SUR LA SOMME D'UNE PROGRESSION GÉOMÉTRIQUE

On the sum of a geometric progression

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From: Serge Cohen 8/04/08 10:56

"L'expose se doit d'etre assez general ..."

1. Basics

- The main (easy) theorem
- The ballot problem...
- The coin-fountain problem

A PIECE OF TRIVIALITY

Quasi-inverses. Let $\mathcal{A}$ be a **ring**; by *telescoping*:

$$(1-f)(1+f+\cdots+f^n) = 1-f^{n+1}.$$

Let $\mathcal{A}$ be "topological"; *if* $f^n \to 0$, *then*

$$\frac{1}{1-f} = 1+f+f^2+f^3+\cdots .$$

$\boxed{\textbf{Sum of a geometric progression}}$

- Works analytically with $\mathbb{R}, \mathbb{C}$.
- Works formally with $\mathbb{C}[\![z]\!]$ (needs $f(0)=0$), as well as $\mathbb{C}[\![x,y,z,\ldots]\!]$, $\mathbb{C}\langle\!\langle x,y,z,\rangle\!\rangle$.

ANOTHER PIECE OF TRIVIALITY

Distributivity. Let $\mathcal{A}$ be a ring. Then

$$\begin{cases} (a+b)^3 = aaa + aab + aba + abb + baa + bab + bba + bbb \\ (a+b)^n = \sum_{|w|=n} w \qquad \text{(all words of length } n\text{).} \end{cases}$$

Corollary A. If $a = \nearrow$, $b = \searrow$, $c = \rightarrow$, then

$$(c+ab)^3 = \overbrace{\rightarrow\rightarrow\rightarrow + \nearrow\searrow\rightarrow\rightarrow + \rightarrow\nearrow\searrow\rightarrow + \nearrow\searrow\nearrow\searrow\nearrow\searrow + \cdots}^{\text{3 blocks}}.$$

Corollary B. Combining with the *sum of a geometric progression*

$$\frac{1}{1-c-ab} = \sum \overbrace{(\rightarrow \nearrow\searrow \rightarrow\rightarrow \nearrow\searrow)}^{\text{any \# blocks}}.$$

TWO PIECES OF TRIVIALITY TOGETHER ...

Recall: $$\frac{1}{1-c-ab} = \sum \overbrace{\left(\rightarrow \nearrow\searrow \rightarrow\rightarrow \nearrow\searrow\right)}^{\text{any \# blocks}}.$$

Substitute further $ab \mapsto a\frac{1}{1-d}b$. Then:

Corollary C. With $a = \nearrow$, $b = \searrow$, $c = \rightarrow$, $d = \Rightarrow$, in $\mathbb{C}[\![a,b,c,d]\!]$:

$$\frac{1}{1-c-a\dfrac{1}{1-d}b} = \sum(\text{all diagrams of height} \leq 1)$$

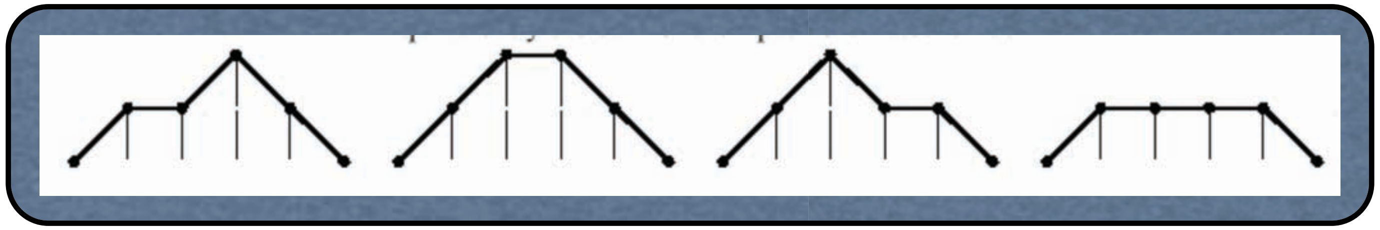
$$= \sum \rightarrow \nearrow \Rightarrow\Rightarrow\Rightarrow \searrow \rightarrow\rightarrow \nearrow \Rightarrow \searrow .$$

And get **Corollaries D, E, F, G, H, I, J, K, L, M,...**

... GIVE A THEOREM(?!)

Theorem [The main continued fraction theorem]

$$\cfrac{1}{1-c_0-\cfrac{a_0b_1}{1-c_1-\cfrac{a_1b_2}{1-c_2-\ddots}}} = \sum(\text{all lattice paths})$$

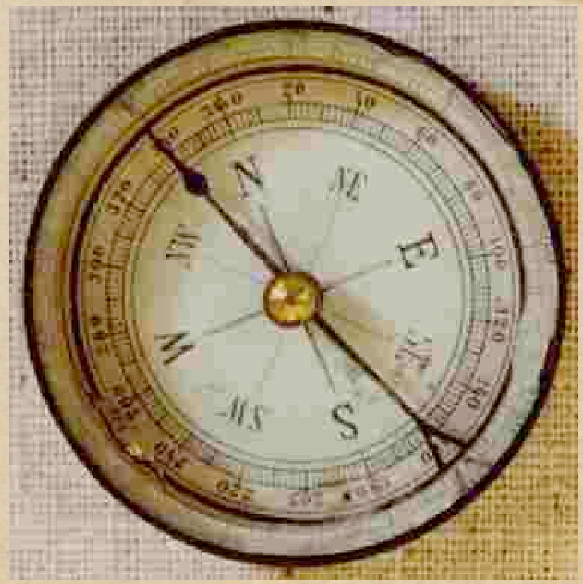


$$\cfrac{1}{1-c_0-\cfrac{a_0b_1}{1-c_1-\cfrac{a_1b_2}{\ddots \atop 1-c_2-\cfrac{}{1-c_h}}}} = \sum\left(\begin{array}{c}\text{all lattice paths}\\ \text{with height} \le h\end{array}\right)$$

Continued Fraction = a cascade of geom. progressions
= Sum of all lattice paths

Algebraic continued fractions

- 1. Cont'ed: simple applications (ballot, coins)
- 2. Orthogonal polynomials
- 3. Arches
- 4. Snakes
- 5. Addition formulae
- 6. Elliptic stuff



A SOLUTION (?!) TO THE BALLOT PROBLEM

"Two candidates, Alice and Bob, with (eventually) each n votes. What is the probability that Alice is always ahead or tied?"

