HACS Inverse Hyperbolic Cosine

HACS.1 Introduction

Let x be a complex variable of $\mathbb{C}\setminus\{\infty\}$.The function Inverse Hyperbolic Cosine (noted arccosh) is defined by the following second order differential equation

$$x\frac{\partial y(x)}{\partial x}+\left(x^2-1\right)\frac{\partial^2 y(x)}{\partial x^2}=0. \tag{HACS.1.1}$$

The initial conditions of HACS.1.1 are given at 0 by

$$\begin{aligned}\operatorname{arccosh}(0)&=\frac{i}{2}\pi,\\ \frac{\partial \operatorname{arccosh}(x)}{\partial x}(0)&=-i.\end{aligned} \tag{HACS.1.2}$$

Related functions: Inverse Cosine,Inverse Sine

HACS.2 Series and asymptotic expansions

HACS.2.1 Asymptotic expansion at -1.

HACS.2.1.1 First terms.

$$\begin{aligned}\operatorname{arccosh}(x)\approx(i\pi\ldots)+\sqrt{x+1}\Bigg(&-i\sqrt{2}-\frac{i}{12}(x+1)\sqrt{2}-\\ &\frac{3i}{160}(x+1)^2\sqrt{2}-\frac{5i}{896}(x+1)^3\sqrt{2}-\frac{35i}{18432}(x+1)^4\sqrt{2}-\\ &\frac{63i}{90112}(x+1)^5\sqrt{2}-\frac{231i}{851968}(x+1)^6\sqrt{2}-\frac{143i}{1310720}(x+1)^7\sqrt{2}\\ &-\frac{6435i}{142606336}(x+1)^8\sqrt{2}\ldots\Bigg).\end{aligned} \tag{HACS.2.1.1.1}$$

HACS.2.1.2 General form. The general form of is not easy to state and requires to exhibit the basis of formal solutions of ?? (coming soon).

HACS.2.2 Asymptotic expansion at 1.

HACS.2.2.1 First terms.

$$\operatorname{arccosh}(x) \approx \sqrt{x-1}\Bigg(\sqrt{2} - \frac{\sqrt{2}(x-1)}{12} + \frac{3\sqrt{2}(x-1)^2}{160} - \frac{5\sqrt{2}(x-1)^3}{896} + \frac{35\sqrt{2}(x-1)^4}{18432} - \frac{63\sqrt{2}(x-1)^5}{90112} + \frac{231\sqrt{2}(x-1)^6}{851968} - \frac{143\sqrt{2}(x-1)^7}{1310720} + \frac{6435\sqrt{2}(x-1)^8}{142606336} - \frac{12155\sqrt{2}(x-1)^9}{637534208} + \frac{46189\sqrt{2}(x-1)^{10}}{5637144576} - \frac{88179\sqrt{2}(x-1)^{11}}{24696061952} + \frac{676039\sqrt{2}(x-1)^{12}}{429496729600} - \frac{1300075\sqrt{2}(x-1)^{13}}{1855425871872} + \frac{5014575\sqrt{2}(x-1)^{14}}{15942918602752} - \frac{9694845\sqrt{2}(x-1)^{15}}{68169720922112} \dots\Bigg). \tag{HACS.2.2.1.1}$$

HACS.2.2.2 General form.

$$\operatorname{arccosh}(x) \approx \sqrt{x-1}\sum_{n=0}^{\infty} u(n)(x-1)^n. \tag{HACS.2.2.2.1}$$

The coefficients $u(n)$ satisfy the recurrence

$$2u(n)\left(n+\frac{1}{2}\right)n + u(n-1)\left(-\frac{1}{2}+n\right)^2 = 0. \tag{HACS.2.2.2.2}$$

Initial conditions of HACS.2.2.2.2 are given by

$$u(0) = \sqrt{2}. \tag{HACS.2.2.2.3}$$

The recurrence HACS.2.2.2.2 has the closed form solution

$$u(n) = \frac{2^{\left(n+\frac{1}{2}\right)}\Gamma\left(n+\frac{1}{2}\right)(-1)^n}{4^n\sqrt{\pi}\Gamma(n+1)(2n+1)}. \tag{HACS.2.2.2.4}$$

HACS.2.3 Asymptotic expansion at ∞.

HACS.2.3.1 First terms.

$$\operatorname{arccosh}(x) \approx \left(\ln(2) + \frac{1}{4x^2} + \frac{3}{32x^4} + \frac{5}{96x^6} + \frac{35}{1024x^8} + \ln\left(\frac{1}{x}\right)\dots\right). \tag{HACS.2.3.1.1}$$

HACS.2.3.2 General form. The general form of is not easy to state and requires to exhibit the basis of formal solutions of ?? (coming soon).

HACS.2.4 Taylor expansion at 0.

HACS.2.4.1 First terms.

$$\operatorname{arccosh}(x) = \frac{i}{2}\pi - ix - \frac{i}{6}x^3 - \frac{3i}{40}x^5 - \frac{5i}{112}x^7 - \frac{35i}{1152}x^9 - \frac{63i}{2816}x^{11} - \frac{231i}{13312}x^{13} - \frac{143i}{10240}x^{15} + \mathrm{O}\left(x^{16}\right). \tag{HACS.2.4.1.1}$$

HACS.2.4.2 General form.

$$\operatorname{arccosh}(x) = \sum_{n=0}^{\infty} u(n) x^n. \tag{HACS.2.4.2.1}$$

The coefficients $u(n)$ satisfy the recurrence

$$n^2 u(n) - -\left(-n^2 - 3n - 2\right) u(n+2) = 0. \tag{HACS.2.4.2.2}$$

Initial conditions of HACS.2.4.2.2 are given by

$$u(0) = \frac{i}{2}\pi, \quad u(1) = -i. \tag{HACS.2.4.2.3}$$

HACS.3 Graphs

HACS.3.1 Real axis.

HACS.3.2 Complex plane.