H2 Hankel H2

H2.1 Introduction

Let x be a complex variable of $\mathbb{C} \setminus \{0, \infty\}$ and let ν denote a parameter (independent of x).The function Hankel H2 (noted $\mathrm{H}_\nu^{(2)}$) is defined by the following second order differential equation

$$\left(x^2 - \nu^2\right) y(x) + x \frac{\partial y(x)}{\partial x} + x^2 \frac{\partial^2 y(x)}{\partial x^2} = 0. \tag{H2.1.1}$$

Although 0 is a singularity of H2.1.1, the initial conditions can be given by

$$\begin{aligned} \left[x^{(-\nu)}\right] \mathrm{H}_\nu^{(2)}(x) &= \frac{i\Gamma(\nu)}{\frac{\pi}{2^\nu}}, \\ \left[x^\nu\right] \mathrm{H}_\nu^{(2)}(x) &= \frac{i\,\mathrm{e}^{i\pi\nu}\,\Gamma(-\nu)}{\pi 2^\nu}. \end{aligned} \tag{H2.1.2}$$

The formulae of this document are valid for $2\nu \notin \mathbb{Z}$.
Related functions: Hankel H1,Bessel Y,Bessel J

H2.2 Series and asymptotic expansions

H2.2.1 Asymptotic expansion at ∞.

H2.2.1.1 First terms.

$$\mathrm{H}_\nu^{(2)}(x) \approx \mathrm{e}^{\left(-\frac{\mathrm{RootOf}_{\xi,1}(1+\xi^2)}{x}\right)} \sqrt{x}\, y_0(x),$$

where

$$\begin{aligned} y_0(x) = {} & \frac{\sqrt{2}\,\mathrm{e}^{\frac{i}{4}\pi(2\nu+1)}}{\sqrt{\pi}} - \frac{-\left(4\nu^2-1\right)\sqrt{2}\,\mathrm{e}^{\frac{i}{4}\pi(2\nu+1)}\,x}{8\sqrt{\pi}\,\mathrm{RootOf}_{\xi,1}\left(1+\xi^2\right)} + \\ & \frac{\left(4\nu^2-9\right)\left(4\nu^2-1\right)\sqrt{2}\,\mathrm{e}^{\frac{i}{4}\pi(2\nu+1)}\,x^2}{128\sqrt{\pi}\,\mathrm{RootOf}_{\xi,1}\left(1+\xi^2\right)^2} - \\ & \frac{-\left(4\nu^2-25\right)\left(4\nu^2-9\right)\left(4\nu^2-1\right)\sqrt{2}\,\mathrm{e}^{\frac{i}{4}\pi(2\nu+1)}\,x^3}{3072\sqrt{\pi}\,\mathrm{RootOf}_{\xi,1}\left(1+\xi^2\right)^3} + 2\ldots \end{aligned}$$

H2.2.1.2 General form.

H2.2.1.2.1 Auxiliary function $y_0(x)$. The coefficients $u(n)$ of $y_0(x)$ satisfy the following recurrence

$$8u(n)n\,\mathrm{RootOf}_{\xi,1}\left(1+\xi^2\right) + u(n-1)\left(-4\nu^2 - 3 + 4n + 4(n-1)^2\right) = 0$$

whose initial conditions are given by

$$u(0) = \frac{\sqrt{2}\,\mathrm{e}^{\frac{i}{4}\pi(2\nu+1)}}{\sqrt{\pi}}$$

This recurrence has the closed form solution

$$u(n) = \left(2^{\left(n+\frac{1}{2}\right)} \operatorname{RootOf}_{\xi,1}\left(1+\xi^2\right)^n \sin\left(\frac{\pi(2\nu+1)}{2}\right)\Gamma\left(n-\nu+\frac{1}{2}\right)\right.$$
$$\left.\Gamma\left(\nu+\frac{1}{2}+n\right)(-2)^n(-1)^n\,\mathrm{e}^{\frac{i}{4}\pi(2\nu+1)}\right)\Big/\left(8^n\Gamma(n+1)\pi^{\frac{3}{2}}\right).$$

H2.2.2 Asymptotic expansion at 0**.**

H2.2.2.1 First terms.

$$\mathrm{H}_\nu^{(2)}(x) \approx \left(\frac{i\Gamma(\nu)}{\frac{\pi}{2^\nu}} + \frac{ix^2\Gamma(\nu)}{\frac{4(\nu-1)\pi}{2^\nu}} + \frac{ix^4\Gamma(\nu)}{\frac{32(\nu-1)(\nu-2)\pi}{2^\nu}} + \frac{ix^6\Gamma(\nu)}{\frac{384(\nu-1)(\nu-2)(\nu-3)\pi}{2^\nu}} + \frac{ix^8\Gamma(\nu)}{\frac{6144(\nu-1)(\nu-2)(\nu-3)(\nu-4)\pi}{2^\nu}} \ldots\right)\Big/ x^\nu + x^\nu\left(\frac{i\,\mathrm{e}^{i\pi\nu}\,\Gamma(-\nu)}{\pi 2^\nu} - \frac{ix^2\,\mathrm{e}^{i\pi\nu}\,\Gamma(-\nu)}{4(\nu+1)\pi 2^\nu} + \frac{ix^4\,\mathrm{e}^{i\pi\nu}\,\Gamma(-\nu)}{32(\nu+1)(\nu+2)\pi 2^\nu} - \frac{ix^6\,\mathrm{e}^{i\pi\nu}\,\Gamma(-\nu)}{384(\nu+1)(\nu+2)(\nu+3)\pi 2^\nu} + \frac{ix^8\,\mathrm{e}^{i\pi\nu}\,\Gamma(-\nu)}{6144(\nu+1)(\nu+2)(\nu+3)(\nu+4)\pi 2^\nu} \ldots\right). \tag{H2.2.2.1.1}$$

H2.2.2.2 General form. The general form of is not easy to state and requires to exhibit the basis of formal solutions of ?? (coming soon).