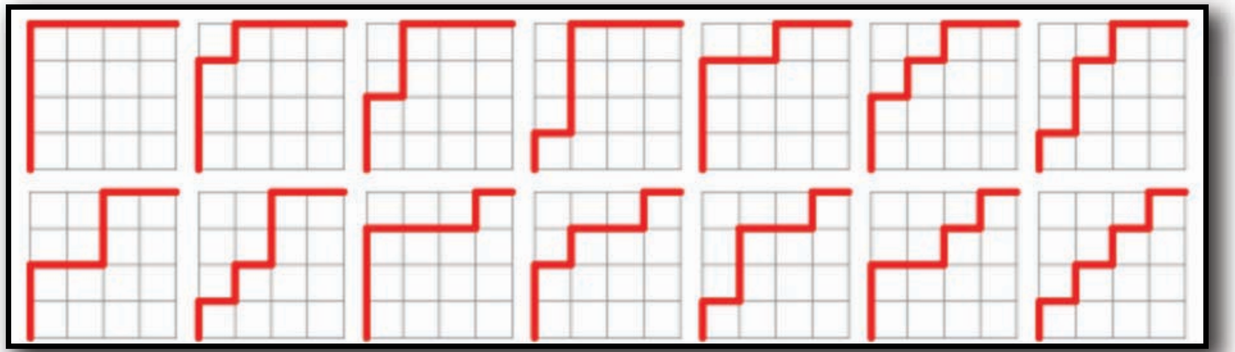
Do $\boxed{a \mapsto z;\ b \mapsto z;\ c \mapsto 0}$. By main theorem:

$$C := \cfrac{1}{1 - \cfrac{z^2}{1 - \cfrac{z^2}{\ddots}}} = \sum_{\beta \text{ ballot sequence}} z^{|\beta|} = \sum_n C_n z^{2n}.$$

We get Catalan numbers [Euler-Segner 1750; Catalan 1850]

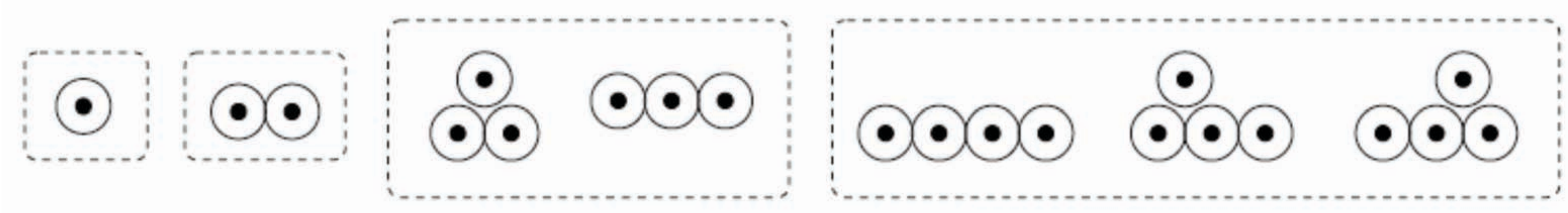
$$C = \frac{1}{1 - z^2 C} \implies C = \frac{1 - \sqrt{1 - 4z^2}}{2z^2} \implies C_n = \frac{1}{n+1}\binom{2n}{n}.$$

The probability is $\dfrac{C_n}{\binom{2n}{n}} = \dfrac{1}{n+1}$.



COIN FOUNTAINS

$$C(q) = 1 + q + q^2 + 2q^3 + 3q^4 + 5q^5 + 9q^6 + 15q^7 + 26q^8 + \cdots$$



Greetings from The On-Line Encyclopedia of Integer Sequences!

Search: **1, 2, 3, 5, 9, 15, 26, 45**

Displaying 1-2 of 2 results found. pag

Format: long | short | internal | text Sort: relevance | references | number Highlight: on | off

A005169 Number of fountains of n coins.
(Formerly M0708)

1, 1, 1, 2, 3, 5, 9, 15, 26, 45, 78, 135, 234, 406, 704, 1222, 2120, 3679, 6385, 11081, 19232, 33379, 57933, 100550, 174519, 302903, 525734, 912493, 1583775, 2748893, 4771144, 8281088, 14373165, 24946955, 43299485, 75153286, 130440740 (list; graph; listen)

OFFSET 0,4

COMMENT A fountain is formed by starting with a row of coins, then stacking additional coins on top so that each new coin touches two in the previous row.

Do: $\boxed{a_j \mapsto 1,\ b_j \mapsto q^j}$. Then:

$$C(q) = \cfrac{1}{1 - \cfrac{q}{1 - \cfrac{q^2}{1 - \cfrac{q^3}{\cdots}}}} = \frac{\sum (-1)^n \dfrac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)}}{\sum (-1)^n \dfrac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}}.$$

Number of coin fountains: $C_n \sim 0.31 \cdot 1.73566^n$.

Ramanujan's fraction:

$$\cfrac{1}{1 + \cfrac{e^{-2\pi\sqrt{5}}}{1 + \cfrac{e^{-4\pi\sqrt{5}}}{1 + \cfrac{e^{-6\pi\sqrt{5}}}{\cdots}}}} = e^{2\pi/\sqrt{5}} \left(\frac{\sqrt{5}}{1 + \sqrt[5]{\frac{5^{3/4}}{\left(1/2 + 1/2\sqrt{5}\right)^{5/2}} - 1}} - \frac{1+\sqrt{5}}{2} \right)$$

$$= 0.\mathbf{999999}2087\cdots.$$

$$\boxed{e^{\pi\sqrt{163}} = 262537412640768743.99999999999\cdots}$$

2. Convergent polynomials

- Revisiting the ballot problem
- Three-term recurrences
- Orthogonality

"Two candidates, Alice and Bob, with (eventually) each n votes. If Alice is always ahead (or tied), what is the probability that she never **leads** *by more than h?"*

The number of favorable cases has generating function (GF), with $z^2 \mapsto z$:

$$C^{[h]}(z) = \left.\frac{1}{1 - \cfrac{z}{\ddots \atop 1 - \cfrac{}{1-z}}}\right\} h \text{ stages.}$$

$$\frac{1}{1}, \quad \frac{1}{1-z}, \quad \frac{1-z}{1-2z}, \quad \frac{1-2z}{1-3z+z^2}, \quad \cdots, \quad \frac{F_{h+1}(z)}{F_{h+2}(z)},$$

where $F_{h+2} = F_{h+1} - zF_h$ are Fibonacci polynomials.

- "Constant-coefficient" recurrence; Lagrange inversion.
- Roots are $1/(4\cos^2\theta)$, $\theta = \frac{k\pi}{h}$; partial fractions.

