed all been found. □

It is pleasant to note that the elliptic urns correspond to the *crystallographic groups* of the Euclidean plane, that is, groups of isometries acting discontinuously (in fact, the ones which admit a compact fundamental domain). As is well known, these groups themselves describe the possible regular tesselations of the plane by polygonal tiles and are in finite number (see, e.g., [7, 53]).

The stronger properties of the $\mathcal{T}_{2,3}$ model extend to all the six elliptic cases. In particular, like in Corollary 3, the moment of order r is of the form $P_r(n)$ for each n large enough where the P_r are polynomials that constitute a binomial family.

[10] Take $x \leq y \leq z$. One has $z \leq 2y$ since $z \leq x + y$ by the third divisibility condition in (83); then note the stronger property that x, y, z are pairwise coprime (proof: a contrario). Then set $z + x = qy$, where $q \leq 3$; the first divisibility condition then implies $x \mid (q+1)y$, and since x and y are coprime, $x \mid 4$ so that $x \in \{1, 2, 4\}$. This in turn implies $y \leq z \leq y + 4$ while z must divide $y + x$. Combining this with the second and third divisibility conditions of (83), we see that there are only finitely many possibilities which are then easily listed.

5.4. **Urns without subtraction: Two classical models.** This subsection is only meant to verify the applicability of the methods underlying Theorem 3 to the simplest models of urns without subtraction, that is, involving positive diagonal entries.

- We rederive easily elementary properties of the original Pólya-Eggenberger distribution.
- In the case of the dual "altruistic" model considered earlier by Friedman [22], we get explicit expressions related to the Eulerian numbers, which, as far as we know, are new.

Note that no tenability condition arises in this case as balls are never pulled out of the urn.

5.4.1. *The contagion model* $(a, 0, 0, a)$. This is precisely the original Pólya-Eggenberger model of 1923. Under this model, a balls of the type drawn are placed back into the urn, balls of the other colour remaining unaffected. Thus, drawing a ball of one colour reinforces the preponderance of that particular colour. The matrix is then

$$T = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

Although this involves no subtraction (the diagonal entries are all positive), the model is amenable to the operator method. The operator, which is $\Gamma = u^a\theta_u + v^a\theta_v$, and the homogeneity conditions lead to the partial differential equation

$$\left[(u^{a+1} - u)\partial_u + (az - 1)\partial_z + t_0 I\right] \circ H(z, u) = 0,$$

which is easily integrated like before using the identity

$$\int \frac{du}{u^{a+1} - u} = \log \frac{\delta(u)}{u}, \qquad \delta(u) := (1 - u^a)^{1/a}.$$

This gives

$$\psi(z) = z^{-b_0}(1 + z^a)^{-a_0/a}.$$

All computations done, one finds the exact form of the BGF of evolutions, namely

$$H(z, u) = \frac{u^{a_0}}{(1 - az)^{b_0/a}(1 - au^a z)^{a_0/a}}. \tag{84}$$

(Note that this formula is otherwise expected combinatorially: since balls of different colours don't interact, the evolution histories are shuffles of the histories of the individual balls, themselves deterministic quantities enumerated by factorial numbers.) An expansion then yields back the distribution of the urn at time n; see [30, Sec. 4.2] for related calculations. For instance, with $a = 1$ and starting from $a_0 = b_0 = 1$ at time 0, the PGF of the urn at time n is

$$\frac{u}{n+1}(1 + u + \cdots + u^n),$$

which represents the uniform distribution. In general, one finds

$$\begin{aligned} \mathbb{P}(X_n = a_0 + ja) &= \frac{[z^n u^j]\,(1 - z)^{-b_0/a}(1 - uz)^{-a_0/a}}{[z^n]\,(1 - z)^{-t_0/a}} \\ &= \binom{n + t_0/a - 1}{n}^{-1} \binom{j + a_0/a - 1}{j} \binom{n - j + b_0/a - 1}{n - j}. \end{aligned}$$

This implies in particular (differentiate the bivariate generating function of (84) with respect to u and set $u = 1$) the fact that the mean number of white balls at time n is a_0/t_0, a result that is attribute to Jordan (1956) in [30, p. 182].

5.4.2. *The "altruistic" model model* $(0, a, a, 0)$. This model is defined by the matrix

$$T = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}.$$

Smythe [47] evokes this model for $a = 1$ as a special case of "Bernard Friedman's urn" [22], and mentions its connections with stemma construction in philology [40] as well as with recursive trees [48]. The case of a general a, as above, is described by Friedman [22] as a possible model of a safety campaign: "Every time an accident occur, i.e. a white ball is drawn, the safety campaign is pushed harder. Whenever no accident occurs, i.e., a black ball is drawn, the safety campaign slackens and the probability of an accident increases."

