ERFI Imaginary Error Function

ERFI.1 Introduction

Let x be a complex variable of $\mathbb{C}\setminus\{\infty\}$.The function Imaginary Error Function (noted erfi) is defined by the following second order differential equation

$$-2x\frac{\partial y(x)}{\partial x}+\frac{\partial^2 y(x)}{\partial x^2}=0. \tag{ERFI.1.1}$$

The initial conditions of ERFI.1.1 are given at 0 by

$$\begin{aligned}\operatorname{erfi}(0)&=0,\\ \frac{\partial\operatorname{erfi}(x)}{\partial x}(0)&=\frac{2}{\sqrt{\pi}}.\end{aligned} \tag{ERFI.1.2}$$

ERFI.2 Series and asymptotic expansions

ERFI.2.1 Taylor expansion at 0.

ERFI.2.1.1 First terms.

$$\begin{aligned}\operatorname{erfi}(x)=&\frac{2}{\sqrt{\pi}}x+\frac{2}{3\sqrt{\pi}}x^3+\frac{1}{5\sqrt{\pi}}x^5+\frac{1}{21\sqrt{\pi}}x^7+\frac{1}{108\sqrt{\pi}}x^9+\\ &\frac{1}{660\sqrt{\pi}}x^{11}+\frac{1}{4680\sqrt{\pi}}x^{13}+\frac{1}{37800\sqrt{\pi}}x^{15}+\mathrm{O}\left(x^{16}\right).\end{aligned} \tag{ERFI.2.1.1.1}$$

ERFI.2.1.2 General form.

$$\operatorname{erfi}(x)=\sum_{n=0}^{\infty}u(n)x^n. \tag{ERFI.2.1.2.1}$$

The coefficients $u(n)$ satisfy the recurrence

$$-2nu(n)+\left(n^2+3n+2\right)u(n+2)=0. \tag{ERFI.2.1.2.2}$$

Initial conditions of ERFI.2.1.2.2 are given by

$$\begin{aligned}u(0)&=0,\\ u(1)&=\frac{2}{\sqrt{\pi}}.\end{aligned} \tag{ERFI.2.1.2.3}$$

ERFI.2.2 Asymptotic expansion at ∞.

ERFI.2.2.1 First terms.

$$\operatorname{erfi}(x) \approx \mathrm{e}^{x^{(-2)}} x y_0(x) + ser_{\left[1,1,\left[\left[0,\left[[0,-i]\right]\right]\right]\right]},$$

where

$$y_0(x) = \pi^{\left(-\frac{1}{2}\right)} + \frac{x^2}{2\sqrt{\pi}} + 2\ldots$$
$$y_1(x) = terms_{\left[1,1,\left[\left[0,\left[[0,-i]\right]\right]\right]\right]} + \ldots$$

ERFI.2.2.2 General form.

ERFI.2.2.2.1 Auxiliary function $y_0(x)$. The coefficients $u(n)$ of $y_0(x)$ satisfy the following recurrence

$$-2nu(n) + u(n-2)\left(-4 + 3n + (n-2)^2\right) = 0$$

whose initial conditions are given by

$$u(0) = \pi^{\left(-\frac{1}{2}\right)}$$
$$u(1) = 0$$

This recurrence has the closed form solution

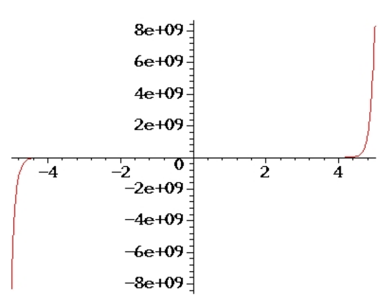
$$u(2n) = \frac{\Gamma\left(n + \frac{1}{2}\right)}{\pi},$$
$$u(2n+1) = 0.$$

ERFI.2.2.2.2 Auxiliary function $y_1(x)$. The auxiliary function $y_1(x)$ has the exact form

$$y_1(x) = -i$$

ERFI.3 Graphs

ERFI.3.1 Real axis.



ERFI.3.2 Complex plane.