SI Sine Integral

SI.1 Introduction

Let x be a complex variable of $\mathbb{C} \setminus \{\infty\}$.The function Sine Integral (noted Si) is defined by the following third order differential equation

(SI.1.1) $$x\frac{\partial y(x)}{\partial x} + 2\frac{\partial^2 y(x)}{\partial x^2} + x\frac{\partial^3 y(x)}{\partial x^3} = 0.$$

Although 0 is a singularity of SI.1.1, the initial conditions can be given by

(SI.1.2) $$\frac{\partial \frac{\mathrm{Si}(x)}{x}}{\partial x} = 1.$$

Related function: Cosine Integral

SI.2 Series and asymptotic expansions

SI.2.1 Asymptotic expansion at 0.

SI.2.1.1 First terms.

(SI.2.1.1.1) $$\mathrm{Si}(x) \approx x\left(1 - \frac{x^2}{18} + \frac{x^4}{600} - \frac{x^6}{35280} + \frac{x^8}{3265920} - \frac{x^{10}}{439084800} + \frac{x^{12}}{80951270400} - \frac{x^{14}}{19615115520000}\dots\right).$$

SI.2.1.2 General form.

(SI.2.1.2.1) $$\mathrm{Si}(x) \approx x\sum_{n=0}^{\infty} u(n)x^n.$$

The coefficients $u(n)$ satisfy the recurrence

(SI.2.1.2.2) $$u(n)(n+1)\left(-1 - n + (n+1)^2\right) + u(n-2)(-1+n) = 0.$$

Initial conditions of SI.2.1.2.2 are given by

(SI.2.1.2.3) $$u(1) = 0,\quad u(0) = 1,\quad u(2) = -\frac{1}{18}.$$

The recurrence SI.2.1.2.2 has the closed form solution

(SI.2.1.2.4) $$u(2n+1) = 0,\quad u(2n) = \frac{(-1)^n}{(2n+1)\Gamma(2n+2)}.$$

SI.2.2 Asymptotic expansion at ∞.

SI.2.2.1 First terms.

$$\operatorname{Si}(x) \approx ser_{\left[1,1,\left[\left[0,\left[\left[0,\frac{\pi}{2}\right]\right]\right]\right]\right]} + \mathrm{e}^{\left(-\frac{\mathrm{RootOf}_{\xi,2}(1+\xi^2)}{x}\right)} x y_1(x) + \mathrm{e}^{\left(-\frac{\mathrm{RootOf}_{\xi,1}(1+\xi^2)}{x}\right)} x y_2(x),$$

where

$$y_0(x) = terms_{\left[1,1,\left[\left[0,\left[\left[0,\frac{\pi}{2}\right]\right]\right]\right]\right]} + \dots$$

$$y_1(x) = -\frac{1}{2} - \frac{\mathrm{RootOf}_{\xi,2}\left(1+\xi^2\right)x}{2} - \\ - \left(-\frac{1}{4} - \frac{5\,\mathrm{RootOf}_{\xi,2}\left(1+\xi^2\right)^2}{4}\right)x^2 - -\Bigg(-\mathrm{RootOf}_{\xi,2}\left(1+\xi^2\right) + \\ 4\,\mathrm{RootOf}_{\xi,2}\left(1+\xi^2\right)\left(-\frac{1}{4} - \frac{5\,\mathrm{RootOf}_{\xi,2}\left(1+\xi^2\right)^2}{4}\right)\Bigg)x^3 + 2\dots$$

$$y_2(x) = -\frac{1}{2} - \frac{\mathrm{RootOf}_{\xi,1}\left(1+\xi^2\right)x}{2} - \\ - \left(-\frac{1}{4} - \frac{5\,\mathrm{RootOf}_{\xi,1}\left(1+\xi^2\right)^2}{4}\right)x^2 - -\Bigg(-\mathrm{RootOf}_{\xi,1}\left(1+\xi^2\right) + \\ 4\,\mathrm{RootOf}_{\xi,1}\left(1+\xi^2\right)\left(-\frac{1}{4} - \frac{5\,\mathrm{RootOf}_{\xi,1}\left(1+\xi^2\right)^2}{4}\right)\Bigg)x^3 + 2\dots$$

SI.2.2.2 General form.

SI.2.2.2.1 Auxiliary function $y_0(x)$. The auxiliary function $y_0(x)$ has the exact form

$$y_0(x) = \frac{\pi}{2}$$

SI.2.2.2.2 Auxiliary function $y_1(x)$. The coefficients $u(n)$ of $y_1(x)$ satisfy the following recurrence

$$2u(n)n + u(-1+n)\Big(-2\,\mathrm{RootOf}_{\xi,2}\left(1+\xi^2\right) - \\ 5\,\mathrm{RootOf}_{\xi,2}\left(1+\xi^2\right)(-1+n) - 3(-1+n)^2\,\mathrm{RootOf}_{\xi,2}\left(1+\xi^2\right)\Big) + \\ u(n-2)\left(8 - 5n - 4(n-2)^2 - (n-2)^3\right) = 0$$

whose initial conditions are given by

$$u(1) = \frac{-\mathrm{RootOf}_{\xi,2}\left(1+\xi^2\right)}{2}$$
$$u(0) = -\frac{1}{2}$$

SI.2.2.2.3 Auxiliary function $y_2(x)$. The coefficients $u(n)$ of $y_2(x)$ satisfy the following recurrence

$$2u(n)n + u(-1+n)\Big(-2\,\mathrm{RootOf}_{\xi,1}\big(1+\xi^2\big) - 5\,\mathrm{RootOf}_{\xi,1}\big(1+\xi^2\big)(-1+n) - 3(-1+n)^2\,\mathrm{RootOf}_{\xi,1}\big(1+\xi^2\big)\Big) + u(n-2)\big(8-5n-4(n-2)^2-(n-2)^3\big) = 0$$

whose initial conditions are given by

$$u(0) = -\frac{1}{2}$$
$$u(1) = \frac{-\,\mathrm{RootOf}_{\xi,1}\big(1+\xi^2\big)}{2}$$