- Lagrange [1775] & Lord Kelvin & De Bruijn, Knuth, Rice [1973]

$$C_n^{[h]} = \sum_k \cdots 4^n \cos^{2n}\left(\frac{k\pi}{h}\right) = \sum_k \cdots \binom{2n}{n-kh}.$$

- Related to Kolmogorov–Smirnov tests in statistics:

  Compare $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$? "Sort and vote!"

- Pólya [1927]: **totally elementary proof of elliptic-theta transformation**:

$$\sum_{\nu=-\infty}^{\infty} e^{-\nu^2 t^2} = \sqrt{\frac{\pi}{t^2}} \sum_{\nu=-\infty}^{\infty} e^{-\pi^2\nu^2/t^2}.$$

$= \boxed{\textit{Do multisection of } (1+z)^{2n}\textit{, with } h = t\sqrt{n}\textit{, in two ways!}}$

Orthogonal polynomials

Linear fractional transformations [homographies] get composed like 2×2 matrices:

$$\frac{ax+b}{cx+d} \qquad \mapsto \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

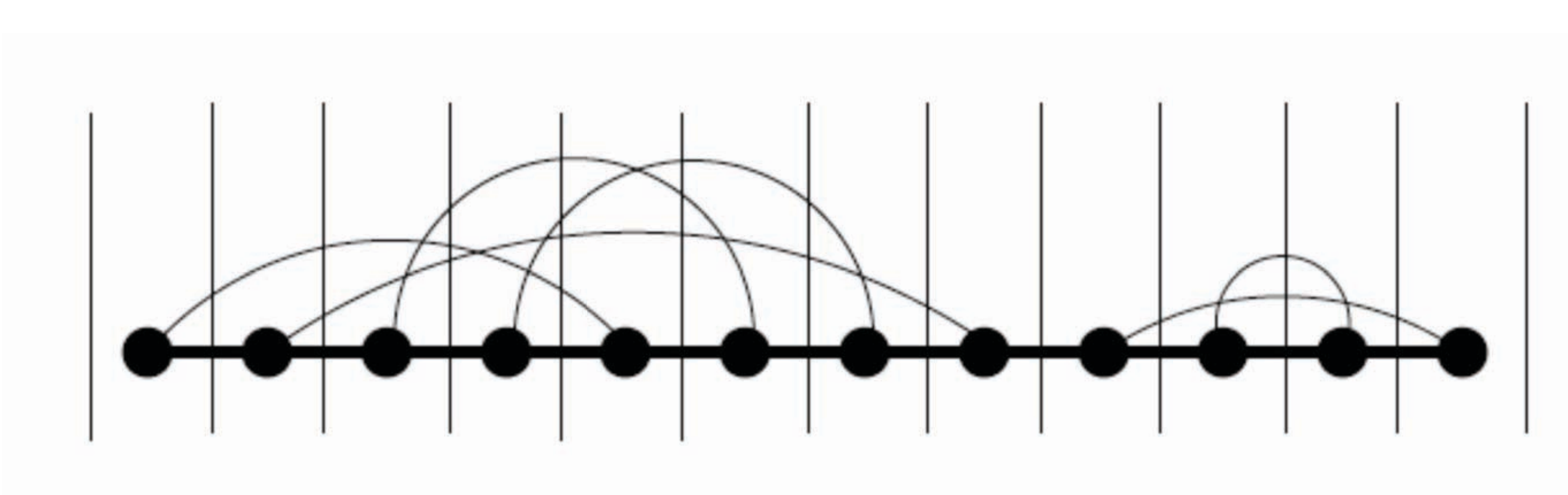
- Convergent polynomials $\dfrac{P_h(z)}{Q_h(z)}$ satisfy a three-term recurrence with numers/denoms of the continued fraction.
- Reciprocals of convergent polynomials are **orthogonal** with respect to $\langle f, g \rangle = \langle f \cdot g \rangle$, where moments $\langle z^n \rangle$ are coefficients in the expansion of the continued fraction.

The formal theories of **continued fractions** *and* **orthogonal polynomials** *are two aspects of one and the same thing.*

3. Arches and such

- Colouring rules
- An interconnection problem from industry
- Hermite polynomials

In how many ways can one join $2n$ points on the line in pairs?



$\nearrow \quad \nearrow \quad \nearrow \quad \nearrow \quad \searrow \quad \searrow \quad \searrow \quad \searrow \quad \nearrow \quad \nearrow \quad \searrow \quad \searrow$

1 2 2 1 2 1

A descent from altitude j has j possibilities: $d_j \mapsto jz, \quad a_j \mapsto z$.

$$\sum_{n\geq 1}(1\cdot 3\cdots(2n-1))z^{2n} = \cfrac{1}{1-\cfrac{1\cdot z^2}{1-\cfrac{2\cdot z^2}{1-\cfrac{3\cdot z^2}{\ddots}}}}.$$

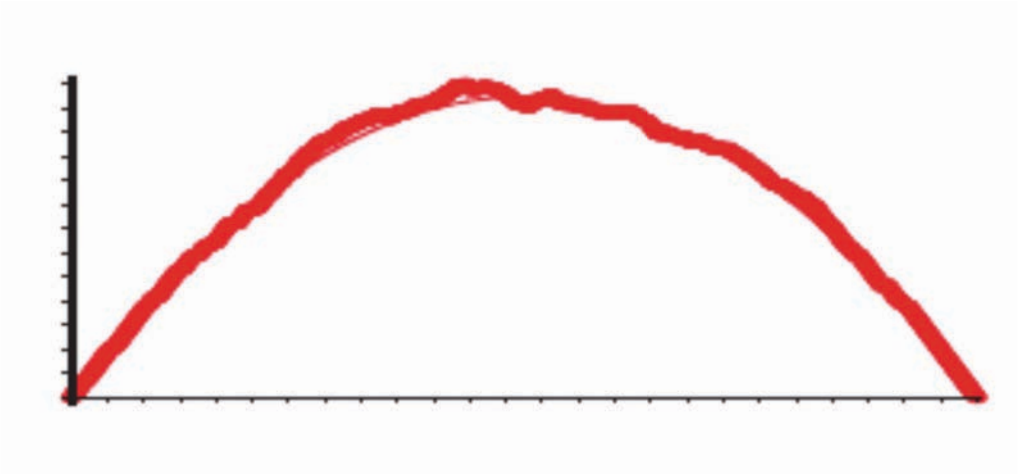
$\boxed{\textbf{Gauss !!}}$

Lagarias-Odlyzko-Zagier [1985]: Which capacity do we need to arrange pairwise connections between 2n points, with high probability?

