WM Whittaker M

WM.1 Introduction

Let x be a complex variable of $\mathbb{C}\setminus\{0,\infty\}$ and let μ,ν denote a set of parameters (independent of x).The function Whittaker M (noted $\mathrm{WM}_{\mu,\nu}$) is defined by the following second order differential equation

$$-x^2 - 4\mu x - 1 + 4\nu^2 y(x) + 4x^2 \frac{\partial^2 y(x)}{\partial x^2} = 0. \tag{WM.1.1}$$

Although 0 is a singularity of WM.1.1, the initial conditions can be given by

$$\frac{\partial \frac{\mathrm{WM}_{\mu,\nu}(x)}{x^{\left(\nu+\frac{1}{2}\right)}}}{\partial x} = 1. \tag{WM.1.2}$$

The formulae of this document are valid for $2\nu \notin \mathbb{Z}$.
Related function: Whittaker W

WM.2 Series and asymptotic expansions

WM.2.1 Asymptotic expansion at 0.

WM.2.1.1 First terms.

(WM.2.1.1.1)
$$\mathrm{WM}_{\mu,\nu}(x) \approx x^{\left(\nu+\frac{1}{2}\right)}\left(1-\frac{\mu x}{2\nu+1}+\frac{\left(2\nu+1+4\mu^2\right)x^2}{(8\nu+8)(4\nu+2)}+\frac{-\mu\left(6\nu+5+4\mu^2\right)x^3}{12(8\nu+12)(2\nu+1)(\nu+1)}+\frac{\left(12\nu^2+24\nu+9+48\mu^2\nu+56\mu^2+16\mu^4\right)x^4}{192(8\nu+16)(2\nu+1)(\nu+1)(2\nu+3)}+\frac{-\mu\left(60\nu^2+160\nu+89+80\mu^2\nu+120\mu^2+16\mu^4\right)x^5}{1920(8\nu+20)(2\nu+1)(\nu+1)(2\nu+3)(\nu+2)}+\left(\left(120\nu^3+540\nu^2+690\nu+225+720\mu^2\nu^2+2400\mu^2\nu+1756\mu^2+480\mu^4\nu+880\mu^4+64\mu^6\right)x^6\right)\Big/\left(46080(8\nu+24)(2\nu+1)(\nu+1)(2\nu+3)(\nu+2)(2\nu+5)\right)+\left(-\mu\left(840\nu^3+4620\nu^2+7518\nu+3429+1680\mu^2\nu^2+6720\mu^2\nu+6076\mu^2+672\mu^4\nu+1456\mu^4+64\mu^6\right)x^7\right)\Big/\left(645120(8\nu+28)(2\nu+1)(\nu+1)(2\nu+3)(\nu+2)(2\nu+5)(\nu+3)\right)\dots\right).$$

WM.2.1.2 General form.

(WM.2.1.2.1)
$$\mathrm{WM}_{\mu,\nu}(x) \approx x^{\left(\nu+\frac{1}{2}\right)}\sum_{n=0}^{\infty} u(n)x^n.$$

The coefficients $u(n)$ satisfy the recurrence

(WM.2.1.2.2)
$$u(n)\left(\left(\nu+\frac{1}{2}+n\right)^2-4\nu-1-4n-4\nu^2\right)+4u(n-1)\mu-u(n-2)=0.$$

Initial conditions of WM.2.1.2.2 are given by

(WM.2.1.2.3)
$$u(1)=\frac{-4\mu}{8\nu+4},\quad u(0)=1.$$

The recurrence WM.2.1.2.2 has the closed form solution

(WM.2.1.2.4)
$$u(n)=0.$$

WM.2.2 Asymptotic expansion at ∞.

WM.2.2.1 First terms.

$$\mathrm{WM}_{\mu,\nu}(x) \approx \mathrm{e}^{\frac{1}{2x}}\, x^{\mu} y_0(x),$$

where

$$y_0(x) = \frac{\Gamma(2\nu+1)}{\Gamma\left(\frac{1}{2}+\nu-\mu\right)} + \frac{-\left(4\nu^2-4\mu-4\mu^2-1\right)\Gamma(2\nu+1)x}{4\Gamma\left(\frac{1}{2}+\nu-\mu\right)} + \frac{\left(4\nu^2-12\mu-4\mu^2-9\right)\left(4\nu^2-4\mu-4\mu^2-1\right)\Gamma(2\nu+1)x^2}{32\Gamma\left(\frac{1}{2}+\nu-\mu\right)}$$
$$+ \left(-\left(4\nu^2-20\mu-4\mu^2-25\right)\left(4\nu^2-12\mu-4\mu^2-9\right)\right.$$
$$\left.\left(4\nu^2-4\mu-4\mu^2-1\right)\Gamma(2\nu+1)x^3\right) \Big/ \left(384\Gamma\left(\frac{1}{2}+\nu-\mu\right)\right) + 2\dots$$

WM.2.2.2 General form.

WM.2.2.2.1 Auxiliary function $y_0(x)$. The coefficients $u(n)$ of $y_0(x)$ satisfy the following recurrence

$$-4u(n)n+$$
$$u(n-1)\left(-4\nu^2+4\mu+4\mu^2-3+4n+8(n-1)\mu+4(n-1)^2\right)=0$$

whose initial conditions are given by

$$u(0) = \frac{\Gamma(2\nu+1)}{\Gamma\left(\frac{1}{2}+\nu-\mu\right)}$$

This recurrence has the closed form solution

$$u(n) = \left(\sin\left(\frac{\pi(1+2\nu-2\mu)}{2}\right)(-2)^n 2^n (-1)^n \Gamma\left(n-\nu+\frac{1}{2}+\mu\right)\right.$$
$$\left.\Gamma\left(n+\nu+\frac{1}{2}+\mu\right)\Gamma(2\nu+1)\right) \Big/ \left(4^n \pi \Gamma\left(\nu+\frac{1}{2}+\mu\right)\Gamma(n+1)\right).$$