The calculations are a variant of the previous ones, and one finds a similar PDE

$$\left[(u - u^{a+1})\partial_u + (au^a z - 1)\partial_z + u^a t_0 I\right] \circ H(z, u) = 0.$$

The solution is obtained by the usual devices and one obtains the formula:

$$H(z, u) = u^{a_0} e^{a_0 z(1-u^a)} \frac{(1 - u^a)^{t_0/a}}{(1 - ue^{az(1-u^a)})^{t_0/a}},$$

which seems to be new. The coefficients are thus generalized Eulerian numbers. In particular, the case $a = a_0 = t_0 = 1$ gives

$$\begin{aligned} H(z, u) &= \frac{u(1 - u)e^{z(1-u)}}{1 - ue^{z(1-u)}} \\ &= u + uz + u(u + 1)\frac{z^2}{2!} + u(1 + 4u + u^2)\frac{z^3}{3!} + \cdots, \end{aligned}$$

where one recognizes the standard Eulerian numbers [14].

In the latter case $a = a_0 = t_0 = 1$, there exists a direct combinatorial connection between this urn model and certain trees that explains the appearance of Eulerian numbers: let the balls be numbered $0, 1, 2, \ldots$, where each ball has its own time stamp. A ball is linked to the ball that "gave rise to it", that is the ball drawn that determined its colour. The trees one gets in this way are Cayley trees satisfying the "increasing property": labels increase along each branch from the root, itself labelled 0. (These are sometimes curiously called "recursive trees".) The number of such trees with n nodes is well-known to equal $(n - 1)!$ and the trees are in natural correspondence with permutations [8, 11, 39]. Then the number of white balls corresponds to the leaf nodes in the increasing Cayley tree, the statistics of which are known to be given by the Eulerian numbers [8]. (In fact Mahmoud and Smythe have obtained distributional results about increasing Cayley trees going the other way round and reducing tree statistics to urn statistics.)

5.5. **Urns without subtraction: A not so classical model.** We shall content ourselves to examine the case of the matrices

$$\begin{pmatrix} a + 1 & 0 \\ 1 & a \end{pmatrix},$$

which we call the KMR models as they are discussed by Kotz, Mahmoud, and Robert in [33]. Bagchi and Pal [4] write that these models "present some curious technical problems and appear to require a separate treatment". Indeed, the composition of the urn is *not* asymptotically Gaussian.

Following the general methodology, we find automatically

$$H(z,u) = u^{t_0}\left(1 - \frac{1-u^a}{(1-(a+1)u^{a+1}z)^{a/(a+1)}}\right)^{(a_0-t_0)/a}\left(1-(a+1)u^{a+1}z\right)^{-t_0/(a+1)}.$$

Thus the solution is a bivariate algebraic function.

For instance, for $a = 3$ and an initial configuration with one ball of the second type ($a_0 = 0, t_0 = 1$), the mean number of balls of the first type at time n is exactly and asymptotically:

$$4n + 1 - \frac{1}{\binom{n-3/4}{n}} \sim 4n - \frac{\pi\sqrt{2}}{\Gamma(3/4)}n^{3/4} + 1 + O(n^{-1/4}).$$

Similar forms have been computed by Kotz, Mahmoud, and Robert [33] using somewhat heavier recurrence manipulations. Here this results effortlessly from taking partial derivatives of the BGF $H(z,u)$ at $u = 1$,

$$\begin{aligned} &H(z,1) = \frac{1}{\Delta^{1/4}}, \quad H'_u(z,1) = \frac{1}{\Delta^{5/4}} - \frac{1}{\Delta}, \\ &H''_{uu}(z,1) = 5\Delta^{-9/4} - 8\Delta^{-2} + 4\Delta^{-7/4} - 5\Delta^{-5/4} + 4\Delta^{-1}, \end{aligned} \tag{85}$$

where $\Delta = 1 - 4z$, and extracting coefficients. (Such calculations illustrate the workings of Corollary 8 and are easily carried out using a computer algebra system.) As a consequence of the last equation, the variance also admits an explicit form involving binomial coefficients of fractional index, and its asymptotic value is

$$\frac{2}{3}\pi\frac{8\sqrt{2}-3\pi}{\Gamma(3/4)^2}n^{3/2} - \frac{3\pi\sqrt{2}}{\Gamma(3/4)}n^{3/4} + O(\sqrt{n}).$$