- The answer lies in the zeroes of **Hermite polynomials**.

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\cdot g(x)\, e^{-x^2/2}\, dx.$$

Proof. For width h: $$\left.\cfrac{1}{1-\cfrac{1\cdot z^2}{1-\cfrac{2\cdot z^2}{\ddots\, \over 1-h\cdot z^2}}}\right\}\ h \text{ levels.}$$



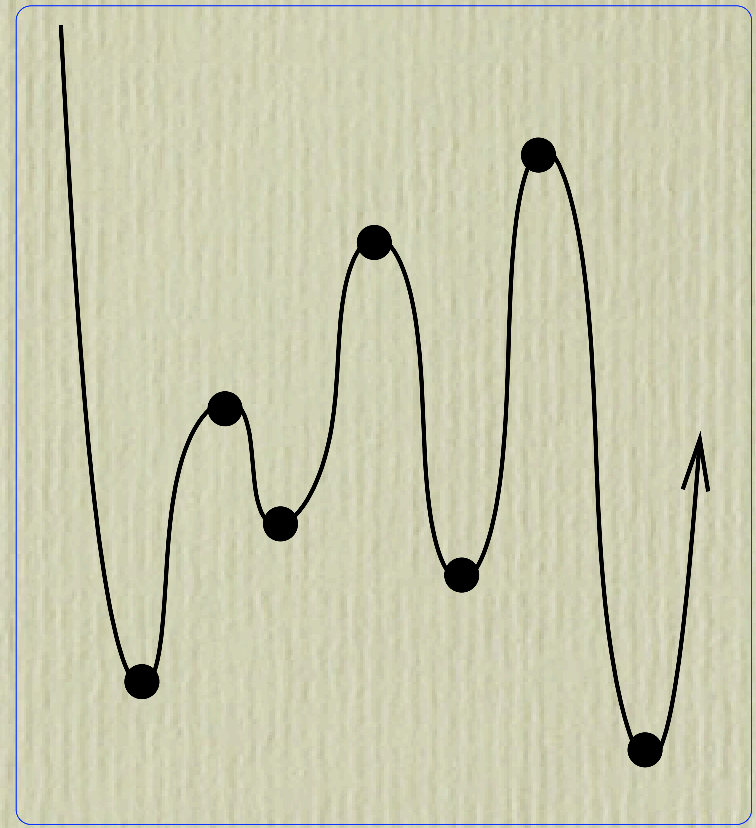
Louchard & Janson: a Gaussian process = deterministic parabola + Brownian noise.

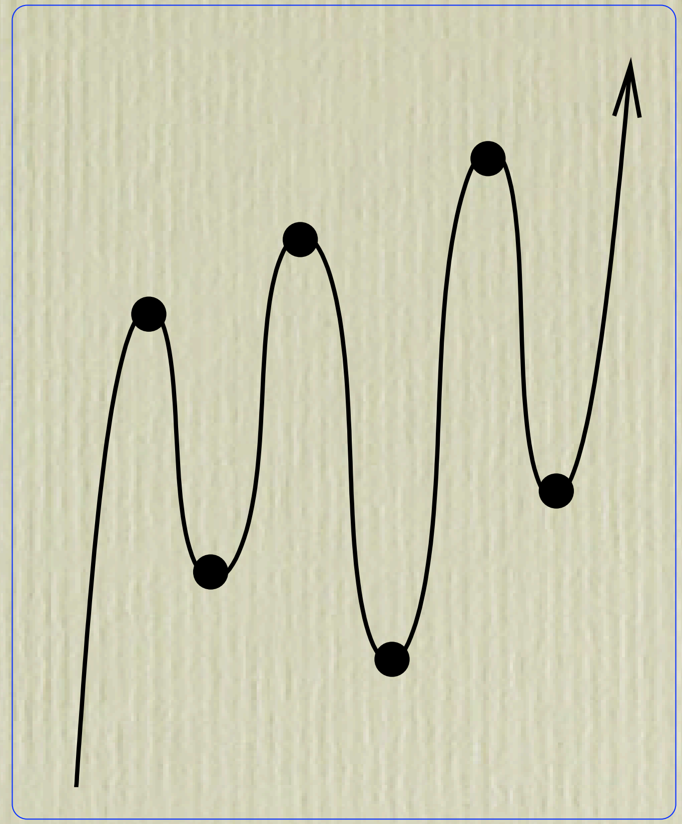
Airy connection?

4. Snakes and curves

- Arnold's snakes
- Stieltjes' fraction
- Postnikov's Morse links

Arnold [1992]: How many types of "open" curves?

