LI2 Dilogarithm

LI2.1 Introduction

Let x be a complex variable of $\mathbb{C} \setminus \{0, \infty\}$.The function Dilogarithm (noted dilog) is defined by the following third order differential equation

$$\frac{\partial y(x)}{\partial x} - -(-1+3x)\frac{\partial^2 y(x)}{\partial x^2} - -\left(-x+x^2\right)\frac{\partial^3 y(x)}{\partial x^3} = 0. \qquad \text{(LI2.1.1)}$$

The initial conditions of LI2.1.1 at 0 are not simple to state, since 0 is a (regular) singular point.

LI2.2 Series and asymptotic expansions

LI2.2.1 Asymptotic expansion at 1.

LI2.2.1.1 First terms.

$$\mathrm{dilog}(x) \approx (x-1)\left(-\frac{5}{4} + \frac{x}{4} - \frac{(x-1)^2}{9} + \frac{(x-1)^3}{16} - \frac{(x-1)^4}{25} + \frac{(x-1)^5}{36} - \frac{(x-1)^6}{49} + \frac{(x-1)^7}{64} - \frac{(x-1)^8}{81} + \frac{(x-1)^9}{100} - \frac{(x-1)^{10}}{121} + \frac{(x-1)^{11}}{144} - \frac{(x-1)^{12}}{169} + \frac{(x-1)^{13}}{196} - \frac{(x-1)^{14}}{225} + \frac{(x-1)^{15}}{256} \dots\right). \qquad \text{(LI2.2.1.1.1)}$$

LI2.2.1.2 General form.

$$\mathrm{dilog}(x) \approx (x-1)\sum_{n=0}^{\infty} u(n)(x-1)^n. \qquad \text{(LI2.2.1.2.1)}$$

The coefficients $u(n)$ satisfy the recurrence

$$u(n)(n+1)^2 n + u(n-1)n^3 = 0. \qquad \text{(LI2.2.1.2.2)}$$

Initial conditions of LI2.2.1.2.2 are given by

$$u(0) = -1, \quad u(1) = \frac{1}{4}. \qquad \text{(LI2.2.1.2.3)}$$

The recurrence LI2.2.1.2.2 has the closed form solution

$$u(n) = -\frac{(-1)^n}{(n+1)^2}. \qquad \text{(LI2.2.1.2.4)}$$

LI2.2.2 Asymptotic expansion at ∞.

LI2.2.2.1 First terms.

$$\operatorname{dilog}(x) \approx \left(\frac{1-8\ln\left(\frac{1}{x}\right)}{64x^8} + \frac{1-7\ln\left(\frac{1}{x}\right)}{49x^7} + \frac{1-6\ln\left(\frac{1}{x}\right)}{36x^6} + \frac{1-5\ln\left(\frac{1}{x}\right)}{25x^5} - \frac{-4\ln\left(\frac{1}{x}\right)+1}{16x^4} + \frac{1-3\ln\left(\frac{1}{x}\right)}{9x^3} + \frac{1-2\ln\left(\frac{1}{x}\right)}{4x^2} + \frac{1-\ln\left(\frac{1}{x}\right)}{x} - \frac{\pi^2}{6} - \frac{\ln\left(\frac{1}{x}\right)^2}{2} \dots\right). \tag{LI2.2.2.1.1}$$

LI2.2.2.2 General form. The general form of is not easy to state and requires to exhibit the basis of formal solutions of ?? (coming soon).

LI2.2.3 Asymptotic expansion at 0.

LI2.2.3.1 First terms.

$$\operatorname{dilog}(x) \approx \left(-\frac{15}{64} + \frac{\ln(x)}{8}x^8 - -\left(-\frac{13}{49} - \frac{\ln(x)}{7}\right)x^7 - \left(-\frac{11}{36} - \frac{\ln(x)}{6}\right)x^6 - -\left(-\frac{9}{25} - \frac{\ln(x)}{5}\right)x^5 - -\left(-\frac{7}{16} - \frac{\ln(x)}{4}\right)x^4 - -\left(-\frac{5}{9} - \frac{\ln(x)}{3}\right)x^3 - -\left(-\frac{3}{4} - \frac{\ln(x)}{2}\right)x^2 - -\left(-1 - \ln(x)\right)x + \frac{\pi^2}{6} \dots\right). \tag{LI2.2.3.1.1}$$

LI2.2.3.2 General form. The general form of is not easy to state and requires to exhibit the basis of formal solutions of ?? (coming soon).