WW Whittaker W

WW.1 Introduction

Let x be a complex variable of $\mathbb{C}\setminus\{0,\infty\}$ and let μ,ν denote a set of parameters (independent of x).The function Whittaker W (noted $\mathrm{WW}_{\mu,\nu}$) is defined by the following second order differential equation

(WW.1.1) $$-x^2-4\mu x-1+4\nu^2 y(x)+4x^2\frac{\partial^2 y(x)}{\partial x^2}=0.$$

Although 0 is a singularity of WW.1.1, the initial conditions can be given by

(WW.1.2)
$$\left[x^{\left(\nu+\frac{1}{2}\right)}\right]\mathrm{WW}_{\mu,\nu}(x)=\frac{\pi}{\sin\big(\pi(2\nu+1)\big)\Gamma(2\nu+1)\Gamma\left(\frac{1}{2}-\nu-\mu\right)},$$
$$\left[x^{\left(-\nu+\frac{1}{2}\right)}\right]\mathrm{WW}_{\mu,\nu}(x)=\frac{\pi}{\sin\big(\pi(2\nu+1)\big)\Gamma(1-2\nu)\Gamma\left(\frac{1}{2}+\nu-\mu\right)}.$$

The formulae of this document are valid for $2\nu\notin\mathbb{Z}$.
Related function: Whittaker M

WW.2 Series and asymptotic expansions

WW.2.1 Asymptotic expansion at ∞.

WW.2.1.1 First terms.

$$\mathrm{WW}_{\mu,\nu}(x)\approx\frac{\mathrm{e}^{\left(\frac{-1}{2x}\right)}y_0(x)}{x^{\mu}},$$

where

$$y_0(x)=1+\left(\nu^2+\mu-\mu^2-\frac{1}{4}\right)x-$$
$$\frac{-\big(4\nu^2+12\mu-4\mu^2-9\big)\big(4\nu^2+4\mu-4\mu^2-1\big)x^2}{32}+$$
$$\big(4\nu^2+20\mu-4\mu^2-25\big)\big(4\nu^2+12\mu-4\mu^2-9\big)$$
$$\big(4\nu^2+4\mu-4\mu^2-1\big)x^3\Big/384+2\ldots$$

WW.2.1.2 General form.

WW.2.1.2.1 Auxiliary function $y_0(x)$. The coefficients $u(n)$ of $y_0(x)$ satisfy the following recurrence

$$4u(n)n+$$
$$u(n-1)\big(-4\nu^2-4\mu+4\mu^2-3+4n-8(n-1)\mu+4(n-1)^2\big)=0$$

whose initial conditions are given by

$$u(0) = 1$$

This recurrence has the closed form solution

$$u(n) = \frac{(-2)^n \Gamma\left(n + \frac{1}{2} - \nu - \mu\right) 2^n \Gamma\left(n + \frac{1}{2} + \nu - \mu\right)}{4^n \Gamma(n+1) \Gamma\left(\frac{1}{2} - \nu - \mu\right) \Gamma\left(\frac{1}{2} + \nu - \mu\right)}.$$

WW.2.2 Asymptotic expansion at 0**.**

WW.2.2.1 First terms.

$$\mathrm{WW}_{\mu,\nu}(x) \approx x^{\left(\nu+\frac{1}{2}\right)}\Bigg(\frac{\pi}{\sin\big(\pi(2\nu+1)\big)\Gamma(2\nu+1)\Gamma\left(\frac{1}{2}-\nu-\mu\right)} - \frac{\mu x\pi}{(2\nu+1)\sin\big(\pi(2\nu+1)\big)\Gamma(2\nu+1)\Gamma\left(\frac{1}{2}-\nu-\mu\right)} + \frac{(2\nu+1+4\mu^2)x^2\pi}{16(2\nu+1)(\nu+1)\sin\big(\pi(2\nu+1)\big)\Gamma(2\nu+1)\Gamma\left(\frac{1}{2}-\nu-\mu\right)} - \Big(\mu\big(6\nu+5+4\mu^2\big)x^3\pi\Big)\Big/\Big(48(2\nu+1)(\nu+1)(2\nu+3)\sin\big(\pi(2\nu+1)\big)\Gamma(2\nu+1)\Gamma\left(\frac{1}{2}-\nu-\mu\right)\Big) + \Big(\big(12\nu^2+24\nu+9+48\mu^2\nu+56\mu^2+16\mu^4\big)x^4\pi\Big)\Big/\Big(1536(2\nu+1)(\nu+1)(2\nu+3)(\nu+2)\sin\big(\pi(2\nu+1)\big)\Gamma(2\nu+1)\Gamma\left(\frac{1}{2}-\nu-\mu\right)\Big) - \Big(\mu\big(60\nu^2+160\nu+89+80\mu^2\nu+120\mu^2+16\mu^4\big)x^5\pi\Big)\Big/\Big(7680(2\nu+1)(\nu+1)(2\nu+3)(\nu+2)(2\nu+5)\sin\big(\pi(2\nu+1)\big)\Gamma(2\nu+1)\Gamma\left(\frac{1}{2}-\nu-\mu\right)\Big) + \Big(\big(120\nu^3+540\nu^2+690\nu+225+720\mu^2\nu^2+2400\mu^2\nu+1756\mu^2+480\mu^4\nu+880\mu^4+64\mu^6\big)x^6\pi\Big)\Big/\Big(368640(2\nu+1)(\nu+1)(2\nu+3)(\nu+2)(2\nu+5)(\nu+3)\sin\big(\pi(2\nu+1)\big)\Gamma(2\nu+1)\Gamma\left(\frac{1}{2}-\nu-\mu\right)\Big) - \Big(\mu\big(840\nu^3+4620\nu^2+7518\nu+3429+1680\mu^2\nu^2+6720\mu^2\nu+6076\mu^2+672\mu^4\nu+1456\mu^4+64\mu^6\big)x^7\pi\Big)\Big/\Big(2580480(2\nu+1)(\nu+1)(2\nu+3)(\nu+2)(2\nu+5)(\nu+3)(2\nu+7)\sin\big(\pi(2\nu+1)\big)\Gamma(2\nu+1)\Gamma\left(\frac{1}{2}-\nu-\mu\right)\Big) + \Big(\big(1680\nu^4+13440\nu^3+36120\nu^2+36960\nu+11025+13440\mu^2\nu^3+87360\mu^2\nu^2+172256\mu^2\nu+99760\mu^2+13440\mu^4\nu^2+62720\mu^4\nu+67424\mu^4+3584\mu^6\nu+8960\mu^6+256\mu^8\big)x^8\pi\Big)\Big/\Big(165150720(2\nu+1)(\nu+1)(2\nu+3)(\nu+2)(2\nu+5)(\nu+3)(2\nu+7)(\nu+4)\sin\big(\pi(2\nu+1)\big)\Gamma(2\nu+1)\Gamma\left(\frac{1}{2}-\nu-\mu\right)\Big)\ldots\Bigg) + \Bigg(\frac{\pi}{\sin\big(\pi(2\nu+1)\big)\Gamma(1-2\nu)\Gamma\left(\frac{1}{2}+\nu-\mu\right)} + \tag{WW.2.2.1.1}$$

WW.2.2.2 General form. The general form of is not easy to state and requires to exhibit the basis of formal solutions of ?? (coming soon).