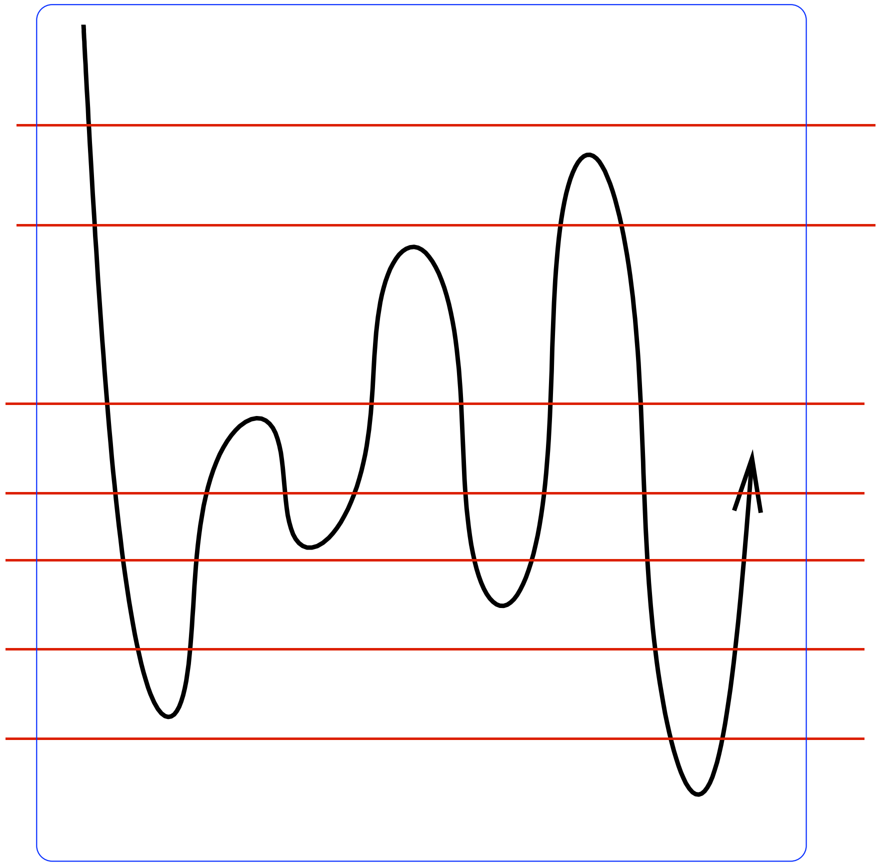


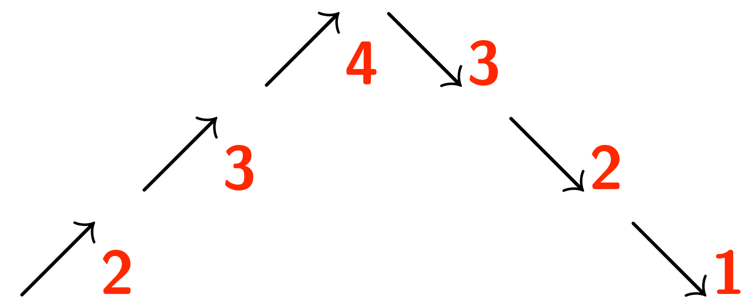


D. André [1881]: alternating perms = the coefficients of tan(z) and sec(z).

$$\begin{cases} \tan z &= \mathbf{1}\frac{z}{1!} + \mathbf{2}\frac{z^3}{3!} + \mathbf{16}\frac{z^5}{5!} + \mathbf{272}\frac{z^7}{7!} + \cdots \\ \sec z &= \mathbf{1} + \mathbf{1}\frac{z^2}{2!} + \mathbf{5}\frac{z^4}{4!} + \mathbf{61}\frac{z^6}{6!} + \cdots . \end{cases}$$

Proof. Decompose according to minimum and get ODE: $Y' = 1 + Y^2$ via recurrence.





$$\sum (\tan)_{2n+1} z^{2n+1} = \left\langle\!\left\langle \int_0^\infty e^t \tan(zt)\, dt \right\rangle\!\right\rangle = \cfrac{z}{1 - \cfrac{\mathbf{1 \cdot 2}\, z^2}{1 - \cfrac{\mathbf{2 \cdot 3}\, z^3}{\cdots}}}.$$

1, 2, 16, 272, ...

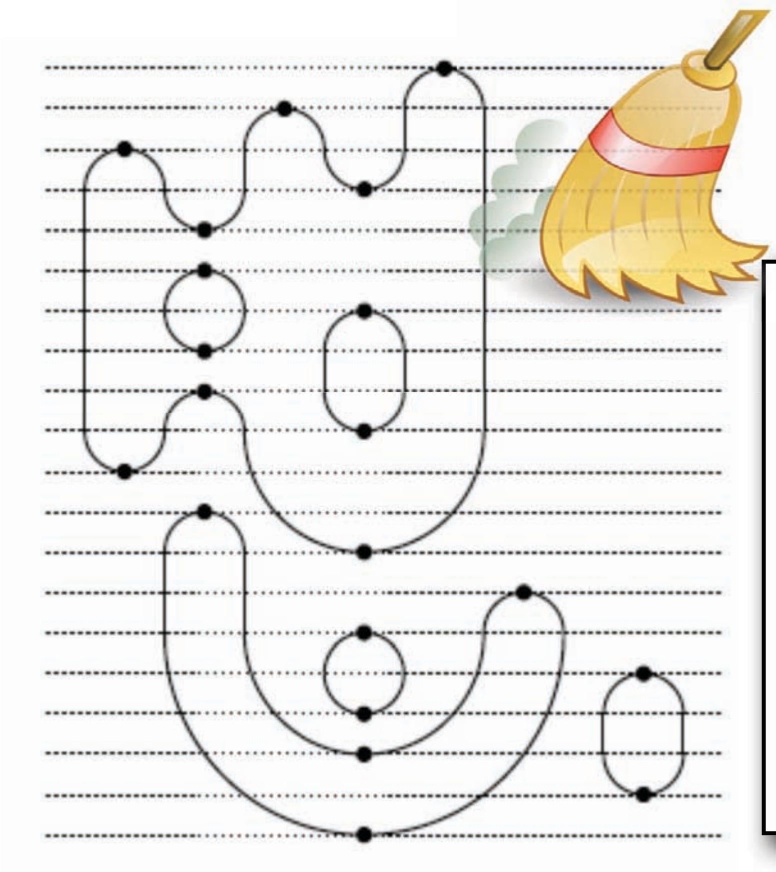
Related to a bijection of Françon and Viennot

= A continued fraction of Stieltjes

Ann. Fac. Sci. Toulouse, 1894

A continued fraction of Postnikov (2000)

= *Morse links (systems of closed Morse curves)*



$$\frac{\Gamma\left(x-\frac{1}{2}a+\frac{1}{4}\right)\Gamma\left(x+\frac{1}{2}a+\frac{1}{4}\right)}{\Gamma\left(x-\frac{1}{2}a+\frac{3}{4}\right)\Gamma\left(x+\frac{1}{2}a+\frac{3}{4}\right)}= \cfrac{4}{4x+\cfrac{1^2-4a^2}{8x+\cfrac{3^2-4a^2}{8x+\cfrac{5^2-4a^2}{8x+\cfrac{7^2-4a^2}{8x+\cdots}}}}}$$

Theorem [F.2008]. *The Morse–Postnikov numbers satisfy*

$$L_n \sim \widehat{L}_n, \qquad \text{where} \quad \widehat{L}_n = \frac{1}{2}(2n-1)!\left(\frac{4}{\pi}\right)^{2n+1}.$$