The number of balls of the second colour has thus been found to be of average order $n^{3/4}$, which coincides with the order of growth of the standard deviation. This is a strong indication of the presence of a non-Gaussian law[11]. Finally, consider the situation where the urn is initialized with $a_0 = 0$, $b_0 = t_0 = 1$. After rescaling, it is seen that the law parameterized by a coincides with the law defined by the bivariate generating function

$$\overline{H}(z,u) = \left(1 - u(1-(1-z)^{a/(a+1)})\right)^{-1/a}. \tag{86}$$

Precisely, let Y_n be the random variable defined by

$$\mathbb{P}(Y_n = k) = \frac{[z^n u^k]\overline{H}(z,u)}{[z^n]\overline{H}(z,1)}.$$

[11] We are grateful to Dr Philippe Robert for asking us to determine the corresponding distribution. The non-Gaussian character of the limit in such cases was observed by Smythe in [47]. However, apparently the martingale arguments of [47] do not lead to a determination of the limit law. Gouet [23] shows the *existence* of a limit distribution and remarks that "information on the moments of Z [the limit] can be obtained from the difference equations characterizing the moments of W_n [our X_n]", but he does not appear to have available a complete characterization of this limit.

The generating function $\overline{H}$ is obtained from H by the sequence of transformations,

$$(a+1)z \mapsto z, \quad u^{a+1}z \mapsto z, \quad u \mapsto u^{-1}, \quad u \mapsto u^{1/a},$$

which corresponds to the fact that

$$Y_n = \frac{1}{a}\left(n(a+1) - X_n\right).$$

In other words, Y_n is linearly related to X_n and, in fact, it represents the number of times a ball of the second colour is drawn. The expansion of (86) is now immediate:

Corollary 11. *In the urn with matrix $(a+1,0,1,a)$ initialized with $b_0 = 1$, $a_0 = 0$, the number Y_n of times a ball of the second colour is drawn has the explicit expression:*

$$\mathbb{P}(Y_n = k) = \frac{\binom{k+1/a-1}{k}}{\binom{n+1/(a+1)-1}{n}} \sum_{j=0}^{k} (-1)^{n-j} \binom{k}{j} \binom{n - j\frac{a}{a+1} - 1}{n}.$$

For the particular case of the urn $\binom{2\ 0}{1\ 1}$, i.e., $a = 1$, the formula simplifies to

$$\mathbb{P}(Y_n = k) = \frac{k2^k}{n} \frac{\binom{2n-k-1}{n-1}}{\binom{2n}{n}}.$$

These formulæ are somewhat simpler than the ones deriving from general expressions given by Kotz *et al.* (Theorem 1 and Proposition 1 in [33]) as they involve a *single* summation, in contrast to those of [33] that are k–fold summations, and hence become of arbitrarily high complexity as k varies. The simplification in the case $a = 1$ corresponds to ballot numbers and is readily checked by Lagrange inversion.

Analytically, the type of singularity change of $\overline{H}$ as u crosses the value 1 corresponds exactly to composition schemas which have been systematically studied by Banderier *et al.* [5] by means of singularity analysis; see Appendix A of that article, where stable laws are detected. A direct consequence of [5, Th. 12] is then the following:

Corollary 12. *In the urn with matrix $(a+1,0,1,a)$ initialized with $b_0 = 1$, $a_0 = 0$, the number $Y_n = \frac{1}{a}(n(a+1) - X_n)$ of times a ball of type II is drawn satisfies a local limit law which is an elementary variant of the stable law of index $a/(a+1)$. Precisely, for x in any compact set of $(0, +\infty)$ such that $xn^{a/(a+1)}$ is an integer, one has:*

$$\mathbb{P}\left(\frac{Y_n}{n^{a/(a+1)}} = x\right) \sim \frac{g_a(x)}{n^{a/(a+1)}},$$

where

$$\begin{cases} g_a(x) & := \dfrac{\Gamma(1/(a+1))}{\Gamma(1/a)} x^{1/a-1} G\left(x; \dfrac{a}{a+1}\right) \\ G(x;\lambda) & := -\dfrac{1}{\pi} \displaystyle\sum_{j\geq 1} \dfrac{(-x)^j}{j!} \Gamma(1+\lambda j) \sin(j\pi\lambda). \end{cases}$$

