H1 Hankel H1

H1.1 Introduction

Let x be a complex variable of $\mathbb{C} \setminus \{0, \infty\}$ and let ν denote a parameter (independent of x).The function Hankel H1 (noted $\mathrm{H}_{\nu}^{(1)}$) is defined by the following second order differential equation

$$(x^2 - \nu^2)y(x) + x\frac{\partial y(x)}{\partial x} + x^2\frac{\partial^2 y(x)}{\partial x^2} = 0. \tag{H1.1.1}$$

Although 0 is a singularity of H1.1.1, the initial conditions can be given by

$$\begin{aligned} \left[x^{(-\nu)}\right] \mathrm{H}_{\nu}^{(1)}(x) &= \frac{-i\Gamma(\nu)}{\frac{\pi}{2^{\nu}}}, \\ \left[x^{\nu}\right] \mathrm{H}_{\nu}^{(1)}(x) &= \frac{-i\,\mathrm{e}^{(-i\pi\nu)}\,\Gamma(-\nu)}{\pi 2^{\nu}}. \end{aligned} \tag{H1.1.2}$$

The formulae of this document are valid for $2\nu \notin \mathbb{Z}$.
Related functions: Hankel H2,Bessel Y,Bessel J

H1.2 Series and asymptotic expansions

H1.2.1 Asymptotic expansion at ∞.

H1.2.1.1 First terms.

$$\mathrm{H}_{\nu}^{(1)}(x) \approx \mathrm{e}^{\left(-\frac{\mathrm{RootOf}_{\xi,2}(1+\xi^2)}{x}\right)}\sqrt{x}y_0(x),$$

where

$$\begin{aligned} y_0(x) = &\frac{\sqrt{2}\,\mathrm{e}^{\left(-\frac{i}{4}\pi(2\nu+1)\right)}}{\sqrt{\pi}} - \frac{-\left(4\nu^2-1\right)\sqrt{2}\,\mathrm{e}^{\left(-\frac{i}{4}\pi(2\nu+1)\right)}\,x}{8\sqrt{\pi}\,\mathrm{RootOf}_{\xi,2}\left(1+\xi^2\right)} + \\ &\frac{\left(4\nu^2-9\right)\left(4\nu^2-1\right)\sqrt{2}\,\mathrm{e}^{\left(-\frac{i}{4}\pi(2\nu+1)\right)}\,x^2}{128\sqrt{\pi}\,\mathrm{RootOf}_{\xi,2}\left(1+\xi^2\right)^2} - \\ &\frac{-\left(4\nu^2-25\right)\left(4\nu^2-9\right)\left(4\nu^2-1\right)\sqrt{2}\,\mathrm{e}^{\left(-\frac{i}{4}\pi(2\nu+1)\right)}\,x^3}{3072\sqrt{\pi}\,\mathrm{RootOf}_{\xi,2}\left(1+\xi^2\right)^3} + \\ &2\ldots \end{aligned}$$

H1.2.1.2 General form.

H1.2.1.2.1 Auxiliary function $y_0(x)$. The coefficients $u(n)$ of $y_0(x)$ satisfy the following recurrence

$$8u(n)n\,\mathrm{RootOf}_{\xi,2}\left(1+\xi^2\right)+u(n-1)\left(-4\nu^2-3+4n+4(n-1)^2\right)=0$$

whose initial conditions are given by

$$u(0)=\frac{\sqrt{2}\,\mathrm{e}^{\left(-\frac{i}{4}\pi(2\nu+1)\right)}}{\sqrt{\pi}}$$

This recurrence has the closed form solution

$$u(n)=\left(\Gamma\left(\nu+\frac{1}{2}+n\right)2^{\left(n+\frac{1}{2}\right)}\,\mathrm{RootOf}_{\xi,2}\left(1+\xi^2\right)^n\Gamma\left(n-\nu+\frac{1}{2}\right)\right.$$
$$\left.\sin\left(\frac{\pi(2\nu+1)}{2}\right)(-2)^n(-1)^n\,\mathrm{e}^{\left(-\frac{i}{4}\pi(2\nu+1)\right)}\right)\Big/$$
$$\left(\Gamma(n+1)8^n\pi^{\frac{3}{2}}\right).$$

H1.2.2 Asymptotic expansion at 0.

H1.2.2.1 First terms.

$$\mathrm{H}_\nu^{(1)}(x)\approx\left(\frac{-i\Gamma(\nu)}{\frac{\pi}{2^\nu}}-\frac{ix^2\Gamma(\nu)}{\frac{4(\nu-1)\pi}{2^\nu}}-\frac{ix^4\Gamma(\nu)}{\frac{32(\nu-1)(\nu-2)\pi}{2^\nu}}-\right.$$
$$\frac{ix^6\Gamma(\nu)}{\frac{384(\nu-1)(\nu-2)(\nu-3)\pi}{2^\nu}}-\frac{ix^8\Gamma(\nu)}{\frac{6144(\nu-1)(\nu-2)(\nu-3)(\nu-4)\pi}{2^\nu}}$$
$$\left.\ldots\right)\Big/x^\nu+x^\nu\left(\frac{-i\,\mathrm{e}^{(-i\pi\nu)}\,\Gamma(-\nu)}{\pi2^\nu}+\frac{ix^2\,\mathrm{e}^{(-i\pi\nu)}\,\Gamma(-\nu)}{4(\nu+1)\pi2^\nu}-\right.\tag{H1.2.2.1.1}$$
$$\frac{ix^4\,\mathrm{e}^{(-i\pi\nu)}\,\Gamma(-\nu)}{32(\nu+1)(\nu+2)\pi2^\nu}+\frac{ix^6\,\mathrm{e}^{(-i\pi\nu)}\,\Gamma(-\nu)}{384(\nu+1)(\nu+2)(\nu+3)\pi2^\nu}-$$
$$\left.\frac{ix^8\,\mathrm{e}^{(-i\pi\nu)}\,\Gamma(-\nu)}{6144(\nu+1)(\nu+2)(\nu+3)(\nu+4)\pi2^\nu}\ldots\right).$$

H1.2.2.2 General form. The general form of is not easy to state and requires to exhibit the basis of formal solutions of ?? (coming soon).