E.g.: $\dfrac{L_4}{\widehat{L}_4} \doteq 0.99949.$

5. Addition formulae &c.

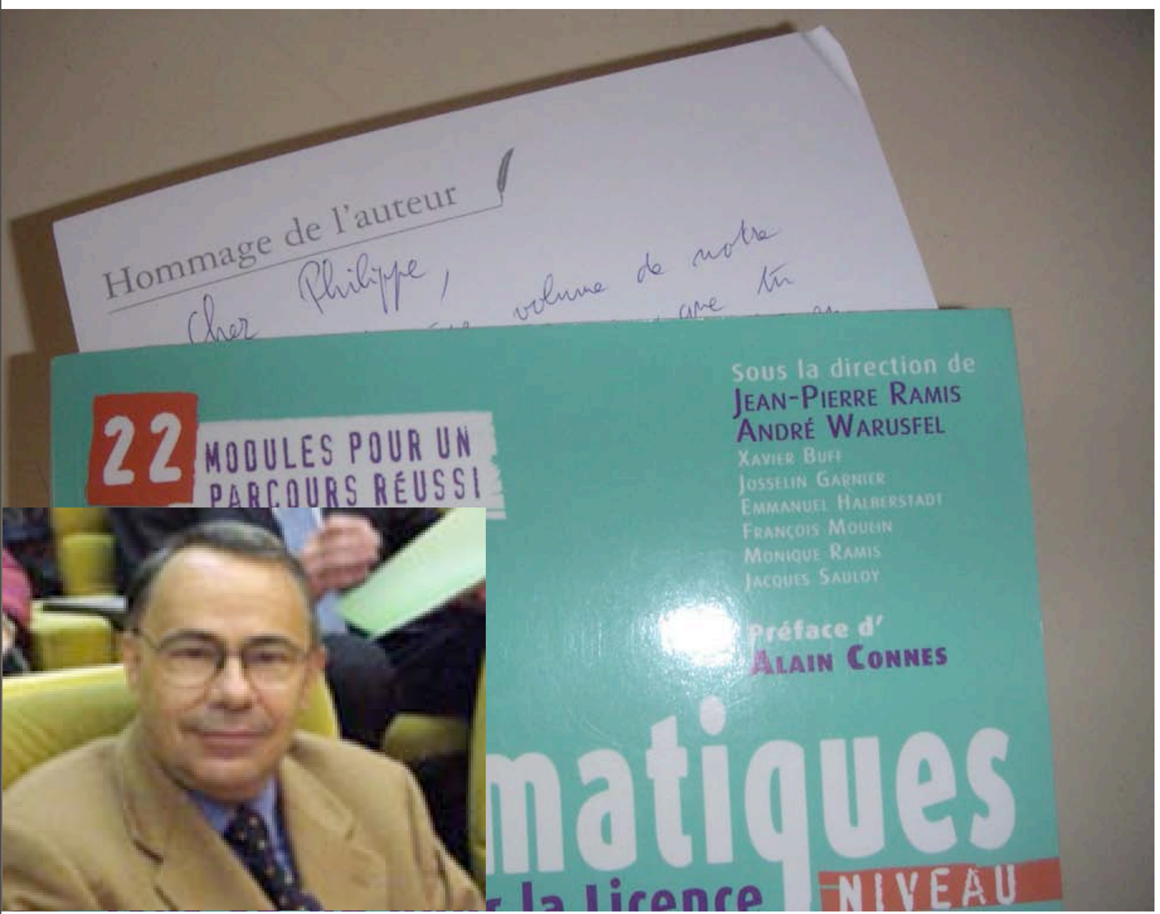
- Stieltjes-Rogers
- Addition formulae, paths, and OPs are all belong to a single family of identities
- Applications to "processes"...

The Stieltjes–Rogers Theorem

Definition. $\phi(z) = \sum_{n=0}^{\infty} \phi_n \frac{z^n}{n!}$ *satisfies an addition formula if*

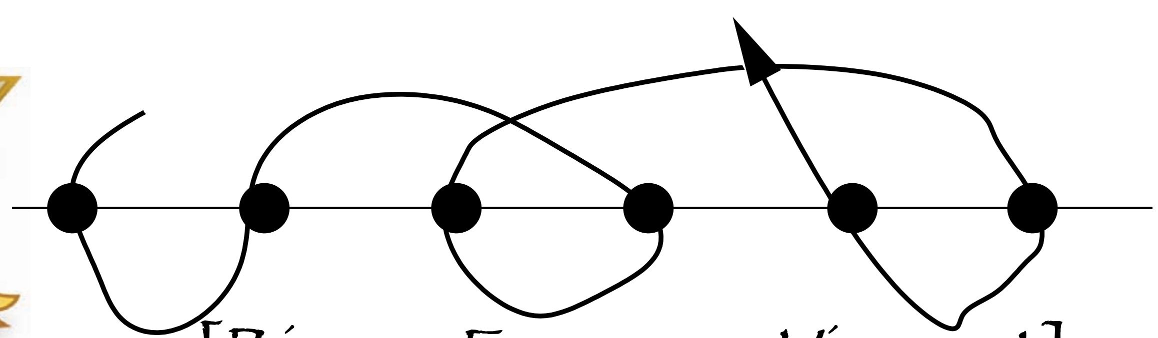
$$\phi(x+y) = \sum_k \omega_k \phi_k(x)\phi_k(y), \text{ where } \phi_k(x) = \frac{x^k}{k!} + O(x^{k+1}).$$

Theorem. *An addition formula gives automatically a continued fraction for* $f(z) = \sum_{n=0}^{\infty} \phi_n z^n = \langle\!\langle \int_0^{\infty} e^t \phi(zt)\, dt \rangle\!\rangle$.

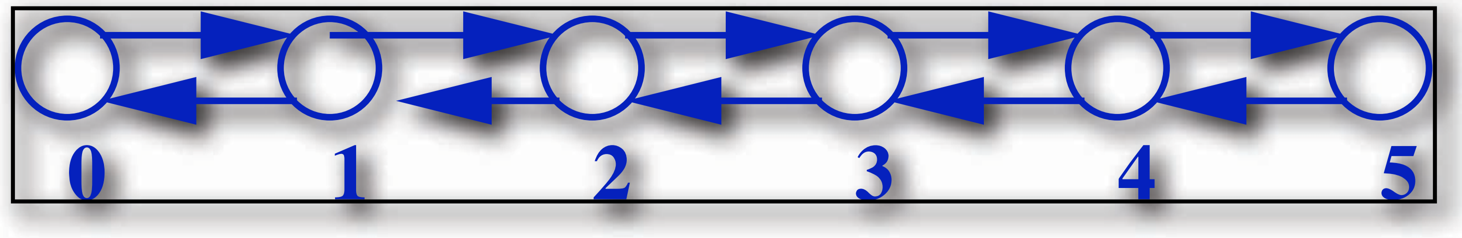


$$\frac{1}{1-x-y} = \sum_k (k!)^2 \frac{x^k/k!}{(1-x)^{k+1}} \frac{y^k/k!}{(1-y)^{k+1}}$$

$$\sum n! z^n = \cfrac{1}{1 - z - \cfrac{1^2 z^2}{1 - 3z - \cfrac{2^2 z^2}{\ddots}}}$$



[Biane, Françon-Viennot]



Systems of paths and **birth–death processes**:
Number & probability of weighted paths from a to b;

- Discrete time processes: I.J. Good [1950's];
- Continuous time processes: Karlin–McGregor; F–Guillemin [AAP 2000];
- Combinatorial processes = "file histories", [F–Françon–Vuillemin–Puech, 1980+]

$\leadsto$ Number of paths from 0 to absorbing state, equiv. probability of traversal, is $\propto \dfrac{1}{Q_h}$; "Keilson's Theorem".
$\leadsto$ Paths from 0 to k have exp. gen. function φ_k of addition formula.