In the particular case of the $(2,0,1,1)$ urn, the density is of the Rayleigh type:

$$g_a(x) = \frac{1}{2} x e^{-x^2/4}.$$

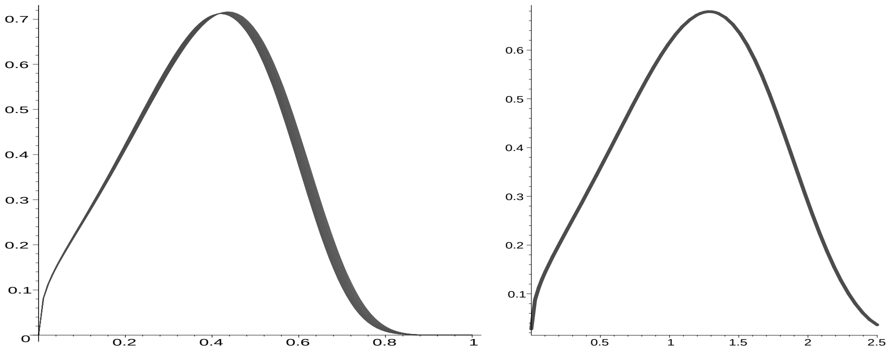

FIGURE 13. The KMR urn with $a = 3$: (left) Sedgewick plots of the finite distributions for $n = 80\,..\,100$ with the horizontal axis scaled to n, the vertical axis scaled to $n^{-a/(a+1)}$; (right) the limit density $g_3(x)$.

The quantity $x^{-1}G(x^{-\lambda};\lambda)$ is exactly the density of a stable law of index λ for $0 < \lambda < 1$. Thus, in probabilistic terms, the statement means that Y_n obeys in the limit a "size-biased" stable law of index $a/(a+1)$. Figure 13 displays the distribution in the case of $a = 3$.

Proof (Sketch; see [5] *for details).* We summarize the proof here for completeness. By expanding the outer radical $(\cdot)^{-1/a}$ in the expression (86) of $\overline{H}$, one finds

$$\mathbb{P}(X_n = k) = \frac{\binom{k-1+1/a}{k}}{\binom{n-1+1/(a+1)}{n}}[z^n]\left(1-(1-z)^{a/(a+1)}\right)^k.$$

The coefficient of z^n is estimated by means of Cauchy's coefficient formula,

$$[z^n]\left(1-(1-z)^{a/(a+1)}\right)^k = \frac{1}{2i\pi}\int_{(0)+}\left(1-(1-z)^{a/(a+1)}\right)^k \frac{dz}{z^{n+1}},$$

using as contour of integration a loop around 1, scaled according to the general principles of singularity analysis [20]. We thus set $z = (1 - t/n)$ and get with $k = xn^{a/(a+1)}$ the approximation,

$$\mathbb{P}(X_n = k) \sim x^{1/a-1}\cdot\frac{1}{2i\pi n}\int_{\mathcal{H}} e^{-xt^{a/(a+1)}+t}\,dt,$$

where $\mathcal{H}$ is a clockwise loop around the negative real axis. Next, keep the "kernel" e^t and expand the other exponential. This gives, with $\lambda = a/(a+1)$,

$$\begin{aligned}\frac{1}{2i\pi}\int_{\mathcal{H}} e^{-xt^{\lambda}+t}\,dt &= \frac{1}{2i\pi}\sum_{k\geq 0}\frac{(-x)^k}{k!}\int_{\mathcal{H}} t^{k\lambda}e^t\,dt \\ &= -\frac{1}{\pi}\sum_{k\geq 1}\frac{(-x)^k}{k!}\Gamma(1+k\lambda)\sin(k\pi\lambda),\end{aligned}$$

where the last equality derives from Hankel's classical representation of the gamma function [51, Sec. 12.22]. The connection with stable laws results from a trite comparison with the classical expression of their densities as found in the books of Feller [18, p. 583] or Zolotarev [54]. □

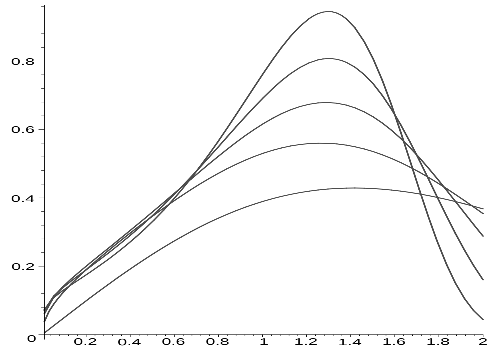

FIGURE 14. The limit densities $g_a(x)$ of the $(a+1, 0, 1, a)$ urn corresponding to values $a = 1\,..\,5$ (top graphs correspond to larger values of a).

