

ERFC Complementary Error Function

ERFC.1 Introduction

Let x be a complex variable of $\mathbb{C} \setminus \{\infty\}$. The function Complementary Error Function (noted erfc) is defined by the following second order differential equation

$$(ERFC.1.1) \quad 2x \frac{\partial y(x)}{\partial x} + \frac{\partial^2 y(x)}{\partial x^2} = 0.$$

The initial conditions of ERFC.1.1 are given at 0 by

$$(ERFC.1.2) \quad \begin{aligned} \operatorname{erfc}(0) &= 1, \\ \frac{\partial \operatorname{erfc}(x)}{\partial x}(0) &= \frac{-2}{\sqrt{\pi}}. \end{aligned}$$

Related function: Error Function

ERFC.2 Series and asymptotic expansions

ERFC.2.1 Taylor expansion at 0.

ERFC.2.1.1 First terms.

$$(ERFC.2.1.1.1) \quad \begin{aligned} \operatorname{erfc}(x) &= 1 - \frac{2}{\sqrt{\pi}}x + \frac{2}{3\sqrt{\pi}}x^3 - \frac{1}{5\sqrt{\pi}}x^5 + \frac{1}{21\sqrt{\pi}}x^7 - \frac{1}{108\sqrt{\pi}}x^9 + \\ &\quad \frac{1}{660\sqrt{\pi}}x^{11} - \frac{1}{4680\sqrt{\pi}}x^{13} + \frac{1}{37800\sqrt{\pi}}x^{15} + O(x^{16}). \end{aligned}$$

ERFC.2.1.2 General form.

$$(ERFC.2.1.2.1) \quad \operatorname{erfc}(x) = \sum_{n=0}^{\infty} u(n)x^n.$$

The coefficients $u(n)$ satisfy the recurrence

$$(ERFC.2.1.2.2) \quad 2nu(n) + (n^2 + 3n + 2)u(n+2) = 0.$$

Initial conditions of ERFC.2.1.2.2 are given by

$$(ERFC.2.1.2.3) \quad \begin{aligned} u(1) &= \frac{-2}{\sqrt{\pi}}, \\ u(0) &= 1. \end{aligned}$$

ERFC.2.2 Asymptotic expansion at ∞ .

ERFC.2.2.1 First terms.

$$\operatorname{erfc}(x) \approx e^{\left(-\frac{1}{x^2}\right)} xy_0(x),$$

where

$$y_0(x) = \pi^{\left(-\frac{1}{2}\right)} - \frac{x^2}{2\sqrt{\pi}} + 2 \dots$$

ERFC.2.2.2 General form.

ERFC.2.2.2.1 Auxiliary function $y_0(x)$. The coefficients $u(n)$ of $y_0(x)$ satisfy the following recurrence

$$2nu(n) + u(n-2)(-4 + 3n + (n-2)^2) = 0$$

whose initial conditions are given by

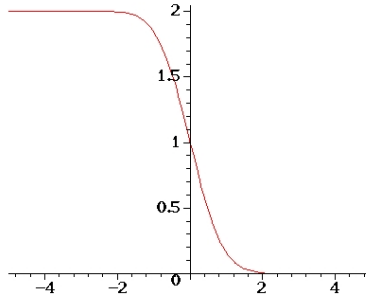
$$\begin{aligned} u(1) &= 0 \\ u(0) &= \pi^{\left(-\frac{1}{2}\right)} \end{aligned}$$

This recurrence has the closed form solution

$$\begin{aligned} u(2n+1) &= 0, \\ u(2n) &= \frac{(-1)^n \Gamma\left(n + \frac{1}{2}\right)}{\pi}. \end{aligned}$$

ERFC.3 Graphs

ERFC.3.1 Real axis.



ERFC.3.2 Complex plane.