Meixner's class of special OP's

Classical orthogonal polynomials seem to share many properties.

Theorem [Meixner 1934]: *If the exponential generating function satisfies a strong decomposability property,*

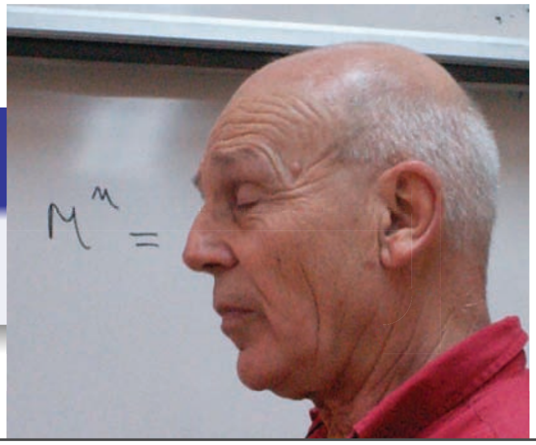
$$\sum_h \overline{Q}_h(z)\frac{t^n}{n!} = A(t)e^{zB(t)},$$

then there are only five possibilities.

Laguerre	Hermite	Poisson-Charlier	Meixner I	Meixner II
Perms	Arcs	Set partitions	Snakes	Pref. arrang.
$\frac{1}{1-z}$	$e^{z^2/2}$	e^{e^z-1}	$\sec(z)$	$\frac{1}{2-e^z}$.

Computations are automatic.

G. Letac & librairies?

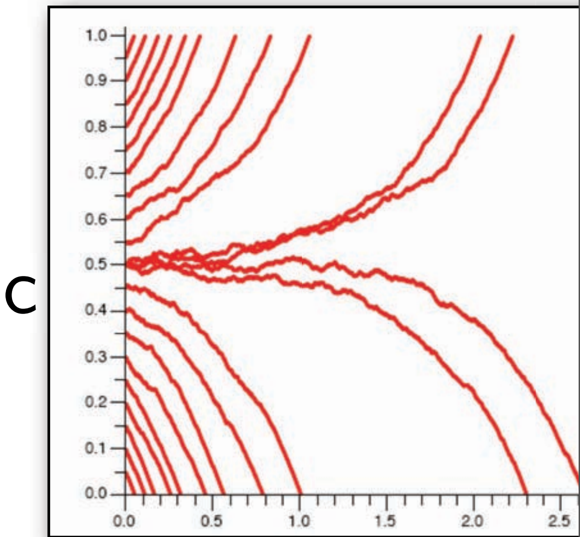


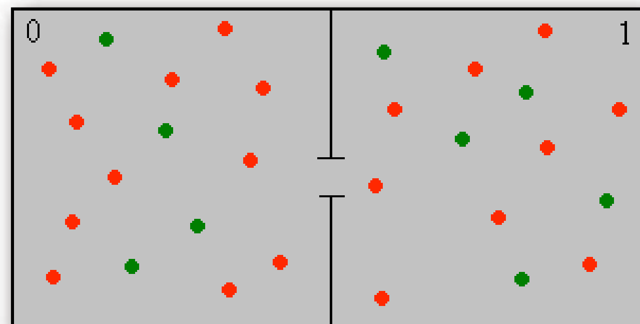
The Mabinogion urn model

Spread of influence in populations:

$A \Longrightarrow (B \longrightarrow A), \; B \Longrightarrow (A \longrightarrow B).$

Analytic duality from Ehrenfest urn solved by M. Kac





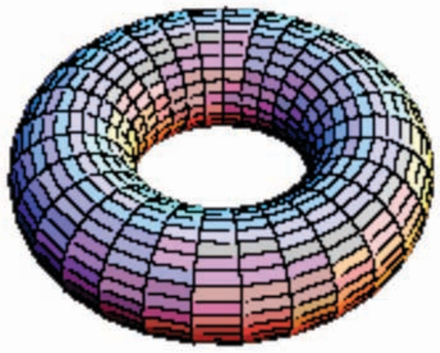
Theorem [F–Huillet 2008]. *Fair urn: absorption time is* $\sim \frac{1}{4} N \log N$, *with limit distribution of density* $\asymp e^{-t} e^{-e^{-2t}}$.

Stieltjes suggests considering addition formulae and continued fractions related to $\boxed{\sinh^k(z) \cosh^{N-k}(z),}$ which have combinatorial significance ($\neq$Markov) and relate to

$$\cfrac{1}{1 - \cfrac{1 \cdot N z^2}{1 - \cfrac{2 \cdot (N-1) z^2}{\cdots}}}.$$

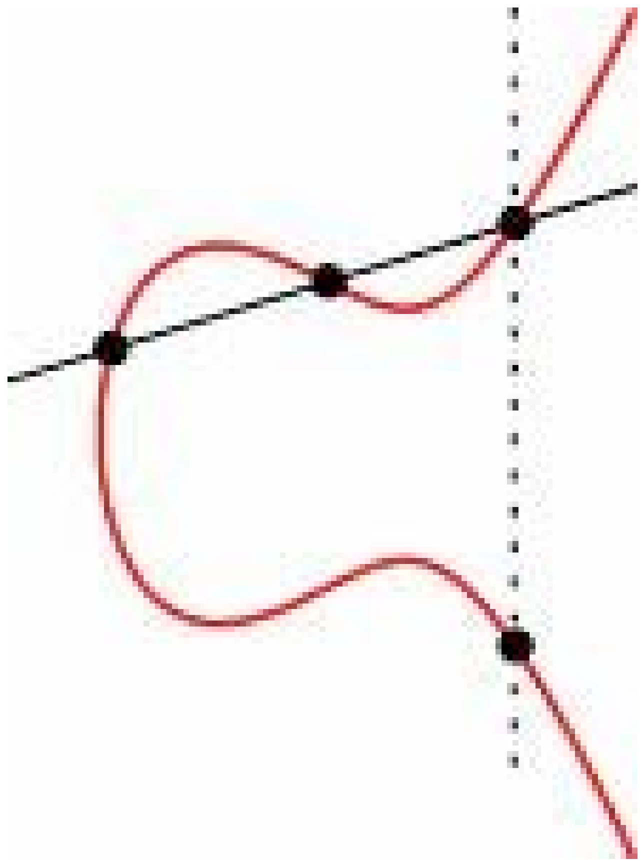
5. Some Elliptic matters

- Jacobian functions
- Dixonian functions
- Bacher's numbers



- **Algebraic curves of genus 1** are doughnuts. The integrals have two "periods". The inverse functions are **elliptic functions**; i.e., doubly periodic meromorphic.
- Weierstraß $\wp$ arises from $y^2 = P_3(z)$;
- Jacobian sn, cn arise from $y^2 = (1 - z^2)(1 - k^2 z^2)$;
- Dixonian sm, cm arise from $y^3 + z^3 = 1$.