The shape of the limit laws $g_a(x)$ for small values of a is displayed in Figure 14.

6. DISCUSSION

Starting from the basic partial differential equation, a frontal analytic attack on 2×2 urn models with constant row sum succeeds and leads to a concise description of the probabilistic composition of the urn at various epochs. The bivariate generating functions obtained are invariably "constructible by quadratures" in Taylor's terms [49, p. 86] as they only involve (i) elementary algebraic operations, (ii) integration, (iii) taking inverses of maps.

As concerns the special functions involved, the solution is expressible in terms of specific integrals (and their inverses) that can be equivalently regarded as Abelian integrals over the Fermat curves, as Schwarz–Christoffel maps, as incomplete Beta integrals, or as ${}_2F_1$–hypergeometric functions. Although the original problem has an apparent genus that is quite high, namely $((h-1)(h-2)/2)$, there seems to be an "effective genus"[12] representing the complexity of integrals associated with each problem which tends to be much smaller. In particular, several cases are recognized to be expressible in terms of algebraic functions of genus 0: this covers the classical Pólya–Eggenberger model of contagion as well as all urns of the type $\binom{a+1\ 0}{1\ \ a}$ discussed in Subsections 5.4 and 5.5. In a sense, the Ehrenfest model $\binom{-1\ \ 0}{0\ -1}$, the Pólya–Eggenberger urn $\binom{a\ 0}{0\ a}$, and the "altruistic" model of Friedmann $\binom{0\ a}{a\ 0}$ are also associated with explicit objects of effective genus 0. It is amusing to note that even the coupon collector problem, when formulated in terms of the urn model $\binom{-1\ 1}{0\ \ 0}$, as Laplace did, succumbs to this approach. The methods cover in passing the special model $\binom{0\ 1}{1\ 0}$ related to Eulerian numbers and increasing Cayley trees. Last but not least, several models resort to elliptic functions, corresponding to genus 1, a fact that can be regarded as widely extending earlier results of Panholzer and Prodinger.

Once closed-form solutions for generating functions have been found, a number of probabilistic properties easily result. Singularity analysis makes it possible to

[12]Arnaud Beauville (private communication, November 2002) points to a possible explanation based on the Jacobian of the Fermat curve together with the corresponding automorphisms and factorizations, referring to the works of Tetsuji Shioda for such questions.

extract asymptotically coefficients from generating functions and does so in a systematic manner, while being tolerant to analytic perturbations, hence applicable to multivariate problems, i.e., limit distributions. In this way, we can view the classical Gaussian laws known to hold for many urn models as deriving from a quasi-powers approximation associated with a smooth displacement of singularity: this is how we have rederived Bagchi and Pal's central limit law known to hold for all urns with subtraction. Speed of convergence estimates are immediate consequences, as could be in principle (but with additional work) local limit laws. A surprising outcome of our methodology is that it uncovers the presence of stable laws in KMR urns of the type $\binom{a+1\ 0}{1\ \ a}$: such laws are found to arise from a nonsmooth behaviour of the singular exponent in a way that analytically parallels properties of random planar maps [5]. A noteworthy consequence of the analytic approach is a large deviation principle for urns with subtraction, where the large deviation function surfaces explicitly as the Legendre transform of the basic Abelian integral of the model.

While we have focussed most of this paper on urns with subtraction (corresponding to negative diagonal entries), we could not resist listing a few scattered results for models with zero or positive diagonal entries, as these illustrate the further applicability of our methods to models like some of Bernard Friedman's urns [22]. Such results complement martingale arguments developed by Gouet [23], where strong functional limits are shown to exist, but some of the involved characteristics remain inaccessible due to nonconstructive aspects of martingale theory. (The case of stable laws in the present paper typically illustrates this point.) It is our belief that, by extending methods of the present paper, all 2×2 urn models with constant row sum can be eventually analysed in terms of: (i) characterizing the limiting distribution of the composition of the urn at each epoch n; (ii) determining explicitly the large deviation function. More generally, randomized schemes where one amongst several possible replacement rules is chosen at random will lead to Abelian integrals, provided they retain constant row sum and have balls of two colours only. The characteristic curve will in general no longer be of the Fermat type, but such randomized models are most probably amenable to a treatment similar to the one presented here.

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