# HATN Inverse Hyperbolic Tangent

## HATN.1 Introduction

Let $x$ be a complex variable of $\mathbb{C} \setminus \{-1, 1\}$.The function Inverse Hyperbolic Tangent (noted arctanh) is defined by the following second order differential equation

(HATN.1.1)
$$2x\frac{\partial y(x)}{\partial x} + \left(x^2 - 1\right)\frac{\partial^2 y(x)}{\partial x^2} = 0.$$

The initial conditions of HATN.1.1 are given at 0 by

(HATN.1.2)
$$\operatorname{arctanh}(0) = 0,$$
$$\frac{\partial \operatorname{arctanh}(x)}{\partial x}(0) = 1.$$

Related function: Inverse Hyperbolic Cotangent

## HATN.2 Series and asymptotic expansions

### HATN.2.1 Taylor expansion at 0.

*HATN.2.1.1 First terms.*

(HATN.2.1.1.1)
$$\operatorname{arctanh}(x) = x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \frac{1}{9}x^9 + \frac{1}{11}x^{11} + \frac{1}{13}x^{13} + \frac{1}{15}x^{15} + \mathrm{O}\left(x^{16}\right).$$

*HATN.2.1.2 General form.*

(HATN.2.1.2.1)
$$\operatorname{arctanh}(x) = \sum_{n=0}^{\infty} u(n)x^n.$$

The coefficients $u(n)$ satisfy the recurrence

(HATN.2.1.2.2)
$$nu(n) - -(-n-2)u(n+2) = 0.$$

Initial conditions of HATN.2.1.2.2 are given by

(HATN.2.1.2.3)
$$u(1) = 1,$$
$$u(0) = 0.$$

### HATN.2.2 Asymptotic expansion at 1.

*HATN.2.2.1 First terms.*

(HATN.2.2.1.1)
$$\operatorname{arctanh}(x) \approx \left(\frac{i}{2}\pi + \frac{\ln(2)}{2} - \frac{x}{4} + \frac{1}{4} + \frac{(x-1)^2}{16} - \frac{(x-1)^3}{48} + \frac{(x-1)^4}{128} - \frac{(x-1)^5}{320} + \frac{(x-1)^6}{768} - \frac{(x-1)^7}{1792} + \frac{(x-1)^8}{4096} + \frac{\ln(x-1)}{2} \dots\right).$$

*HATN.2.2.2 General form.* The general form of is not easy to state and requires to exhibit the basis of formal solutions of ?? (coming soon).

**HATN.2.3 Asymptotic expansion at −1.**

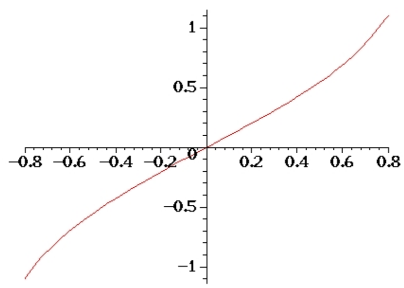
*HATN.2.3.1 First terms.*

$$\operatorname{arctanh}(x) \approx \left( \frac{-\ln(2)}{2} - \frac{x}{4} - \frac{1}{4} - \frac{(x+1)^2}{16} - \frac{(x+1)^3}{48} - \frac{(x+1)^4}{128} - \frac{(x+1)^5}{320} - \frac{(x+1)^6}{768} - \frac{(x+1)^7}{1792} - \frac{(x+1)^8}{4096} - \frac{\ln(x+1)}{2} \dots \right). \tag{HATN.2.3.1.1}$$

*HATN.2.3.2 General form.* The general form of is not easy to state and requires to exhibit the basis of formal solutions of ?? (coming soon).

## HATN.3 Graphs

**HATN.3.1 Real axis.**




**HATN.3.2 Complex plane.**