They satisfy addition formulae!



- **Theorem [F; Dumont 1980].** *Jacobian elliptic functions count alternating perms w/parity of peaks.*
- **Theorem [Conrad+F, 2006].** *Dixonian functions have continued fractions*

$$\int_0^\infty \mathrm{sm}(u)e^{-u/x}\,du = \cfrac{x^2}{1+b_0x^3-\cfrac{1\cdot 2^2\cdot 3^3\cdot 4\,x^6}{1+b_1x^3-\cfrac{4\cdot 5^2\cdot 6^2\cdot 7\,x^6}{\cdots}}};$$

$\equiv$ *levels in trees and an urn model ($\approx$Yule process), &c*

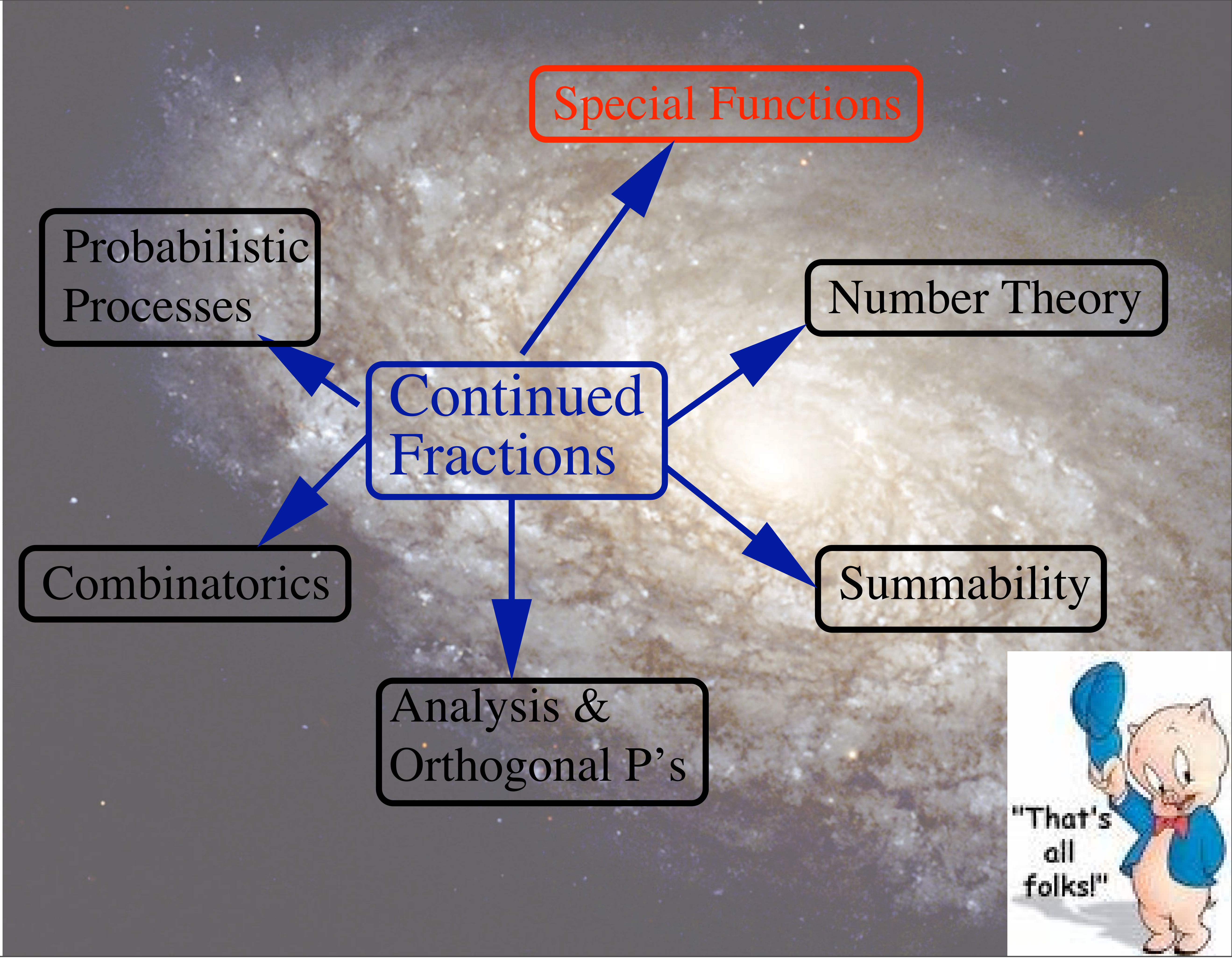
$$\zeta(3) = \cfrac{6}{\varpi(0)-\cfrac{1^6}{\varpi(1)-\cfrac{2^6}{\varpi(2)-\cfrac{3^6}{\ddots}}}},$$

- **Theorem [Bacher+F, 2006].** *Pseudofactorials* $a_{n+1} = (-1)^{n+1}\sum\binom{n}{k}a_ka_{n-k}$ *have a CF*

$$\sum a_nz^n = \cfrac{1}{1+z+\cfrac{3\cdot 1^2\,z^2}{1-z+\cfrac{2^2\,z^2}{1+3z+\ddots}}}.$$

A098777 a(0)=0, a(n+1)=(-1)^(n+1)*sum('binomial(n,k)*a(k)*a(n-k)','k'=0..n), n>=0.

1, -1, -2, 2, 16, -40, -320, 1040, 12160, -52480, -742400, 3872000, 664
411136000, -8202444800, 58479872000, 1335009280000, -10791497728000, -2


Special Functions
Probabilistic Processes
Number Theory
Continued Fractions
Combinatorics
Summability
Analysis & Orthogonal P’s
"That's all folks!"